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CONTROLLABILITY OF NONLINEAR DEGENERATE PARABOLIC CASCADE SYSTEMS

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ABSTRACT. This article studies of null controllability property of nonlinear coupled one dimensional degenerate parabolic equations. These equations form a cascade system, that is, the solution of the first equation acts as a control in the second equation and the control function acts only directly on the first equation. We prove positive null controllability results when the control and a coupling set have nonempty intersection.

1. Introduction and statement of main results

The control of coupled parabolic system is a challenging issue, which has attracted the interest of the control community in the last decade. These parabolic systems arise, for example, in the study of chemical reactions (see e.g. [5, 12]), and in a wide variety of mathematical biology and physical situations (see e.g. [3, 20, 21]). For some examples involving degeneracy, let us recall some applications arising in aeronautics (the Crocco equation, see, e.g., [8]), in physics (boundary layer models, see e.g. [6]), in genetics (Wright-Fisher and Fleming-Viot models, see e.g. [24, 27]) and in mathematical finance (Black-Merton-Scholes models, see e.g. [11, 18, 23]). In [13, 22, 28], the authors developed a functional analytic approach to the construction of Feller semigroups generated by degenerate elliptic operators with Wentzell boundary conditions. In [17], the authors consider degenerate operators with several boundary conditions.

In [26], the authors studied the null controllability properties for two systems of coupled one dimensional degenerate parabolic equations. The first system consists of two forward equations, while the second one consists of one forward equation and one backward equation with $k(x) = x^\alpha$, $0 < \alpha < 2$.

In this article, we study the null controllability of a cascade system of nonlinear coupled one dimensional degenerate parabolic equations, at each fixed time $T > 0$. More exactly, we show that for all $y_0, z_0 \in L^2(\Omega)$ and $T > 0$, there exists a control $h \in L^2(\omega \times (0,T))$ such that the associated solution of (1.1) satisfies

$$y(\cdot,T) \in L^2(\Omega) \quad \text{and} \quad z(x,T) = 0 \quad \text{a.e. in } \Omega$$

In this context of degeneracy and nonlinearity of the system, we will study the null controllability of the linear degenerate coupled system by using the method

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developed in [8] and after, we will use Kakutani's fixed point theorem to deduce the null controllability of the nonlinear system.

We consider the nonlinear coupled system

$$\begin{gathered} y_t-(k(x)y_x)_x+f(y)=h1_\omega \quad \text{in } Q=\Omega\times(0,T),\\ z_t-(k(x)z_x)_x+g(z)=y1_{\mathcal{O}} \quad \text{in } Q,\\ y=z=0 \quad \text{on } \Sigma=\{0,1\}\times(0,T),\\ y(x,0)=y_0(x), \quad z(x,0)=z_0(x) \quad \text{in } \Omega=(0,1), \end{gathered} \tag{1.1}$$

where $f,g:\mathbb{R}\to\mathbb{R}$ are two locally Lipschitz-continuous functions, the initial data y_0,z_0 are given in $L^2(\Omega)$, $h\in L^2(Q)$ is a control function to be determined, k is a diffusion coefficient. Here, $\omega\subset\Omega$ is an arbitrarily small open control set, $\mathcal{O}\subset\Omega$ is an arbitrarily small open coupling set and 1_ω denotes the characteristic function of ω.

In this article, we assume that the coefficient k satisfies the following hypotheses:

$$\begin{gathered} k\in\mathcal{C}([0,1])\cap\mathcal{C}^1((0,1]),\ k>0 \text{ in } (0,1] \text{ and } k(0)=0, \text{ there exists}\\ \lambda\in[0,1) \text{ such that } xk'(x)\le\lambda k(x) \text{ for all } x\in[0,1]. \end{gathered} \tag{1.2}$$

The remainder of this article is organized as follows: In Section 2, we establish the uniqueness of the solution to (2.1). Section 3 is devoted to the proof of our general Carleman's inequality of degenerate cascade system. This inequality is crucial for the proof of the observability inequality that is used to prove the null controllability of the linear system. In the next two Sections, we prove the null controllability results for the linear system, and the last Section is devoted to the proof of Theorem 1.1 and Theorem 1.2, which are the main results of this paper.

Theorem 1.1. *We assume that* (1.2) *holds,* $\omega\cap\mathcal{O}\neq\emptyset$ *and* $f,g:\mathbb{R}\to\mathbb{R}$ *are two locally Lipschitz-continuous functions such that* $f(0)=g(0)=0$ *and that:*

$$\lim_{|s|\to+\infty}\frac{f(s)}{|s|\log^{3/2}(1+|s|)}=0,\quad \lim_{|\sigma|\to+\infty}\frac{g(\sigma)}{|\sigma|\log^{3/2}(1+|\sigma|)}=0.$$

Then, for any $(y_0,z_0)\in(H_k^2(\Omega))^2$ *and* $T>0$, *system* (1.1) *admits (at least) a solution* $y,z\in X=\mathcal{C}([0,T];L^2(\Omega))\cap L^2(0,T;H_k^1(\Omega))$ *and there exists a control* $h\in L^2(Q)$ *such that the solution* (y,z) *satisfies* $z(x,T)=0$ *a.e in* Ω.

Theorem 1.2. *We assume that* (1.2) *holds and* $\omega\cap\mathcal{O}\neq\emptyset$. *For any subset* ω' *such that* $\omega'\Subset\omega\cap\mathcal{O}$, *there exists two positive constants* $C=C(\Omega,\omega,k,\|a\|_\infty,\|b\|_\infty)$ *and* s_0 *such that for every* $s\ge s_0$, *one has*

$$\begin{aligned} &\int_Q\Big(s\theta k(x)|v_x|^2+s^3\theta^3\frac{x^2}{k(x)}|v|^2+s^4\theta^4k(x)|w_x|^2+s^6\theta^6\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\le Cs^7\int_0^T\int_{\omega'}\theta^7e^{2s\Phi}|w|^2. \end{aligned} \tag{1.3}$$

Theorem 1.2, improves the Carleman estimate established in [19] for parabolic degenerate equations. Our proofs use the Carleman estimate established in [19].

2. Well-posedness of the problem

Now, we introduce the Sobolev spaces

$$\begin{gathered} H_k^1(\Omega)=\big\{u\in L^2(\Omega); u \text{ absolute continuous in } [0,1],\\ \sqrt{k}u_x\in L^2(\Omega),\ u(0)=u(1)=0\big\}, \end{gathered}$$

$$H_k^2(\Omega) = \{u \in H_k^1(\Omega);\ k(x)u_x \in H^1(\Omega)\},$$

respectively with the norms

$$\|u\|^2_{H_k^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\sqrt{k}u_x\|^2_{L^2(\Omega)}, \quad \forall u \in H_k^1(\Omega);$$
$$\|u\|^2_{H_k^2(\Omega)} = \|u\|^2_{H_k^1(\Omega)} + \|(ku_x)_x\|^2_{L^2(\Omega)}, \quad \forall u \in H_k^2(\Omega).$$

Consider the unbounded operator $A : D(A) = H_k^2(\Omega) \to L^2(\Omega)$ defined by $Au = (k(x)u_x)_x$, $u \in D(A)$. Recall that the operator $Au = (ku_x)_x$, $u \in D(A) = H_k^2(\Omega) \subset H_k^1(\Omega)$, is a closed, symmetric, self-adjoint and negative operator in addition, $D(A)$ is dense in $L^2(\Omega)$ (see [1, 7, 8]). Moreover, it is infinitesimal generator of a strongly continuous semi-group. This permits to deduce the following result.

Theorem 2.1. *Under hypothesis* (1.2), *for all $a \in L^\infty(Q)$ and $f \in L^2(Q)$, the system*

$$\begin{gathered} y_t - (k(x)y_x)_x + ay = f \quad \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(x,0) = y_0(x) \text{ in } \Omega \end{gathered} \tag{2.1}$$

admits a unique solution $y \in X = \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H_k^1(\Omega))$ and we have

$$\sup_{[0,T]} \|y(t)\|^2_{L^2(\Omega)} + \int_0^T \|y(t)\|^2_{H_k^1(\Omega)}\,\mathrm{d}t \le C\Big(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{L^2(\Omega)}\Big). \tag{2.2}$$

If moreover $y_0 \in D(A)$ then, $y \in X_1 = H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H_k^2(\Omega)) \cap \mathcal{C}([0,T]; H_k^1(\Omega))$ and

$$\sup_{[0,T]} \|y(t)\|^2_{H_k^1(\Omega)} + \int_0^T (\|y_t\|^2_{L^2(\Omega)} + \|(ky_x)_x\|^2_{L^2(\Omega)})\,\mathrm{d}t \le C(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{H_k^1(\Omega)}). \tag{2.3}$$

Proof. The existence and uniqueness of the solution of system (1.1) comes from the theory of continuous semi-group see [7] and [25]. Moreover, the solution is

$$y(t) = T(t)y_0 + \int_0^t T(t-s)(h(s) - f(s))\,\mathrm{d}s, \tag{2.4}$$

where $(T(t))_{t\ge0}$ is the semigroup generated by the operator A.

Multiplying the first equation of (2.1) by y and integrating by parts then integrate on Ω, we obtain

$$\frac{d}{2dt}\|y(t)\|^2_{L^2(\Omega)} + \|\sqrt{k}y_x(t)\|^2_{L^2(\Omega)} = \int_\Omega y(f(t) - a(t)y(t))\,\mathrm{d}x. \tag{2.5}$$

Thus,

$$\begin{aligned} &\frac{d}{2dt}\|y(t)\|^2_{L^2(\Omega)} + \|\sqrt{k}y_x(t)\|^2_{L^2(\Omega)} \\ &\le (\frac{1}{2} + \|a(t)\|_{L^\infty(\Omega)})\|y(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|f(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.6}$$

Now, integrating this inequality on $(0,t)$ we find

$$\begin{aligned} &\frac{1}{2}\|y(t)\|^2_{L^2(\Omega)} + \int_0^t \|\sqrt{k}y_x(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\ &\le (\frac{T}{2} + \|a\|_{L^\infty(Q)})\int_0^t \|y(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{1}{2}\|f\|^2_{L^2(Q)} + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.7}$$

Using Gronwall's inequality, we obtain

$$\|y(t)\|^2_{L^2(\Omega)} \le e^{(T(T+2\|a\|_{L^\infty(Q)}))}(\|f(t)\|^2_{L^2(\Omega)} + \|y_0\|^2_{L^2(\Omega)}). \tag{2.8}$$

From (2.8) and by integrating (2.6) on $(0,T)$, we obtain

$$\begin{aligned}\|\sqrt{k}y_x\|^2_{L^2(Q)} \le \frac{1}{2}[1 + (T+2\|a\|_{L^\infty(Q)})e^{(T(T+2\|a\|_{L^\infty(Q)}))}]\\ \times (\|f\|^2_{L^2(Q)} + \|y_0\|^2_{L^2(\Omega)}).\end{aligned} \tag{2.9}$$

Thus, (2.8) and (2.9) give (2.2).

Now, multiplying the first equation of system (2.1), by $-(ky_x)_x$ and integrating by parts on Ω, we have

$$\begin{aligned}&\frac{d}{2dt}\|\sqrt{k}y_x(t)\|^2_{L^2(\Omega)} + \|(ky_x)_x(t)\|^2_{L^2(\Omega)}\\ &= \int_0^1 ((ky_x)_x(t))(a(t)y(t) - f(t))\\ &\le \frac{1}{2}\|(ky_x)_x(t)\|^2_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)} + \|a(t)\|^2_{L^\infty(\Omega)}\|y(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating the last inequality on $(0,T)$ and using (2.8), one gets

$$\begin{aligned}&\|\sqrt{k}y_x(T)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \|(ky_x)_x(t)\|^2_{L^2(\Omega)}\\ &\le \|f\|^2_{L^2(Q)} + \|\sqrt{k}y_x(0))\|^2_{L^2(\Omega)}\\ &\quad + \|a\|^2_\infty e^{(T(T+2\|a\|_{L^\infty(Q)}))}(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{L^2(\Omega)})\\ &\le (1 + \|a\|^2_{L^\infty(Q)}e^{(T(T+2\|a\|_{L^\infty(Q)}))})(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{H^1_k(\Omega)}),\end{aligned} \tag{2.10}$$

with $\|y_0\|^2_{H^1_k(\Omega)} = \|y_0\|^2_{L^2(\Omega)} + \|\sqrt{k}y_x(0))\|^2_{L^2(\Omega)}$. From (2.1), we have

$$y_t^2(t) = ((ky_x(t))_x - a(t)y(t) + f(t))^2 \le 3[((ky_x(t))_x)^2 + (a(t)y(t))^2 + f^2(t)]. \tag{2.11}$$

Hence, by integrating on Q and using the inequalities (2.8) and (2.10), we obtain

$$\int_0^T \|y_t(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C(1+\|a\|^2_\infty e^{(T(T+2\|a\|_{L^\infty(Q)}))})(\|f\|^2_{L^2(Q)} + \|y_0\|^2_{H^1_k(\Omega)}). \tag{2.12}$$

Now, (2.8), (2.10) and (2.12) give (2.3), for all $y_0 \in H^2_k(\Omega)$. $\square$

3. Carleman inequality for degenerate systems

The main result of this section is the following.

For $\omega = (a,b)$ let $\alpha = \frac{2a+b}{3}$, $\beta = \frac{a+2b}{3}$ and let $\rho \in \mathcal{C}^2(\mathbb{R})$ be such that $0 \le \rho \le 1$ and

$$\rho(x) = \begin{cases} 1 & \text{if } x \in (0,\alpha)\\ 0 & \text{if } x \in (\beta,1)\end{cases}$$

Let us define

$$\theta(t) = \frac{1}{(t(T-t))^4} \quad \forall t \in (0,T), \quad \psi(x) = c_1\Big(\int_0^x \frac{r}{k(r)}\,\mathrm{d}r - c_2\Big)$$

with $c_1 > 0$ and $c_2 > \frac{1}{k(1)(2-\lambda)}$. Let $\Psi(x) = e^{r\sigma(x)} - e^{2r\|\sigma\|_\infty}$ be where $r > 0$ and σ a function which satisfies: $\sigma \in \mathcal{C}^2([0,1])$, $\sigma > 0$ in Ω, $\sigma = 0$ on $\partial\Omega$ and $\sigma_x(x) \ne 0$ in $[0,1] \setminus \omega_0$ where ω_0 is an open set of Ω such that $\omega_0 \Subset \omega$.

Let us define $\Phi(x,t) = \theta(t)[\rho(x)\psi(x) + (1-\rho(x))\Psi(x)]$. The existence of the function σ is proved in [16]. Consider the adjoint system where $G \in L^2(Q)$:

$$\begin{gathered} w_t + (k(x)w_x)_x - aw = G \quad \text{in } Q, \\ w = 0 \quad \text{on } \Sigma, \\ w(x,T) = w_T(x) \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

We have the following result.

Theorem 3.1. *Under hypothesis* (1.2), *for all $T > 0$ and $l \in \mathbb{N}$, there exists two constants $C = C(\Omega, \omega, \|a\|_\infty) > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and all solutions w of* (3.1), *we have*

$$\begin{aligned} &\int_Q \Big(s^{l+1}\theta^{l+1}k(x)|w_x|^2 + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\Big(s^l\int_Q \theta^l e^{2s\Phi}|G|^2\,\mathrm{d}x\,\mathrm{d}t + s^{l+3}\int_0^T\int_\omega \theta^{l+3}e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t\Big). \end{aligned} \tag{3.2}$$

The proof of Theorem 3.1 will be given at the end of this section as a consequence of the following result. Consider the adjoint system

$$\begin{gathered} w_t + (k(x)w_x)_x = G \quad \text{in } Q, \\ w = 0 \quad \text{on } \Sigma, \\ w(x,T) = w_T(x) \quad \text{in } \Omega. \end{gathered} \tag{3.3}$$

We have

Theorem 3.2. *Under hypothesis* (1.2), *for all $T > 0$ and $l \in \mathbb{N}$, there exists two constants $C = C(\Omega, \omega) > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and all solutions w of* (3.3), *we have*

$$\begin{aligned} &\int_Q \Big(s^{l+1}\theta^{l+1}k(x)|w_x|^2 + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(s^l\int_Q \theta^l e^{2s\Phi}|G|^2\,\mathrm{d}x\,\mathrm{d}t + s^{l+3}\int_0^T\int_\omega \theta^{l+3}e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t). \end{aligned} \tag{3.4}$$

For the proof of Theorem 3.2 we follow the ideas of [1] and [19]. We prove first a Carleman inequality for the degenerate part and combine it with a classical Carleman inequality for the non degenerate part.

Proposition 3.3. *Under the hypothesis of theorem 3.1 and for all $l \in \mathbb{N}$, there exists two constants $C = C(\Omega, l)$ and s_0 such that for all $s \geq s_0$ and all solutions w of* (3.3), *we have the inequality*

$$\begin{aligned} &\int_Q \Big(s^{l-1}\theta^{l-1}(|(k(x)w_x)_x|^2 + |w_t|^2) + s^{l+1}\theta^{l+1}k(x)|w_x|^2 \\ &+ s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\Big(s^l\int_Q \theta^l e^{2s\varphi}|G|^2\,\mathrm{d}x\,\mathrm{d}t + k(1)s^{l+1}\int_0^T \theta^{l+1}e^{2s\varphi(1,t)}|w_x(1,t)|^2\,\mathrm{d}t\Big), \end{aligned} \tag{3.5}$$

where $\varphi(x,t) = \theta(t)\psi(x)$.

Proof. Let $y = (s\theta)^{1/2} e^{s\varphi} w$. We obtain from the first equation of (3.3):

$$P_s^+ y + P_s^- y = (s\theta)^{1/2} e^{s\varphi} f, \tag{3.6}$$

with

$$P_s^+ y = (ky_x)_x - s\varphi_t y + s^2\varphi_x^2 ky$$

$$P_s^- y = y_t - s(k\varphi_x)_x y - 2s\varphi_x k y_x - \frac{l}{2}\theta_t\theta^{-1} y$$

The inner product in $L^2(Q)$ gives

$$\|P_s^+ y\|^2 + \|P_s^- y\|^2 + 2\langle P_s^+ y, P_s^- y\rangle = \|(s\theta)^{1/2} e^{s\varphi} f\|^2.$$

The following result is useful for the proof of proposition 3.3.

Lemma 3.4. *There exists two constants $m > 0$ and $m' > 0$ such that*

$$\begin{aligned} \|(s\theta)^{1/2} e^{s\varphi} f\|^2 \geq \|P_s^+ y\|^2 + \|P_s^- y\|^2 + ms \int_Q \theta k(x)|y_x|^2 \\ + ms^3 \int_Q \theta^3 \frac{x^2}{k(x)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t - m' k(1) s \int_0^T \theta |y_x(1,t)|^2 \,\mathrm{d}t \end{aligned} \tag{3.7}$$

Proof. We have

$$\begin{aligned} &\langle P_s^+ y, P_s^- y\rangle \\ &= \frac{s}{2} \int_Q \psi(x)\theta_{tt}|y|^2 \,\mathrm{d}x\,\mathrm{d}t + sc_1 \int_Q \theta(2k(x) - xk'(x))|y_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{l}{2} \int_Q k(x)\theta_t\theta^{-1}|y_x|^2 \,\mathrm{d}x\,\mathrm{d}t - c_1^2 s(1 + s + \frac{l}{2}) \int_Q \theta_t\theta \frac{x^2}{k(x)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{sl}{2} \int_Q \theta_t^2\theta^{-1}\psi(x)|y|^2 \,\mathrm{d}x\,\mathrm{d}t + c_1^3 s^3 \int_Q \theta^3 (\frac{x}{k(x)})^2 (2k(x) - xk'(x))|y|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - c_1 s k(1) \int_0^T \theta |y_x(1,t)|^2 \,\mathrm{d}t. \end{aligned}$$

Using the conditions on k, one obtains

$$\begin{aligned} &\langle P_s^+ y, P_s^- y\rangle \\ &\geq \frac{s}{2} \int_Q \psi(x)\theta_{tt}|y|^2 \,\mathrm{d}x\,\mathrm{d}t + sc_1 \int_Q \theta k(x)|y_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{l}{2} \int_Q k(x)\theta_t\theta^{-1}|y_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - c_1^2 s(1 + s + \frac{l}{2}) \int_Q \theta_t\theta \frac{x^2}{k(x)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{sl}{2} \int_Q \theta_t^2\theta^{-1}\psi(x)|y|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + c_1^3 s^3 \int_Q \theta^3 \frac{x^2}{k(x)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t - c_1 s k(1) \int_0^T \theta |y_x(1,t)|^2 \,\mathrm{d}t. \end{aligned}$$

Notice that there exists a constant $C > 0$ such that $\theta_t\theta^{-1} \leq C\theta$, $\theta_t^2\theta^{-1} \leq T\theta^{3/2}$, $\theta_t\theta \leq C\theta^3$ and $\theta_{tt} \leq C\theta^{3/2}$. Then, we obtain the inequality

$$\big|\frac{s}{2}(1 + l) \int_Q \psi(x)(\theta_{tt} + \theta_t^2\theta^{-1}) y^2\big| \,\mathrm{d}x\,\mathrm{d}t \leq sC(l, T, c_1) \int_Q \theta^{3/2}|y|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3.8}$$

This gives by Höder's and Hardy's type inequalities (see [1])

$$\big|\frac{s}{2}(1 + l) \int_Q \psi(x)(\theta_{tt} + \theta_t^2\theta^{-1}) y^2\big| \,\mathrm{d}x\,\mathrm{d}t$$

$$\le s\delta_0 C(l,T,c_1)\int_Q \theta k(x)|y_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{C(l,T,c_1)}{\delta_0}s^3\int_Q \theta^3\frac{x^2}{k(x)}|y|^2\,\mathrm{d}x\,\mathrm{d}t.$$

We have

$$\Big|-c_1^2 s^2\big(1+\frac{1}{s}+\frac{l}{2s}\big)\int_Q \theta_t\theta\frac{x^2}{k(x)}y^2\Big|\,\mathrm{d}x\,\mathrm{d}t \le C(l,T,c_1)s^3\int_Q \theta^3\frac{x^2}{k(x)}|y|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Next,

$$\frac{l}{2}\int_Q \theta_t\theta^{-1}k(x)|y_x|^2\,\mathrm{d}x\,\mathrm{d}t \le sC(l,T,c_1)\int_Q \theta k(x)|y_x|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.9}$$

Using the inequalities above, we infer that there exists a constant $M>0$ such that

$$\begin{aligned}&2\langle P_s^+y, P_s^-y\rangle\\ &\ge M\int_Q (s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2)\,\mathrm{d}x\,\mathrm{d}t - 2c_1 sk(1)\int_0^T \theta|y_x(1,t)|^2\,\mathrm{d}t.\end{aligned}$$

As $\|(s\theta)^{1/2}e^{s\varphi}f\|^2 \ge 2\langle P_s^+y, P_s^-y\rangle$, one deduces inequality 3.7. □

Proof of Proposition 3.3 (continued). From (3.6), we obtain

$$\int_Q \frac{|(k(x)y_x)_x|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t \le C_0\int_Q (\frac{|P_s^+y|^2}{s\theta}+s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.10}$$

with $C_0>0$, and

$$\int_Q \frac{|y_t|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t \le C_1\int_Q (\frac{|P_s^-y|^2}{s\theta}+s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.11}$$

with $C_1>0$. Then, the lemma 3.4 the inequalities (3.10) and (3.11) give

$$\begin{aligned}&C\|(s\theta)^{l/2}e^{s\varphi}f\|^2\\ &\ge \int_Q \frac{|(k(x)y_x)_x|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{|y_t|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t + m'sk(1)\int_0^T \theta|y_x(1,t)|^2\,\mathrm{d}t\\ &\quad - m\int_Q (s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.12}$$

where, $C=\max(C_0,C_1)$, $m>0$ and $m'=2c_1C$.

Recalling now that $w=(s\theta)^{-l/2}e^{-s\varphi}y$, one obtains

$$(s\theta)^{l-1}e^{2s\varphi}|w_t|^2 \le 2\frac{|y_t|^2}{s\theta}+\frac{2}{s\theta}\Big(\frac{l}{2}\theta_t\theta^{-1}+s\varphi_t\Big)^2|y|^2$$

and $(s\theta)^{l+1}e^{2s\varphi}|w_x|^2 \le 2s\theta|y_x|^2+2s^3\theta\varphi_x^2|y|^2$. Then, from the Höder's and Hardy's inequalities and the previous inequalities, we obtain

$$\begin{aligned}&s^{l-1}\int_Q \theta^{l-1}e^{2s\varphi}|w_t|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2\int_Q \frac{|y_t|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t + c_0\int_Q \Big(s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&s^{l+1}\int_Q \theta^{l+1}k(x)e^{2s\varphi}|w_x|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le c_1\int_Q \Big(s\theta k(x)|y_x|^2+s^3\theta^3\frac{x^2}{k(x)}|y|^2\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.14}$$

$$\begin{aligned} & s^{l-1}\int_Q \theta^{l-1}e^{2s\varphi}|(k(x)w_x)_x|^2\,\mathrm{d}x\,\mathrm{d}t \\ & \le 2\int_Q \frac{|(k(x)y_x)_x|^2}{s\theta}\,\mathrm{d}x\,\mathrm{d}t + c_2 s^3\int_Q \theta^3\frac{x^2}{k(x)}|y|^2\,\mathrm{d}x\,\mathrm{d}t \\ & \quad + c_2 s\int_Q \theta k(x)|y_x|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

We have $(s\theta)^{l+1}e^{2s\varphi}|w_x(1,t)|^2 = s\theta|y_x(1,t)|^2$ since $y(1,t) = 0$. Combining the inequalities (3.12) to (3.15), we infer that the estimate (3.5) of the proposition 3.3 holds. □

Back to the proof of Theorem 3.2. We need the following result that we omit the proof (see [16] for instance).

Proposition 3.5. *We assume that $k \in \mathcal{C}^1([0,1])$ is a strictly positive function and $k(0) \neq 0$. Let $l \in \mathbb{N}$. Then, there exists two positive constants $C > 0$ and $s_0 > 0$ such that for all $s \ge s_0$ and every solution z of (3.3), we have*

$$\begin{aligned} & \int_Q \big((s\theta)^{l-1}|(k(x)z_x)_x|^2 + (s\theta)^{l-1}|z_t|^2 + (s\theta)^{l+1}|z_x|^2 + (s\theta)^{l+3}|z|^2\big)e^{2s\eta}\,\mathrm{d}x\,\mathrm{d}t \\ & \le C\Big(s^l\int_Q \theta^l e^{2s\eta}|g|^2\,\mathrm{d}x\,\mathrm{d}t + s^{l+3}\int_0^T\int_\omega \theta^{l+3}e^{2s\eta}|z|^2\,\mathrm{d}x\,\mathrm{d}t\Big), \end{aligned}$$

where $\eta(x,t) = \theta(t)\Psi(x)$.

Combining Propositions 3.3 and 3.5, it suffices to verify that Theorem 3.1 is true for system (3.3). Thus, by using these propositions, we find the following inequalities:

$$\begin{aligned} & \int_0^T\int_0^\alpha \Big(s^{l-1}\theta^{l-1}(|(k(x)w_x)_x|^2 + |w_t|^2) + s^{l+1}\theta^{l+1}k(x)|w_x|^2 \\ & + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ & \le C\Big(s^l\int_Q \theta^l e^{2s\varphi}|f|^2 + s^{l+3}\int_0^T\int_\alpha^\beta \theta^{l+3}e^{2s\varphi}(|w_x|^2 + |w|^2)\Big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} & \int_0^T\int_\beta^1 \Big(s^{l-1}\theta^{l-1}(|(k(x)w_x)_x|^2 + |w_t|^2) + s^{l+1}\theta^{l+1}k(x)|w_x|^2 \\ & + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ & \le C'\Big(s^l\int_Q \theta^l e^{2s\eta}|f|^2 + s^{l+3}\int_0^T\int_\alpha^\beta \theta^{l+3}e^{2s\eta}(|w_x|^2 + |w|^2)\Big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.17}$$

We recall that for every $x \in (\alpha,\beta)$, φ, η and Φ are equivalent. As $(\alpha,\beta) \Subset \omega$, Caccioppoli's type inequality (see [16, 9]) leads us to the inequality

$$s^{l+1}\int_Q \theta^{l+1}(|w_x|^2 + |w|^2)\,\mathrm{d}x\,\mathrm{d}t \le C''(s^l\int_Q \theta^l|f|^2 + s^{l+3}\int_\omega \theta^{l+3}|w|^2)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.18}$$

Thus, combining (3.15) and (3.16), and adding to both sides of the inequality the term

$$\int_0^T\int_\alpha^\beta \Big(s^{l-1}\theta^{l-1}(|(k(x)w_x)_x|^2+|w_t|^2)+s^{l+1}\theta^{l+1}k(x)|w_x|^2 \\ + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t,$$

and using (3.17) leads from any solution w of (3.3) to

$$\begin{aligned}&\int_Q (s^{l-1}\theta^{l-1}(|(k(x)w_x)_x|^2+|w_t|^2)\,\mathrm{d}x\,\mathrm{d}t+s^{l+1}\theta^{l+1}k(x)|w_x|^2\\ &+s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\Big(s^l\int_Q\theta^l e^{2s\Phi}|f|^2+s^{l+3}\int_0^T\int_\omega\theta^{l+3}e^{2s\Phi}|w|^2\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.19}$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.1. Now, applying inequality (3.19) to (3.1) with $\bar f=f+aw$ as right hand term, one obtains from the proof of Theorem 3.2, the inequality (3.2) of Theorem 3.1. □

4. Null controllability of a linear system

The aim of this section is to prove the null controllability for the linear system

$$\begin{gathered}y_t-(k(x)y_x)_x+ay=h1_\omega\quad\text{in } Q,\\ z_t-(k(x)z_x)_x+bz=y1_{\mathcal{O}}\quad\text{in } Q,\\ y=z=0\quad\text{on }\Sigma,\\ y(x,0)=y_0(x)\quad z(x,0)=z_0(x)\quad\text{in }\Omega,\end{gathered}\tag{4.1}$$

where $a,b\in L^\infty(Q)$, $y_0,z_0\in L^2(\Omega)$ and $h\in L^2(Q)$ is the control of the system. The adjoint of system (4.1) is

$$\begin{gathered}-w_t-(k(x)w_x)_x+aw=v1_{\mathcal{O}}\quad\text{in } Q,\\ -v_t-(k(x)v_x)_x+bv=0\quad\text{in } Q,\\ w=v=0\quad\text{on }\Sigma,\\ w(x,T)=0\;\; v(x,T)=v_T(x)\quad\text{in }\Omega.\end{gathered}\tag{4.2}$$

The main results of this section reads as follows.

Theorem 4.1. *Under hypothesis (1.2), $a,b\in L^\infty(Q)$ and for $y_0,z_0\in L^2(\Omega)$, there exists $h\in L^2(Q)$ such that the corresponding solution to system (4.1) satisfies $z(.,T)=0$ a.e. in Ω.*

First, we give some results useful for the proof of Theorem 4.1. For any $\varepsilon>0$, we set

$$J_\varepsilon(h)=\frac12\int_q\theta(t)^{-7}e^{-2s\Phi}h^2(x,t)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2\varepsilon}\int_\Omega z^2(x,T)\,\mathrm{d}x,\tag{4.3}$$

where $q=\omega\times(0,T)$. We have the following results.

Lemma 4.2. *For any $\varepsilon > 0$, there exists a control $h_\varepsilon \in L^2(q_\varepsilon)$ such that $J_\varepsilon(h_\varepsilon) \le J_\varepsilon(h)$, for all $h \in L^2(q)$. One has $h_\varepsilon(x,t) = \theta(t)^7 e^{2s\Phi} w(x,t) 1_\omega$. Moreover, we have the inequality*

$$\int_q \theta(t)^{-7} e^{-2s\Phi} h_\varepsilon^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega |y_0(x)w(x,0)|\,\mathrm{d}x + \int_\Omega |z_0(x)v(x,0)|\,\mathrm{d}x. \tag{4.4}$$

Proof. The continuity and the strict convexity of the functional J_ε are obvious. Moreover,

$$J_\varepsilon(h) \ge \frac{1}{2}\int_q \theta(t)^{-7} e^{-2s\Phi} h^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Let, $h \in L^2((0,T)\times\Omega)$ and consider (h_n) the sequence $h_n = h 1_{]1/n, T-1/n[}$. Then, $h_n(\cdot,t) \to h(\cdot,t)$, a.e. on $(0,1)\times(0,T)$ and we have $\|h_n\|^2_{L^2((0,1)\times(0,T))} \le \|h\|^2_{L^2((0,1)\times(0,T))}$, for all $n \in \mathbb{N}$. From the Lebesgue dominated convergence theorem,

$$\int_0^T \int_\omega h_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\omega h^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies

$$\int_0^T \int_\omega h_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \ge \frac{1}{2}\|h\|^2_{L^2(\omega\times(0,T))},$$

i.e.

$$\int_{1/n}^{T-1/n} \int_\omega h_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \ge \frac{1}{2}\|h\|^2_{L^2(\omega\times(0,T))}, \quad \forall n \ge n_0.$$

As, $\theta^{-1}(t)e^{-2s\Phi} \ge \eta > 0$ on $\omega\times]1/n, T-1/n[$, we have

$$\frac{1}{2}\|h\|^2_{L^2(\omega\times(0,T))} \le \frac{1}{\eta}\int_0^T \int_\omega \theta^{-1}(t)e^{-2s\Phi} h^2(x,t)\,\mathrm{d}x\,\mathrm{d}t, \quad \text{for } n \ge n_0.$$

Therefore,

$$J_\varepsilon(h) \ge \int_0^T \int_\omega \theta^{-1}(t)e^{-2s\Phi} h^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \ge \frac{\eta}{2}\|h\|^2_{L^2(q)}.$$

The functional J_ε is therefore coercive. As a consequence, there exists $h_\varepsilon \in L^2(q)$ such that $J_\varepsilon(h_\varepsilon) \le J_\varepsilon(h)$ for all $h \in L^2(q)$.

Now, using Euler's condition on J_ε, we obtain

$$\int_q \theta(t)^{-7} e^{-2s\Phi} h_\varepsilon(x,t) h(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{\varepsilon}\int_\Omega z_\varepsilon(x,T) z(x,T)\,\mathrm{d}x. \tag{4.5}$$

On the other hand, we consider the following two systems

$$\begin{gathered}
\overline{y}_t - (k(x)\overline{y}_x)_x + a\overline{y} = h1_\omega \quad \text{in } Q = \Omega\times(0,T),\\
\overline{z}_t - (k(x)\overline{z}_x)_x + b\overline{z} = \overline{y}1_{\mathcal{O}} \quad \text{in } Q,\\
\overline{y} = \overline{z} = 0 \quad \text{on } \Sigma = \{0,1\}\times(0,T),\\
\overline{y}(x,0) = 0 \quad \overline{z}(x,0) = 0 \quad \text{in } \Omega = (0,1),
\end{gathered} \tag{4.6}$$

and

$$\begin{gathered}
-w_t-(k(x)w_x)_x+aw=v1_{\mathcal{O}}\quad \text{in } Q=\Omega\times(0,T),\\
-v_t-(k(x)v_x)_x+bv=0\quad \text{in } Q,\\
w=v=0\quad \text{on } \Sigma=\{0,1\}\times(0,T),\\
w(x,T)=0\quad v(x,T)=-\frac{1}{\varepsilon}z(x,T)\quad \text{in } \Omega=(0,1).
\end{gathered}\tag{4.7}$$

Multiplying the two equations of system (4.6) by w and v respectively, integrating by parts on Q and using (4.5), we infer that

$$h_\varepsilon(x,t)=\theta(t)^7e^{2s\Phi}w_\varepsilon(x,t)1_\omega. \tag{4.8}$$

Multiplying the two equations of (4.7) by y_ε and z_ε associate to the control h_ε respectively and integrating on Q and using (4.8), we infer that

$$\begin{aligned}
&\int_q\theta(t)^{-7}e^{-2s\Phi}h_\varepsilon^2(x,t)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_\Omega z^2(x,T)\,\mathrm{d}x\\
&=\int_q\theta(t)^7e^{2s\Phi}w_\varepsilon^2(x,t)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_\Omega z^2(x,T)\,\mathrm{d}x,\\
&=-\int_\Omega y_0(x)w(x,0)\,\mathrm{d}x-\int_\Omega z_0(x)v(x,0)\,\mathrm{d}x
\end{aligned}\tag{4.9}$$

As a consequence, we obtain the estimate of the lemma 4.2. □

Proposition 4.3 (Observability inequality)**.** *For any $T>0$ and $s_0\geq s$, there exists a constant $C=C(T,\|a\|_\infty,\|b\|_\infty)$ such that*

$$\|w(\cdot,0)\|^2_{L^2(\Omega)}+\|v(\cdot,0)\|^2_{L^2(\Omega)}\leq C\int_q|w(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.10}$$

Moreover,

$$\|h_\varepsilon\|_{L^2(\Omega)}\leq C(\|y_0\|_{L^2(\Omega)}+\|z_0\|_{L^2(\Omega)}). \tag{4.11}$$

Proof. Multiplying the second equation of system (4.7) by v_t and integrating on $(0,1)$, we obtain

$$\begin{aligned}
&\int_\Omega v_t^2(x,t)\,\mathrm{d}x-\frac{d}{2dt}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x\\
&=\int_\Omega bv_t(x,t)v(x,t)\,\mathrm{d}x\\
&\leq\frac{\|b\|_\infty^2}{2}\int_\Omega v^2(x,t)\,\mathrm{d}x+\frac{1}{2}\int_\Omega v_t^2(x,t)\,\mathrm{d}x\quad\forall t\in[0,T].
\end{aligned}\tag{4.12}$$

The function $x\mapsto\frac{k(x)}{x^2}$ is non increasing on $(0,1]$, we obtain by using Hardy's inequality

$$\int_\Omega v^2(x,t)\,\mathrm{d}x\leq\frac{1}{k(1)}\int_\Omega k(x)(\frac{v^2(x,t)}{x})^2\,\mathrm{d}x\leq\frac{C}{k(1)}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x, \tag{4.13}$$

with $C>0$. Then, combining inequalities (4.12) and (4.13), we obtain

$$\frac{d}{dt}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x+\frac{C\|b\|_\infty^2}{k(1)}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x\geq 0. \tag{4.14}$$

This implies

$$\frac{d}{dt}\Big(e^{\frac{C\|b\|_\infty^2 t}{k(1)}}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x\Big)\geq 0\quad\forall t\in[0,T]. \tag{4.15}$$

Thus, the function

$$t \mapsto e^{\frac{C\|b\|_\infty^2 t}{k(1)}} \int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x$$

is increasing on $[0,T]$. We have

$$\frac{T}{4}\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x \le C\int_{T/2}^{3T/4}\int_\Omega k(x)v_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall t\in[0,T/2]. \tag{4.16}$$

Then

$$\int_\Omega k(x)v_x^2(x,t)\,\mathrm{d}x \le C(T,\|b\|_\infty)\int_{T/2}^{3T/4}\int_\Omega k(x)v_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Taking $s \ge C(\|b\|_\infty)T^8$ and $1 \le s\theta$, the latter inequality and (1.3) give

$$\int_\Omega v^2(x,t)\,\mathrm{d}x \le Cs^7\int_0^T\int_{\omega'}\theta(\tau)^7e^{2s\Phi}|w(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall t\in[0,T/2]. \tag{4.17}$$

Now, multiplying the first equation of (4.7) by w_t and integrating on $(0,1)$, we obtain

$$\begin{aligned}
&\int_\Omega w_t^2(x,t)\,\mathrm{d}x - \frac{d}{2dt}\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x\\
&= \int_\Omega w_t(x,t)(aw(x,t)-v(x,t))\,\mathrm{d}x\\
&\le \|a\|_\infty^2\int_\Omega w^2(x,t)\,\mathrm{d}x + \frac12\int_\Omega w_t^2(x,t)\,\mathrm{d}x + \int_\Omega v^2(x,t)\,\mathrm{d}x \quad \forall t\in[0,T].
\end{aligned} \tag{4.18}$$

Using again that the function $x \mapsto \frac{k(x)}{x^2}$ decreases on $(0,1]$, the inequality (4.17) and Hardy's inequality, we obtain that for all $t\in[0,T/2]$

$$\begin{aligned}
&\int_\Omega w_t^2(x,t)\,\mathrm{d}x - \frac{d}{dt}\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x\\
&\le Cs^7\int_0^T\int_{\omega'}\theta(\tau)^7e^{2s\Phi}|w(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{C\|a\|_\infty^2}{k(1)}\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x.
\end{aligned} \tag{4.19}$$

Hence,

$$-\frac{d}{dt}\Big(e^{\frac{C\|a\|_\infty^2 t}{k(1)}}\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x\Big) \le Ce^{\frac{C\|a\|_\infty^2 t}{k(1)}}s^7\int_0^T\int_{\omega'}\theta(\tau)^7e^{2s\Phi}|w(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau,$$

for all $t\in[0,T/2]$. Thus, for every $0\le s\le t\le T/2$, we obtain

$$\begin{aligned}
&\int_\Omega k(x)w_x^2(x,s)\,\mathrm{d}x\\
&\le C\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x + Cs^7\int_0^T\int_{\omega'}\theta(\tau)^7e^{2s\Phi}|w(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned} \tag{4.20}$$

for all $t\in[0,T/2]$. Integrating on $[T/4,T/2]$ with respect to the variable t, we obtain for all $s\le T/4$

$$\begin{aligned}
&\frac{T}{4}\int_\Omega k(x)w_x^2(x,s)\,\mathrm{d}x\\
&\le C\int_{T/4}^{T/2}\int_\Omega k(x)w_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + Cs^7\int_0^T\int_{\omega'}\theta(\tau)^7e^{2s\Phi}|w(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned} \tag{4.21}$$

for all $t \in [0, T/2]$. Taking $1 \leq (s\theta)^4$ for all $s \geq CT^8$, we find by using the inequality (1.3) that

$$\int_\Omega w^2(x,s)\,\mathrm{d}x \leq Cs^7 \int_0^T \int_{\omega'} \theta(t)^7 e^{2s\Phi} |w(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.22}$$

Combining this result with the inequality (4.17) for $s = t = 0$ we can conclude that

$$\int_\Omega (w^2(x,0) + v^2(x,0))\,\mathrm{d}x \leq Cs^7 \int_0^T \int_{\omega'} \theta(t)^7 e^{2s\Phi} |w(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.23}$$

□

Lemma 4.4. *Let $g \in \mathcal{C}(\overline{\Omega})$ be such that $-g(x) \geq m > 0$ for all $x \in \overline{\Omega}$. Let $\Phi(x,t) = \theta(t)g(x)$, for $(x,t) \in Q$ and $m_0 = \min_{\overline{\Omega}}(-g)$. Then*

$$s^7\theta(t)^7 e^{2s\Phi} \leq (\frac{7}{em_0})^7 \quad \textit{for any } s > \frac{7T^8}{2^9 m_0} \textit{ and } (x,t) \in Q. \tag{4.24}$$

Proof. We have $g(x) = \rho(x)\psi(x) + (1-\rho(x))\Psi(x)$, and for all $x \in (0,1)$, $-g(x) > 0$ and g is a continuous function on $[0,1]$. Thus, for every $x \in (0,1)$, there exists a constant $m > 0$, such that $-g(x) \geq m > 0$. We have $\theta(t) \geq (\frac{4}{T^2})^4$. Then $-g(x)\theta(t) \geq m_0(\frac{4}{T^2})^4$, for all $(x,t) \in Q$. For $s > \frac{7T^8}{2^9 m_0}$, we have $2s\Phi(x,t) \leq -7$ and we obtain

$$e^{2s\Phi(x,t)} \leq e^{-7}, \quad \text{for all } (x,t) \in Q. \tag{4.25}$$

Now, if we suppose that there exists a constant $A > 0$ such that $(s\theta(t))^7 \leq A^7$ for all $t \in (0,T)$ then, we obtain $s \leq A(\frac{T^2}{4})^4$ and as, $s > \frac{7T^8}{2^9 m_0}$, we can choose $A = 7/m_0$. With this choice of A, we obtain

$$(s\theta(t))^7 \leq (\frac{7}{m_0})^7, \quad \text{for any } s > \frac{7T^8}{2^9 m_0}. \tag{4.26}$$

Using the inequalities (4.25) and (4.26), we obtain

$$(s\theta(t))^7 e^{2s\Phi(x,t)} \leq (\frac{7}{em_0})^7, \quad \text{for any } s > \frac{7T^8}{2^9 m_0} \text{ and } (x,t) \in Q. \tag{4.27}$$

This completes the proof of the lemma 4.4. □

Proof of Proposition 4.3 (continued). From (4.23) and lemma 4.4, we obtain

$$\int_\Omega (w^2(x,0) + v^2(x,0))\,\mathrm{d}x \leq C\big(\frac{7}{em_0}\big)^7 \int_0^T \int_{\omega'} |w(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.28}$$

Now, from inequalities (4.4), (4.8), (4.28) and Cauchy-Schwarz's inequality one obtains

$$\begin{aligned}
&\int_q s^{-7}\theta(t)^{-7} e^{-2s\Phi} h_\varepsilon^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \|y_0\|_{L^2(\Omega)} \|w(\cdot,0)\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)} \|v(\cdot,0)\|_{L^2(\Omega)} \\
&\leq \frac{C}{2}(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)})^2 + \frac{1}{2}s^7 \int_0^T \int_{\omega'} \theta(t)^7 e^{2s\Phi} |w(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

This gives

$$\int_q s^{-7}\theta(t)^{-7} e^{-2s\Phi} |h_\varepsilon(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq 2C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)})^2. \tag{4.29}$$

We recall that $w_\varepsilon(x,t) = s^{-7}\theta(t)^{-7}e^{-2s\Phi}h_\varepsilon(x,t)$. Thus, using (1.3), (2.11) and (4.29), we obtain

$$\|h_\varepsilon\|^2_{L^2(q)} \le 2C\big(\frac{7}{em_0}\big)^7(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)})^2. \tag{4.30}$$

This completes the proof of the proposition 4.3. □

Proof of Theorem 4.1. Notice that the solution (y,z) of (4.1), can be decomposed as follows

$$y = \overline{y} + y^o, \quad z = \overline{z} + y^o,$$

where $(\overline{y},\overline{z})$, and (y^o,z^o) are the solutions of the systems

$$\begin{gathered}
\overline{y}_t - (k(x)\overline{y}_x)_x + a\overline{y} = h1_\omega \quad \text{in } Q = \Omega\times(0,T),\\
\overline{z}_t - (k(x)\overline{z}_x)_x + b\overline{z} = \overline{y}1_{\mathcal{O}} \quad \text{in } Q,\\
\overline{y} = \overline{z} = 0 \quad \text{on } \Sigma = \{0,1\}\times(0,T),\\
\overline{y}(x,0) = 0 \quad \overline{z}(x,0) = 0, \quad \text{in } \Omega = (0,1)
\end{gathered} \tag{4.31}$$

and

$$\begin{gathered}
y^o_t - (k(x)y^o_x)_x + ay^o = 0 \quad \text{in } Q = \Omega\times(0,T),\\
z^o_t - (k(x)z^o_x)_x + bz^o = y^o1_{\mathcal{O}} \quad \text{in } Q,\\
y^o = z^o = 0 \quad \text{on } \Sigma = \{0,1\}\times(0,T),\\
y^o(x,0) = y_0(x) \quad z^o(x,0) = z_0(x) \quad \text{in } \Omega = (0,1).
\end{gathered} \tag{4.32}$$

Let us define $L : L^2(Q) \to L^2(\Omega)\times L^2(\Omega)$ by $L(h) = (\overline{y}(x,T),\overline{z}(x,T))$ where $(\overline{y},\overline{z})$ is the solution corresponding to (4.31). Also let $M : L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)\times L^2(\Omega)$ be defined by

$$M(y_0,z_0) = (y^o(x,T), z^o(x,T)),$$

where $(y^o(x,T), z^o(x,T))$ is the corresponding solution to (4.32). Thus, Theorem 4.1 is equivalent to the inclusion

$$R(M) \subset R(L). \tag{4.33}$$

Both M and L are $L^2(\Omega)\times L^2(\Omega)$-valued, bounded linear operators. Consequently (4.33) holds if and only if, for every $(w_T,v_T) \in L^2(\Omega)\times L^2(\Omega)$ there exists a constant $C > 0$ such that

$$\|M^*(w_T,v_T)\|_{(L^2(\Omega))^2} \le C\|L^*(w_T,v_T)\|_{L^2(Q)}. \tag{4.34}$$

Now, by multiplying the two equations of (4.31) by w and v and integrating respectively, where (w,v) solves the adjoint system (4.2) and using the fact that these systems are duals, we obtain

$$L^*(w_T,v_T) = w1_\omega. \tag{4.35}$$

On the other hand, multiply the two equations of system (4.32) by w and v where (w,v) solves the adjoint system (4.2) and integrate respectively on Q, and by using the fact that these systems are duals, we obtain

$$M^*(w_T,v_T) = (w(x,0),(v(x,0)). \tag{4.36}$$

Hence, Theorem 4.1 is proved, since (4.33) is just (4.10). □

5. Proof of main results

Proof of Theorem 1.2. Let $\mathcal{O}' \Subset \omega \cap \mathcal{O}$. Applying Theorem 3.1 and taking $l = 3$ for the first equation of (4.2) and $l = 0$ for the second equation of the same system; we find the following inequalities:

$$\begin{aligned} &\int_Q (s^4\theta^4 k(x)|w_x|^2 + s^6\theta^6 \frac{x^2}{k(x)}|w|^2)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(s^3 \int_Q \theta^3 e^{2s\Phi}|v1_{\mathcal{O}}|^2\,\mathrm{d}x\,\mathrm{d}t + s^6 \int_0^T \int_{\mathcal{O}'} \theta^6 e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t), \end{aligned} \tag{5.1}$$

for all $s \geq s_1$; and

$$\begin{aligned} &\int_Q (s\theta k(x)|v_x|^2 + s^3\theta^3 \frac{x^2}{k(x)}|v|^2)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq Cs^3 \int_0^T \int_{\mathcal{O}'} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t, \quad \text{for all } s \geq s_2. \end{aligned} \tag{5.2}$$

Adding these two inequalities, one obtains that for every $s \geq s_1 + s_2$,

$$\begin{aligned} &\int_Q \Big(s\theta k(x)|v_x|^2 + s^3\theta^3 \frac{x^2}{k(x)}|v|^2 + s^4\theta^4 k(x)|w_x|^2 + s^6\theta^6 \frac{x^2}{k(x)}|w|^2\Big)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq Cs^3 \int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^3 \int_0^T \int_{\mathcal{O}'} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + Cs^6 \int_0^T \int_{\mathcal{O}'} \theta^6 e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.3}$$

We will absorb the term $\int_0^T \int_{\mathcal{O}'} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t$ by the term $\int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t$ by using the properties of the integral. We know that $\mathcal{O}' \Subset \mathcal{O}$ and therefore, there exists a constant $C_1 > 0$ such that

$$\int_0^T \int_{\mathcal{O}'} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_1 \int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, the inequality (5.3) gives

$$\begin{aligned} &\int_Q \Big(s\theta k(x)|v_x|^2 + s^3\theta^3 \frac{x^2}{k(x)}|v|^2 + s^4\theta^4 k(x)|w_x|^2 + s^6\theta^6 \frac{x^2}{k(x)}|w|^2)e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq Cs^3 \int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi}|v|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^6 \int_0^T \int_{\mathcal{O}'} \theta^6 e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.4}$$

Now, let $\mathcal{O}' \Subset \omega' \Subset \omega \cap \mathcal{O}$, we define and the function $\zeta \in \mathcal{C}^\infty(\Omega)$ as follows

$$\begin{gathered} 0 \leq \zeta(x) \leq 1 \quad \forall x \in \Omega \\ \zeta(x) = 1 \quad \text{if } x \in \mathcal{O}' \\ \zeta(x) = 0 \quad \text{if } x \in \Omega \setminus \omega' \end{gathered} \tag{5.5}$$

In addition it is assumed that

$$\frac{\nabla\zeta}{\zeta^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\Delta\zeta}{\zeta^{1/2}} \in L^\infty(\Omega). \tag{5.6}$$

Indeed, just take $\zeta = \zeta_0^4$ where $\zeta_0 \in \mathcal{C}^\infty(\Omega)$, to satisfy the two previous conditions on ζ.

We set $\chi = s^3\theta^3 e^{2s\Phi}$ and multiply the first equation of (4.2) by $\zeta\chi v$ and integrate the result on Q. We obtain

$$\begin{aligned}\int_Q v^2 1_{\mathcal{O}}\zeta\chi &= 2\int_Q k(x)v_x w_x \zeta\chi - \int_Q vw(k(x)(\zeta\chi)_x)_x \\ &\quad + \int_Q (a+b)vw\zeta\chi + \int_Q vw\zeta\chi_t, \\ \int_Q v^2 1_{\mathcal{O}}\zeta\chi &= \sum_{i=1}^4 I_i.\end{aligned}$$

Note that there exists a positive constant $C_2 > 0$ such that

$$|\chi_t| \le C_2 s^4\theta^5 e^{2s\Phi} \quad \text{and} \quad |\chi_x| \le C_2 s^4\theta^4 \frac{x}{k(x)} e^{2s\Phi} \tag{5.7}$$

and, using Höder's and Young's inequalities, one obtains

$$\begin{aligned} I_1 &= 2s^3 \int_Q k(x)\theta v_x w_x e^{2s\Phi}\zeta \\ &\le 2\delta_0 s\int_Q k(x)\theta|v_x|^2 e^{2s\Phi}\zeta + \frac{2}{\delta_0}s^5\int_Q k(x)\theta^5|w_x|^2 e^{2s\Phi}\zeta, \quad \delta_0 > 0.\end{aligned}$$

From inequality (5.2), one obtains

$$2\delta_0 s\int_Q k(x)\theta|v_x|^2 e^{2s\Phi}\zeta \le C\delta_0 s^3\int_0^T\int_{\mathcal{O}}\theta^3|v|^2 e^{2s\Phi}\zeta. \tag{5.8}$$

It results that

$$I_1 \le C\delta_0 s^3\int_0^T\int_{\mathcal{O}}\theta^3|v|^2 e^{2s\Phi}\zeta + \frac{2}{\delta_0}s^5\int_Q k(x)\theta^5|w_x|^2 e^{2s\Phi}\zeta, \tag{5.9}$$

$$I_3 = \int_Q (a+b)vw\zeta\chi \le \delta_1 s^3\int_Q \theta^3|v|^2 e^{2s\Phi}\zeta + \frac{\|a+b\|_\infty^2}{\delta_1}s^3\int_Q\theta^3|w|^2 e^{2s\Phi}\zeta, \tag{5.10}$$

with $\delta_1 > 0$. As the function $x \mapsto \frac{x^2}{k(x)}$ is increasing on $(0,1]$, we have

$$\frac{\|a+b\|_\infty^2}{\delta_1}s^3\int_Q\theta^3|w|^2 e^{2s\Phi}\zeta \le \frac{\|a+b\|_\infty^2}{k(1)\delta_1}s^3\int_0^T\theta^3\Big[\int_0^1 k(x)\zeta\Big(\frac{we^{s\Phi}}{x}\Big)^2\,\mathrm{d}x\Big]\,\mathrm{d}t.$$

Using Hardy's type inequality to the function $we^{s\Phi}$ with Fubini-Tonelli's theorem, we obtain

$$\begin{aligned}&\frac{\|a+b\|_\infty^2}{\delta_1}s^3\int_Q\theta^3|w|^2 e^{2s\Phi}\zeta \\ &\le \frac{C_4\|a+b\|_\infty^2}{k(1)\delta_1}s^3\int_Q\theta^3 k(x)\zeta(s^2\Phi_x^2|w|^2 + |w_x|^2)e^{2s\Phi})\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using that $\Phi_x = \frac{c_1\theta x}{k(x)}$, we have

$$\begin{aligned}&\frac{\|a+b\|_\infty^2}{\delta_1}s^3\int_Q\theta^3|w|^2 e^{2s\Phi}\zeta \\ &\le \frac{C_5\|a+b\|_\infty^2}{k(1)\delta_1}\int_Q\Big(s^5\theta^5\frac{x^2}{k(x)}|w|^2 + s^3\theta^3 k(x)|w_x|^2\Big)\zeta e^{2s\Phi}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From inequality (5.4), we find

$$\begin{aligned}
&\frac{\|a+b\|_\infty^2}{\delta_1} s^3 \int_Q \theta^3 |w|^2 e^{2s\Phi}\zeta \\
&\le \frac{C_5 C\|a+b\|_\infty^2}{k(1)\delta_1} s^3 \int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_5 C\|a+b\|_\infty^2}{k(1)\delta_1} s^6 \int_0^T \int_{\mathcal{O}'} \theta^6 e^{2s\Phi} |w|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

It results that

$$\begin{aligned}
I_3 &\le \delta_1 s^3 \int_Q \theta^3 |v|^2 e^{2s\Phi}\zeta + \frac{C_5 C\|a+b\|_\infty^2}{k(1)\delta_1} s^3 \int_0^T \int_{\mathcal{O}} \theta^3 e^{2s\Phi} |v|^2 \zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_5 C\|a+b\|_\infty^2}{k(1)\delta_1} s^6 \int_0^T \int_{\mathcal{O}'} \theta^6 e^{2s\Phi} |w|^2 \zeta \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
I_4 = \int_Q vw\zeta\chi_t \,\mathrm{d}x\,\mathrm{d}t &\le \delta_2 s^3 \int_Q \theta^3 |v|^2 e^{2s\Phi}\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_1^2}{\delta_2} s^5 \int_Q k(x)\theta^7 |w|^2 e^{2s\Phi}\zeta \,\mathrm{d}x\,\mathrm{d}t, \quad \delta_2 > 0
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
I_2 &= -\int_Q vw(k(x)(\zeta\chi)_x)_x \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_Q |k'(x)||vw||(\chi\zeta)_x| \,\mathrm{d}x\,\mathrm{d}t + \int_Q k(x)|vw||(\chi\zeta)_{xx}| \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.13}$$

Notice that

$$\begin{gathered}
k \in \mathcal{C}([0,1]) \cap \mathcal{C}^1((0,1]), \\
(\zeta\chi)_x = \zeta_x\chi + \zeta\chi_x \le C_2 s^4\theta^4 e^{2s\Phi}\zeta^{1/2}, \\
(\chi\zeta)_{xx} = \zeta_{xx}\chi + 2\zeta_x\chi_x + \zeta\chi_{xx} \le C_2 (s\theta)^5 e^{2s\Phi}\zeta^{1/2}.
\end{gathered} \tag{5.14}$$

Using inequality (5.14), one gets

$$\begin{aligned}
I_2^1 &= \int_Q |k'(x)||vw||(\chi\zeta)_x| \,\mathrm{d}x\,\mathrm{d}t \\
&\le C_3 s^4 \int_Q \theta^4 e^{2s\Phi} |wv| \zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t. \\
&\le \delta_3 s^3 \int_Q \theta^3 e^{2s\Phi} |v|^2 \zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t + \frac{C_3^2}{\delta_3} s^5 \int_Q \theta^5 |w|^2 e^{2s\Phi}\zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t, \quad \delta_3 > 0
\end{aligned}$$

and

$$\begin{aligned}
I_2^2 &= \int_Q |k(x)||vw||(\chi\zeta)_{xx}| \,\mathrm{d}x\,\mathrm{d}t \\
&\le C_3 s^5 \int_Q \theta^5 e^{2s\Phi} |wv| \zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t. \\
&\le \delta_4 s^3 \int_Q \theta^3 e^{2s\Phi} |v|^2 \zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t + \frac{C_3^2}{\delta_4} s^7 \int_Q \theta^7 |w|^2 e^{2s\Phi}\zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t, \quad \delta_4 > 0.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I_2 &\le (\delta_3+\delta_4)s^3\int_Q \theta^3 e^{2s\Phi}|v|^2\zeta^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_3^2}{\delta_3}s^5\int_Q \theta^5|w|^2 e^{2s\Phi}\zeta^{1/2}\,\mathrm{d}x\,\mathrm{d}t + \frac{C_3^2}{\delta_4}s^7\int_Q \theta^7|w|^2 e^{2s\Phi}\zeta^{1/2}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.15}
$$

By summing the I_i, for $1 \le i \le 4$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{4} I_i &\le (C\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\frac{C_5C\|a+b\|_\infty^2}{k(1)\delta_1})s^3\int_Q \theta^3|v|^2 e^{2s\Phi}\zeta\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{2}{\delta_0}s^5\int_Q k(x)\theta^5|w_x|^2 e^{2s\Phi}\zeta\,\mathrm{d}x\,\mathrm{d}t + \frac{C_1^2}{\delta_2}s^5\int_Q k(x)\theta^7|w|^2 e^{2s\Phi}\zeta\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_3^2}{\delta_3}s^5\int_Q \theta^5|w|^2 e^{2s\Phi}\zeta^{1/2}\,\mathrm{d}x\,\mathrm{d}t + \frac{C_3^2}{\delta_4}s^7\int_Q \theta^7|w|^2 e^{2s\Phi}\zeta^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_5C\|a+b\|_\infty^2}{k(1)\delta_1}s^6\int_0^T\int_{\mathcal{O}'}\theta^6 e^{2s\Phi}|w|^2\,\mathrm{d}x\,\mathrm{d}t\zeta\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.16}
$$

Now, we look to increase the term

$$
\frac{2}{\delta_0}s^5\int_Q k(x)\theta^5|w_x|^2 e^{2s\Phi}\zeta\,\mathrm{d}x\,\mathrm{d}t.
$$

To do this, we are going to multiply the first equation of system (4.2) by $s^5\theta^5 e^{2s\Phi}w\zeta$, and integrate on Q. To simplify we set $\overline{\chi} = s^5\theta^5 e^{2s\Phi}$,

$$
\begin{aligned}
&\int_Q v1_{\mathcal{O}}w\zeta\overline{\chi}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_Q (w\overline{\chi}\zeta)w_t\,\mathrm{d}x\,\mathrm{d}t - \int_Q (w\overline{\chi}\zeta)(k(x)w_x)_x\,\mathrm{d}x\,\mathrm{d}t + \int_Q a\overline{\chi}w^2\zeta\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\int_Q w^2\zeta\overline{\chi}_t\,\mathrm{d}x\,\mathrm{d}t + \int_Q k(x)\overline{\chi}\zeta w_x^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_Q w^2(k(\zeta\overline{\chi})_x)_x + \int_Q a\overline{\chi}\zeta w^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_Q k(x)\overline{\chi}\zeta|w_x|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_Q v1_{\mathcal{O}}w\zeta\overline{\chi}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q |w|^2|\zeta\overline{\chi}_t|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q |w|^2\big|(k(\zeta\overline{\chi})_x)_x\big|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q |a\overline{\chi}\zeta||w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{i=1}^{4} J_i.
\end{aligned}
$$

Using Höder-Young's inequality, one obtains

$$
\begin{aligned}
J_1 &= s^5\int_Q \theta^5 e^{2s\Phi}vw\zeta\,\mathrm{d}x\,\mathrm{d}t \\
&\le \gamma s^3\int_Q \theta^3 e^{2s\Phi}|v|^2\zeta\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma}s^7\int_Q \theta^7 e^{2s\Phi}|w|^2\zeta\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.17}
$$

Observe that $\overline{\chi}_t = (s^5\theta^4\theta_t e^{2s\Phi}(5+2s\theta h(x))$ because by definition the function Φ is of the form $\Phi(x,t)=\theta(t)h(x)$ where $h(x)=\rho(x)\psi(x)+(1-\rho(x))\Psi(x)$. Thus, $\overline{\chi}_t \le Cs^6\theta^7 e^{2s\Phi}$ for every $s \ge CT^8$. Therefore, for all $s \ge 1$ we have

$$J_2 \le Cs^6\int_Q \theta^7 e^{2s\Phi}|w|^2\zeta \,\mathrm{d}x\,\mathrm{d}t \le Cs^7\int_Q \theta^7 e^{2s\Phi}|w|^2\zeta \,\mathrm{d}x\,\mathrm{d}t, \tag{5.18}$$

$$J_4 = s^5\int_Q |a|\theta^5 e^{2s\Phi}\zeta|w|^2 \,\mathrm{d}x\,\mathrm{d}t \le \|a\|_\infty s^5\int_Q \theta^5 e^{2s\Phi}\zeta|w|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.19}$$

Observe also that $(\zeta\overline{\chi})_x \le C(s\theta)^6\frac{x}{k(x)}e^{2s\Phi}\zeta^{1/2}$ and from the linearity of the derivative function, we have

$$\begin{aligned} J_3 &= \int_Q w^2\,|(k(x)(\zeta\overline{\chi})_x)_x| \,\mathrm{d}x\,\mathrm{d}t \\ &\le C\int_Q |w|^2 k(x)|(\zeta\overline{\chi})_x| \,\mathrm{d}x\,\mathrm{d}t \\ &\le Cs^6\int_Q \theta^6|w|^2 e^{2s\Phi}\zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.20}$$

Let n,m be natural numbers such that for any $n \ge m$, one has

$$s^m\theta^m \le Cs^n\theta^n, \quad \forall s \ge CT^8. \tag{5.21}$$

In fact

$$s^m\theta^m = s^m\theta^n(\theta^{-1})^{n-m} \le s^m\theta^n(\frac{T^8}{16})^{n-m} \le s^m\theta^n(\frac{C}{16}s)^{n-m} = Cs^n\theta^n.$$

Since $s \ge CT^8$, using inequalities (5.17) to (5.20) and (5.21), we find

$$\begin{aligned} &\int_Q k(x)\overline{\chi}\zeta|w_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \gamma s^3\int_Q \theta^3 e^{2s\Phi}|v|^2\zeta \,\mathrm{d}x\,\mathrm{d}t + (\frac{1}{\gamma}+C\|a\|_\infty + C)s^7\int_Q \theta^7 e^{2s\Phi}|w|^2\zeta \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.22}$$

Finally, combining inequalities (5.16), (5.21), and (5.22), one gets

$$\begin{aligned} &s^3\int_Q \theta^3|v|^2 e^{2s\Phi}\zeta \,\mathrm{d}x\,\mathrm{d}t \\ &\le \Big(C\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\frac{2\gamma}{\delta_0}+\frac{C\|a+b\|_\infty^2}{k(1)\delta_1}\Big)s^3\int_Q \theta^3|v|^2 e^{2s\Phi}\zeta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \Big(\frac{C\|a+b\|_\infty^2}{k(1)\delta_1}+\frac{C}{\delta_2}+\frac{1}{\gamma}+C\|a\|_\infty + C\Big)s^7\int_Q \theta^7|w|^2 e^{2s\Phi}\zeta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \Big(\frac{C}{\delta_4}+\frac{C}{\delta_3}+\frac{2C}{\delta_0}\Big)s^7\int_Q \theta^7|w|^2 e^{2s\Phi}\zeta^{1/2} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.23}$$

We now set $\delta_2=\delta_3=\delta_4=\frac{C'\delta_0}{3}$, $\gamma=\frac{C'\delta_0^2}{2}$ and $\delta_1=\frac{1}{\delta_0}$, with

$$C' = \max(\frac{1}{4}, C, \frac{C\|a+b\|_\infty^2}{k(1)}, C\|a\|_\infty) \ge \frac{1}{4}.$$

Using that $\operatorname{supp}\zeta \subset \overline{\mathcal{O}'}$ we have

$$(-4C'\delta_0^2+\delta_0-1)s^3\int_0^T\int_{\mathcal{O}} \theta^3|v|^2 e^{2s\Phi}$$

$$\leq \frac{C'\delta_0^3 + 2C'\delta_0^2 + (6+2C')\delta_0 + 2}{\delta_0} s^7 \int_0^T \int_{\mathcal{O}'} \theta^7 |w|^2 e^{2s\Phi}.$$

The term $-4C'\delta_0^2 + \delta_0 - 1 > 0$ is equivalent to $16C' \geq 1$ and since $C' \geq \frac{1}{4} > \frac{1}{16}$, for δ_0 fixed, we obtain by taking $C = \frac{C'\delta_0^3+2C'\delta_0^2+(6+2C')\delta_0+2}{-\delta(C'\delta_0^2+\delta_0-1)}$

$$s^3 \int_0^T \int_{\mathcal{O}} \theta^3 |v|^2 e^{2s\Phi} \leq C s^7 \int_0^T \int_{\mathcal{O}'} \theta^7 |w|^2 e^{2s\Phi}. \tag{5.24}$$

Now, by multiplying the inequality (5.24) by C_0 and using inequalities (5.4) and (5.21), we obtain the inequality (1.3) of Theorem 1.2. □

Proof of Theorem 1.1. Consider the functions

$$F(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0 \\ f'(0) & \text{if } s = 0 \end{cases} \quad \text{and} \quad G(\sigma) = \begin{cases} \frac{g(\sigma)}{\sigma} & \text{if } \sigma \neq 0 \\ g'(0) & \text{if } \sigma = 0 \end{cases}$$

We need the following result.

Lemma 5.1. *Under the hypothesis of Theorem 1.1 and any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\begin{aligned} |F(s)|^{2/3} &\leq C_\varepsilon + \varepsilon log(1+|s|) \\ |G(\sigma)|^{2/3} &\leq C_\varepsilon + \varepsilon log(1+|\sigma|). \end{aligned} \tag{5.25}$$

Proof. Indeed, it will be sufficient to prove that for each $\varepsilon > 0$, one has $|F(s)| \leq C_\eta + \varepsilon \log^{3/2}(1+|s|)$ for all $s \in \mathbb{R}$. Let $\varepsilon > 0$. From $\lim_{|s|\to+\infty} \frac{f(s)}{|s|\log^{3/2}(1+|s|)} = 0$, there exists $s(\varepsilon) \geq 1$ such that

$$|\frac{f(s)}{s}| \leq \varepsilon \log^{3/2}(1+|s|) \quad \text{for all } |s| > s(\varepsilon). \tag{5.26}$$

On the other hand, using the fact that f is a locally Lipschitz-continuous function, there exists a constant C_η only depending on ε and f, such that $|f(s) - f(s')| \leq C_\eta |s - s'|$, for all $(s, s') \in [-s(\varepsilon), s(\varepsilon)]^2$. In particular, for $s \neq 0$ and $s' = 0$, we find

$$|f(s)| \leq C_\varepsilon |s| \Longrightarrow |\frac{f(s)}{s}| \leq C_\varepsilon. \tag{5.27}$$

From (5.26) and (5.27), we deduce that for all $s \in \mathbb{R}$, $|\frac{f(s)}{s}| \leq C_\varepsilon + \varepsilon \log^{3/2}(1+|s|)$. Therefore, we obtain $|F(s)|^{2/3} \leq C_\varepsilon + \varepsilon log(1+|s|)$, for all $s \in \mathbb{R}$ with $C_\varepsilon = (Ks(\varepsilon))^{2/3}$ depending only on ε and f, and $K > 0$ is a constant.

The same arguments, allow us to prove that $|G(s)|^{2/3} \leq C_\varepsilon + \varepsilon log(1+|s|)$, for all $s \in \mathbb{R}$. This completes the proof. □

Let $R > 0$ be a constant whose value will be determined below. We will use the truncation continuous function $T_R : \mathbb{R} \to \mathbb{R}$, given by

$$T_R(s) = \begin{cases} s & \text{if } |s| \leq R, \\ R \operatorname{sgn}(s) & \text{otherwise }. \end{cases}$$

For each $(\tilde{y},\tilde{z}) \in X_1$, we will consider the linear cascade system

$$\begin{gathered}
y_t - (ky_x)_x + F(T_R(\tilde{y}))y = h1_\omega \quad \text{in } Q, \\
z_t - (kz_x)_x + G(T_R(\tilde{z}))z = y1_{\mathcal{O}} \quad \text{in } Q, \\
y = z = 0 \quad \text{on } \Sigma, \\
y(x,0) = y_0(x), \quad z(x,0) = z_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{5.28}$$

Obviously, system (5.28) is of the same form as system (4.1), with $a_{\tilde{y}} = F(T_R(\tilde{y})) \in L^\infty(Q)$ and $b_{\tilde{z}} = G(T_R(\tilde{z})) \in L^\infty(Q)$ since F and G are continuous functions. Consequently, we can apply Theorem 4.1 to (5.28). In fact, we apply this result in an adequate (eventually smaller) time interval $(0,T_R)$, where

$$T_R = \min(T, \|a_{\tilde{y}}\|^{-1/2}, \|b_{\tilde{z}}\|^{-1/2}). \tag{5.29}$$

According to Theorem 4.1 and applying the density of $H^2_k(\Omega)$ in $L^2(\Omega)$, if y_0, $z_0 \in H^2_k(\Omega)$ then, there exists a control $\tilde{h} \in L^2(\Omega \times (0,T_R))$ such that the solution of (5.28) in $\Omega \times (0,T_R)$ satisfies $z(x,T_R) = 0$ in Ω and we have

$$\|\tilde{h}\|_{L^2(\omega\times(0,T_R)} \le C_0(\Omega,\omega,T_R,a_{\tilde{y}},b_{\tilde{z}})(\|y_0\|_{H^1_k(\Omega)} + \|z_0\|_{H^1_k(\Omega)}), \tag{5.30}$$

$$\|y\|_{X_R} + \|z\|_{X_R} \le C_1(\Omega,\omega,T_R,a_{\tilde{y}},b_{\tilde{z}})(\|y_0\|_{H^1_k(\Omega)} + \|z_0\|_{H^1_k(\Omega)}). \tag{5.31}$$

We now extend the functions $\tilde{h}$, y and z by zero to the whole Q. Denote such extensions again $\tilde{h}$, y and z respectively. Then, (y,z) lies in X_1 and solves the linearised system (5.28) in Q with control term $\tilde{h} \in L^2(Q)$, and satisfies $z(x,T) = 0$ in Ω. Moreover, we have the estimates

$$\|\tilde{h}\|_{L^2(\omega\times(0,T)} \le C_0(\Omega,\omega,T_R,a_{\tilde{y}},b_{\tilde{z}})(\|y_0\|_{H^1_k(\Omega)} + \|z_0\|_{H^1_k(\Omega)}), \tag{5.32}$$

$$\|y\|_{X} + \|z\|_{X} \le C_1(\Omega,\omega,T_R,a_{\tilde{y}},b_{\tilde{z}})(\|y_0\|_{H^1_k(\Omega)} + \|z_0\|_{H^1_k(\Omega)}). \tag{5.33}$$

For a fixed control $h \in L^2(Q)$, we now denote by (y_h,z_h) the solution of (5.28) associated to h and the potentials $a_{\tilde{y}}, b_{\tilde{z}}$. For any $(\tilde{y},\tilde{z}) \in X_1^2$, one defines the family of controls

$$\mathcal{A}_R(\tilde{y},\tilde{z}) = \{h \in L^2(Q) : (y_h,z_h) \in X_1^2, z_h(x,T) = 0 \text{ in } \Omega \text{ and } h \text{ satisfies (5.32)}\}.$$

Thus, we can introduce the multi-valued mapping

$$\Lambda_R\ (\tilde{y},\tilde{z}) \in X_1^2 \mapsto \Lambda_R(\tilde{y},\tilde{z}) \subset X_1^2,$$

where

$$\Lambda_R(\tilde{y},\tilde{z}) = \{(y_h,z_h) \in X_1^2, \text{ solution of (5.28), satisfying (5.33) and } h \in \mathcal{A}_R(\tilde{y},\tilde{z})\}.$$

We will prove that this mapping admits at least one fixed point (y,z). We will also prove that, for some R, every fixed point of Λ_R satisfie

$$\|y\|_{X_1} + \|z\|_{X_1} \le R. \tag{5.34}$$

Notice that Kakutani's fixed point theorem can be applied to Λ_R thus, ensuring the existence of (at least) one fixed point of Λ_R in X_1^2.

First, from the inequalities (5.32) and (5.33), we deduce that $\Lambda_R(\tilde{y},\tilde{z})$ is for every $(\tilde{y},\tilde{z})$ a nonempty set. Moreover, $\Lambda_R(\tilde{y},\tilde{z})$ is a closed and convex subset for any $(\tilde{y},\tilde{z})$. In fact, if (y,z) and (y',z') are two elements of $\Lambda_R(\tilde{y},\tilde{z})$, solutions of (5.28) associated to controls h and h' respectively. Let $\alpha \in [0,1]$. We are going to show that (Y,Z) with $Y=\alpha y+(1-\alpha)y'$ and $Z=\alpha z+(1-\alpha)z'$ and (Y,Z) is a solution of (5.28). Consider the following two systems

$$\begin{gathered} y_t-(ky_x)_x+F(T_R(\tilde{y}))y=h1_\omega \quad \text{in } (0,T_0)\times\Omega,\\ z_t-(kz_x)_x+G(T_R(\tilde{z}))z=y1_{\mathcal{O}} \quad \text{in } (0,T_0)\times\Omega,\\ y=z=0 \quad \text{on } \Sigma,\\ y(x,0)=y_0(x), \quad z(x,0)=z_0(x) \quad \text{in } \Omega, \end{gathered} \tag{5.35}$$

and

$$\begin{gathered} y'_t-(ky'_x)_x+F(T_R(\tilde{y}))y'=h'1_\omega \quad \text{in } (0,T_1)\times\Omega,\\ z'_t-(kz'_x)_x+G(T_R(\tilde{z}))z'=y'1_{\mathcal{O}} \quad \text{in } (0,T_1)\times\Omega,\\ y'=z'=0 \quad \text{on } \Sigma,\\ y'(x,0)=y'_0(x), \quad z'(x,0)=z'_0(x) \quad \text{in } \Omega. \end{gathered} \tag{5.36}$$

Since systems (5.35) and (5.36) admit solutions at any times T_0 and T_1 respectively, we can deduce that (Y,Z) is a solution of

$$\begin{gathered} Y_t-(kY_x)_x+F(T_R(\tilde{y}))Y=H1_\omega \quad \text{in } Q,\\ Z_t-(kZ_x)_x+G(T_R(\tilde{z}))Z=Y1_{\mathcal{O}} \quad \text{in } Q,\\ Y=Z=0 \quad \text{on } \Sigma,\\ Y(x,0)=Y_0(x), \quad Z(x,0)=Z_0(x) \quad \text{in } \Omega, \end{gathered} \tag{5.37}$$

where $H=\alpha h+(1-\alpha)h'$, $Y_0(x)=y_0(x)+y'_0(x)$ and $Z_0(x)=z_0(x)+z'_0(x)$. Consequently, with $T=\max(T_0,T_1)$, we have $Z(x,T)=0$ in Ω and the inequalities (5.32) and (5.33) are satisfied. We can deduce that $(Y,Z)\in\Lambda_R(\tilde{y},\tilde{z})$ for any $(\tilde{y},\tilde{z})\in X_1^2$. Then, $\Lambda_R(\tilde{y},\tilde{z})$ is the convex subset. On the other hand, from (5.33), $\Lambda_R(\tilde{y},\tilde{z})$ is bounded in X_1^2. Hence, Λ_R maps the whole space X_1^2 in a bounded subset of X_1^2. Now, let $K\subset X_1^2$ be a bounded set. Let us show that for any $(\tilde{y},\tilde{z})\in K$, $\Lambda_R(\tilde{y},\tilde{z})$ is a compact set of X_1^2. Thus, let $\{(y_n,z_n)\}$ be a sequence in $\Lambda_R(\tilde{y},\tilde{z})$. From inequality (5.33), (y_n,z_n) is bounded in X_1^2 there exists a sequence $\{h_n\}$ in $\mathcal{A}_R(\tilde{y},\tilde{z})$ and from (5.32), h_n is bounded in $L^2(Q)$. Thus, there exists a subsequence denoted again $\{(y_n,z_n)\}$ and $\{h_n\}$ such that

$$\begin{gathered} h_n\to h \quad \text{weakly in } L^2(Q),\\ (y_n,z_n)\to(y,z) \quad \text{strongly in } (\mathcal{C}([0,T];H_k^1))^2 \text{ and weakly in } X_1^2. \end{gathered} \tag{5.38}$$

Since (y_n,z_n) is a solution of the system

$$\begin{gathered} (y_n)_t-(k(y_n)_x)_x+F(T_R(\tilde{y}))y_n=(h_n)1_\omega \quad \text{in } Q,\\ (z_n)_t-(k(z_n)_x)_x+G(T_R(\tilde{z}))z_n=(y_n)1_{\mathcal{O}} \quad \text{in } Q,\\ y_n=z_n=0 \quad \text{on } \Sigma,\\ y_n(x,0)=y_0(x), \quad z_n(x,0)=z_0(x) \quad \text{in } \Omega, \end{gathered} \tag{5.39}$$

we conclude by passing to the limit that $(y,z)\in\Lambda_R(\tilde{y},\tilde{z})$ and is associated to the control $h\in\mathcal{A}_R(\tilde{y},\tilde{z})$ for any $(\tilde{y},\tilde{z})\in K$, as claimed. Thus, we can conclude that $\Lambda_R(K)=\cup\{\Lambda_R(\tilde{y},\tilde{z}):(\tilde{y},\tilde{z})\in K\}$ is relatively compact in X_1^2.

Let us now prove that the mapping $(\tilde{y},\tilde{z})\mapsto\Lambda_R(\tilde{y},\tilde{z})$ is upper hemicontinuous, i.e. that the real-valued function $(\tilde{y},\tilde{z})\in X_1^2\mapsto\sup_{(y,z)\in\Lambda_R(\tilde{y},\tilde{z})}\langle\mu,(y,z)\rangle$ is upper semicontinuous for each bounded linear form $\mu\in(X_1^2)'$. In other words, let us see that

$$B_{\alpha,\mu} = \{(\tilde{y},\tilde{z}) \in X_1^2 : \sup_{(y,z)\in\Lambda_R(\tilde{y},\tilde{z})} \langle \mu, (y,z)\rangle \geq \alpha\} \tag{5.40}$$

is a closed subset of X_1^2 for every $\alpha \in \mathbb{R}$ and every $\mu \in (X_1^2)'$. Thus, let $((\tilde{y_n},\tilde{z_n}))_n$ be a sequence in $B_{\alpha,\mu}$ such that $(\tilde{y_n},\tilde{z_n}) \to (\tilde{y},\tilde{z})$ in X_1^2. Our aim is to prove that $(\tilde{y},\tilde{z}) \in B_{\alpha,\mu}$. In view of the continuity of on F, G and T_R, we have

$$\begin{aligned} F(T_R(\tilde{y_n})) \to F(T_R(\tilde{y})) \quad \text{strongly in } L^\infty(Q),\\ G(T_R(\tilde{z_n})) \to G(T_R(\tilde{z})) \quad \text{strongly in } L^\infty(Q). \end{aligned} \tag{5.41}$$

Since all sets $\Lambda_R(\tilde{y_n},\tilde{z_n})$ are compact and satisfy (5.33), we deduce that

$$\alpha \leq \sup_{(y,z)\in\Lambda_R(\tilde{y_n},\tilde{z_n})} \langle \mu, (y,z)\rangle = \langle \mu, (y_n,z_n)\rangle \tag{5.42}$$

for some $(y_n,z_n) \in \Lambda_R(\tilde{y}_n,\tilde{z}_n)$. In fact, the mapping $(\tilde{y}_n,\tilde{z}_n) \mapsto \langle \mu,(y,z)\rangle$ is continuous on the compact set $\Lambda_R(\tilde{y}_n,\tilde{z}_n)$, this upper boundary is achieved; i.e. there exists $(y_n,z_n) \in \Lambda_R(\tilde{y}_n,\tilde{z}_n)$, such that $\sup_{(y,z)\in\Lambda_R(\tilde{y}_n,\tilde{z}_n)}\langle\mu,(y,z)\rangle = \langle\mu,(y_n,z_n)\rangle$, as claimed. From the definitions of $\Lambda_R(\tilde{y}_n,\tilde{z}_n)$ and $\mathcal{A}_R(\tilde{y}_n,\tilde{z}_n)$, there must exist a sequence $(h_n)_n \subset L^2(\omega\times(0,T))$ solution of the system

$$\begin{gathered} (y_n)_t - (k(y_n)_x)_x + F(T_R(\tilde{y_n}))y_n = (h_n)1_\omega \quad \text{in } Q,\\ (z_n)_t - (k(z_n)_x)_x + G(T_R(\tilde{z_n}))z_n = (y_n)1_{\mathcal{O}} \quad \text{in } Q,\\ y_n = z_n = 0 \quad \text{on } \Sigma,\\ y_n(x,0) = y_0(x), \quad z_n(x,0) = z_0(x) \quad \text{in } \Omega. \end{gathered}, \tag{5.43}$$

such that h_n and (y_n,z_n) satisfy the inequalities (5.32) and (5.33) respectively. Hence, (y_n,z_n) and h_n are uniformly bounded in X_1^2 and $L^2(Q)$ respectively. Therefore, we must write the following at least for a subsequence that we are going to denote (y_n,z_n) and h_n again respectively such that $(y_n,z_n) \to (\widehat{y},\widehat{z})$ strongly in X_1^2 and $h_n \to \widehat{h}$ weakly in $L^2(Q)$. Since the subsequence $((y_n,z_n))_n$ is a solution of (5.43), we check by passing to the limit that $(\widehat{y},\widehat{z}) \in \Lambda_R(\tilde{y},\tilde{z})$ and $\widehat{h} \in \mathcal{A}_R(\tilde{y}_n,\tilde{z}_n)$. Consequently, we can take the limit in (5.42) and deduce that $\alpha \leq \sup_{(y,z)\in\Lambda_R(\tilde{y},\tilde{z})}\langle\mu,(y,z)\rangle = \langle\mu,(\widehat{y},\widehat{z})\rangle$, this is to say $(\widehat{y},\widehat{z}) \in B_{\alpha,\mu}$. This proves that $(\tilde{y},\tilde{z}) \mapsto \Lambda_R(\tilde{y},\tilde{z})$ is upper hemicontinuous on X_1^2.

As a consequence, for any fixed $R > 0$, Kakutani's fixed point theorem can be applied ensuring the existence of a fixed point of Λ_R, i.e. that there exists $(y,z) \in X_1^2$ such that $(y,z) \in \Lambda_R(y,z)$.

Now, let us show the existence $R > 0$ such that $\|y\|_{L^\infty} + \|z\|_{L^\infty} \leq R$.

$$\begin{aligned} &\|y\|_{X_1} + \|z\|_{X_1}\\ &\leq C(1+\|a_{\tilde{y}}\|_\infty + \|b_{\tilde{z}}\|_\infty)e^{1/2CT(1+\|a_{\tilde{y}}\|_\infty^2+\|b_{\tilde{z}}\|_\infty^2)}(\|y_0\|_{H_k^1} + \|z_0\|_{H_k^1}),\\ &\leq e^{1/2CT(1+a_{\tilde{y}}\|_\infty+\|b_{\tilde{z}}\|_\infty+\|a_{\tilde{y}}\|_\infty^2+\|b_{\tilde{z}}\|_\infty^2)}(\|y_0\|_{H_k^1} + \|z_0\|_{H_k^1}). \end{aligned} \tag{5.44}$$

Since for any small $\varepsilon > 0$,

$$\begin{gathered} \|a_{\tilde{y}}\|_\infty = F(T_R(\tilde{y})) \leq C_\varepsilon + \varepsilon\log(1+R),\\ \|a_{\tilde{y}}\|_\infty^2 = F(T_R(\tilde{y}))^2 \leq C_\varepsilon + \varepsilon\log(1+R),\\ \|b_{\tilde{z}}\|_\infty = G(T_R(\tilde{y})) \leq C_\varepsilon + \varepsilon\log(1+R),\\ \|b_{\tilde{z}}\|_\infty^2 = G(T_R(\tilde{y}))^2 \leq C_\varepsilon + \varepsilon\log(1+R). \end{gathered}$$

Then (5.44), becomes

$$\begin{aligned}\|y\|_{X_1}+\|z\|_{X_1} &\leq e^{C(1+C_\varepsilon+\varepsilon\log(1+R))}(\|y_0\|_{H^1_k}+\|z_0\|_{H^1_k})\\ &\leq e^{C(1+C_\varepsilon)}(1+R)^{C_\varepsilon}(\|y_0\|_{H^1_k}+\|z_0\|_{H^1_k})\end{aligned}\tag{5.45}$$

with $C>0$. The fact that $X_1\hookrightarrow L^\infty(Q)$ (a consequence of [15, Theorem 5.4]), with continuous embedding then, taking $\varepsilon=(2c)^{-1}$ like in [19], we infer that

$$\|y\|_\infty+\|z\|_\infty\leq C(1+R)^{1/2}\big(\|y_0\|_{H^1_k}+\|z_0\|_{H^1_k}\big),\tag{5.46}$$

and we have $\|y\|_\infty+\|z\|_\infty\leq R$, for $R>0$ large enough. Thus, the proof of Theorem 1.1 is complete. □

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