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INFINITELY MANY SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS DEPENDING ON A PARAMETER

JUNTAO SUN, YONGBAO JI, TSUNG-FANG WU

Abstract. In this article, we study a Kirchhoff type problem with a positive parameter λ,

$$
\begin{gathered}
-K\Big(\int_\Omega |\nabla u|^2dx\Big)\Delta u = \lambda f(x,u), \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $K:[0,+\infty)\to\mathbb{R}$ is a continuous function and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a L^1-Carathéodory function. Under suitable assumptions on $K(t)$ and $f(x,u)$, we obtain the existence of infinitely many solutions depending on the real parameter λ. Unlike most other papers, we do not require any symmetric condition on the nonlinear term $f(x,u)$. Our proof is based on variational methods.

1. Introduction

Consider the Kirchhoff type problem

$$
\begin{gathered}
-K\Big(\int_\Omega |\nabla u|^2dx\Big)\Delta u = \lambda f(x,u), \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $\lambda>0$ is a real parameter, $K:[0,+\infty)\to\mathbb{R}$ is a continuous function, $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ is a nonempty bounded open set with a smooth boundary $\partial\Omega$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a L^1-Carathéodory function.

If $K(t)=a+bt$, then (1.1) is related to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$
u_{tt}-\Big(a+b\int_\Omega|\nabla u|^2dx\Big)\Delta u = g(x,u), \tag{1.2}
$$

where u denotes the displacement, $g(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. Equations of this type were suggested by Kirchhoff [16] in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. Equation (1.2) is often referred to as being nonlocal because of the presence of the integral over the entire domain Ω.

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We wish to point out that similar nonlocal problems also model several physical and biological systems. For example, a parabolic version of (1.2) can be used to describe the growth and movement of a particular species theoretically. The movement, simulated by the integral term, is assumed dependent on the energy of the entire system with u as its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which induces equations of the type

$$u_t - a\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = g(x,u).$$

Chipot-Lovat [12] and Corrêa [14] studied the existence of solutions and their uniqueness for such nonlocal problems and their corresponding elliptic equations.

Since Lions [19] introduced an abstract framework of (1.2), the solvability of (1.2) has been well-studied in general dimensions and domains by various researchers [3, 5, 9]. More precisely, Arosio-Panizzi [5] studied the Cauchy-Dirichlet type problem related to (1.2) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. D'Ancona-Spagnolo [3] obtained the existence of a global classical periodic solution for the degenerate Kirchhoff equation with real analytic data.

Compared with (1.2), its stationary equation, including both the case of bounded domain and the case of unbounded domain, has received more attention. We refer to [1, 2, 4, 6, 7, 10, 11, 13, 15, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29] and the references therein. The research of these papers mainly involve the existence of positive solutions, ground state solutions and multiplicity of positive solutions under various assumptions on $K(t)$ and $f(x,u)$. Let us briefly comment some known results related to our paper.

Alves-Corrêa-Ma [2] studied (1.1) with $\lambda = 1$ and found the conditions of $K(t)$ and $f(x,u)$ that permit the existence of a positive solution. That is, $K(t)$ does not grow too fast in a suitable interval near zero and $f(x,u)$ is locally Lipschitz subject to some prescribed criteria.

Chen-Kuo-Wu [10] studied the following Kirchhoff type problem with concave and convex nonlinearities

$$\begin{gathered} -K\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \mu f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $1 < q < 2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1,2$), $K(t) = a + bt$, $a, b, \mu > 0$ are parameters and the weight functions $f, g \in C(\bar\Omega)$ are allowed to be changing-sign. Using the Nehari manifold method and fibering map, several results on the existence and multiplicity of positive solutions for (1.3) are obtained. It is worth noting that they illustrated the difference in the solution behavior which arises from the consideration of the nonlocal effect.

Ricceri [25] investigated the Kirchhoff type problem with two parameters

$$\begin{gathered} -K\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

Under rather general assumptions on $K(t)$ and $f(x,u)$, he proved that for each $\lambda > \lambda^* > 0$ and for each Carathéodory function $g(x,u)$ with a sub-critical growth, (1.4) admits at least three weak solutions for every $\mu \geq 0$ small enough. Later, using a recent three critical points theorem due to Ricceri [24], Graef-Heidarkhani-Kong [15] improved the results in [25] and also obtained the existence of three weak solutions for (1.4) under some appropriate hypotheses, depending on two real parameters.

Motivated by the above works, in the present paper we shall establish some results on the existence of infinitely many solutions for (1.1). In our results neither symmetric nor monotonic condition on the nonlinear term is assumed. We require that $f(x,u)$ have a suitable oscillating behavior either at infinity or at zero on u. For the first case, we obtain an unbounded sequence of solutions; for the second case, we get a sequence of non-zero solutions strongly converging at zero. It is worth emphasizing that we extend and improve the results in [17].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are introduced. The main results and their proofs are presented in Section 3.

2. Preliminaries

Let $H_0^1(\Omega)$ be the usual Sobolev space endowed with norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}.$$

Throughout this paper, we denote the best Sobolev constant by S_r for the imbedding of $H_0^1(\Omega)$ into $L^r(\Omega)$ with $2 \leq r < 2^*$ and define it by

$$S_r = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|}{|u|_r},$$

where $|\cdot|_r$ stands for the classical L^r norm. We recall that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}$;
(b) the mapping $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1(\Omega)$ such that
$$\sup_{|u|\leq\rho} |f(x,u)| \leq l_\rho(x)$$
for almost every $x \in \Omega$.

We shall prove our results by applying the following smooth version of [8, Theorem 2.1], which is a more precise version of Ricceri's Variational Principle [23, Lemma 2.5].

Theorem 2.1. *Let E be a reflexive real Banach space, let $\Phi, \Psi : E \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semi-continuous. For every $r > \inf_E \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_E \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) For every $r > inf_E \Phi$ and every $\lambda \in (0, 1/\varphi(r))$; the restriction of the functional

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in E.

(b) If $\gamma < +\infty$; then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b1) *I_λ possesses a global minimum, or*

(b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$; then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

3. Main results

The following theorem is our first result.

Theorem 3.1. *Let $K : [0, +\infty) \to \mathbb{R}$ be a continuous function and let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that the following conditions hold:*

(A1) *there exists a constant $m > 0$ such that $\inf_{t\geq 0} K(t) \geq m$;*

(A2) *there exist constants $m_1 > 0$ and $m_2 \geq 0$ such that*

$$\overline{K}(t) \leq m_1 t + m_2, \quad \text{for every } t \in [0, +\infty),$$

where $\overline{K}(t) = \int_0^t K(s)ds$ for $t \geq 0$;

(A3) *there exist two constants $a_1 > 0$, $a_2 \geq 0$ and $1 < \alpha < 2$ such that*

$$|f(x, u)| \leq a_1|u| + a_2|u|^{\alpha-1}, \quad \text{for all } x \in \Omega \text{ and } u \in \mathbb{R};$$

(A4) *there exist $x_0 \in \Omega$ and three constants $\tau > \sigma > 0$ and*

$$\gamma > \frac{a_1 m_1(|B(x_0, \tau)| - |B(x_0, \sigma|)}{mS_2^2(\tau - \sigma)^2|B(x_0, \sigma)|} > 0, \tag{3.1}$$

such that

$$\begin{gathered} B(x_0, \sigma) \subset B(x_0, \tau) \subseteq \Omega,\\ F(x, u) \geq 0, \quad \text{for all } (x, u) \in (\Omega \setminus B(x_0, \sigma)) \times \mathbb{R},\\ F(x, u) \geq \gamma u^2, \quad \text{for all } (x, u) \in B(x_0, \sigma) \times [1, +\infty), \end{gathered}$$

where

$$F(x, u) = \int_0^u f(x, s)ds, \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

and $B(x_0, \sigma)$ denotes the open ball with center at x_0 and radius σ.

Then for every

$$\lambda \in \Big(\frac{m_1(|B(x_0, \tau)| - |B(x_0, \sigma)|)}{2\gamma(\tau - \sigma)^2|B(x_0, \sigma)|}, \frac{mS_2^2}{2a_1}\Big),$$

Equation (1.1) *has a sequence of solutions $\{u_n\}$ in $H_0^1(\Omega)$ satisfying*

$$\lim_{n\to+\infty} \|u_n\| = +\infty.$$

Proof. Let $\Phi, \Psi : H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\overline{K}(\|u\|^2), \quad \Psi(u) = \int_\Omega F(x,u)dx \tag{3.2}$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \text{for all } u \in H_0^1(\Omega).$$

Using the properties of f, it is easy to verify that $\Phi, \Psi \in C^1(H_0^1(\Omega), \mathbb{R})$ and for any $v \in H_0^1(\Omega)$, we have

$$\begin{gathered}\langle \Phi'(u), v\rangle = K\Big(\int_\Omega |\nabla u(x)|^2 dx\Big)\int_\Omega \nabla u(x)\nabla v(x)dx,\\ \langle \Psi'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx.\end{gathered}$$

So by the standard arguments, we deduce that the critical points of the functional I_λ are the weak solutions of (1.1).

Using (A1) and (3.2) gives

$$\Phi(u) \ge \frac{m}{2}\|u\|^2 \quad \text{for all } u \in H_0^1(\Omega), \tag{3.3}$$

which implies that Φ is coercive. Moreover, it is easy to show that Φ is sequentially weakly lower semi-continuous and Ψ is sequentially weakly upper semi-continuous. Therefore, the functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1.

Obviously, it follows from (3.1) that

$$\frac{m_1(|B(x_0,\tau)| - |B(x_0,\sigma|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|} < \frac{mS_2^2}{2a_1}.$$

Now we fix

$$\lambda \in \Big(\frac{m_1(|B(x_0,\tau)| - |B(x_0,\sigma)|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|}, \frac{mS_2^2}{2a_1}\Big).$$

Then for any $r > 0$, using (3.3) leads to

$$\begin{aligned}\Phi^{-1}(-\infty, r) &= \{u \in H_0^1(\Omega) : \Phi(u) < r\}\\ &\subseteq \big\{u \in H_0^1(\Omega) : \|u\| < \sqrt{2r/m}\big\},\end{aligned} \tag{3.4}$$

and combining (A3), we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) &\le \sup_{u\in\Phi^{-1}(-\infty,r)} \int_\Omega |F(x,u(x))|dx\\ &\le \sup_{u\in\Phi^{-1}(-\infty,r)} \Big(\frac{a_1}{2}\int_\Omega |u(x)|^2dx + \frac{a_2}{\alpha}\int_\Omega |u(x)|^\alpha dx\Big)\\ &\le \sup_{u\in\Phi^{-1}(-\infty,r)} \Big(\frac{a_1}{2S_2^2}\|u\|^2 + \frac{a_2}{\alpha S_\alpha^\alpha}\|u\|^\alpha\Big)\\ &< \frac{a_1 r}{mS_2^2} + \frac{a_2}{\alpha S_\alpha^\alpha}(\frac{2r}{m})^{\alpha/2}.\end{aligned}$$

Thus,

$$\varphi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)) - \Psi(u)}{r - \Phi(u)}$$

$$
\begin{aligned}
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}\\
&< \frac{a_1}{mS_2^2}+\frac{a_2}{\alpha S_\alpha^\alpha}(\frac{2}{m})^{\alpha/2}r^{\frac{\alpha-2}{2}},
\end{aligned}
$$

which implies

$$
\gamma := \liminf_{r\to+\infty}\varphi(r)\le \frac{a_1}{mS_2^2}<+\infty.
$$

Now we choose a sequence $\{\eta_n\}$ of positive numbers satisfying $\lim_{n\to+\infty}\eta_n=+\infty$. For every $n\in\mathbb{N}$, we define v_n given by

$$
v_n(x):=\begin{cases}0, & x\in\Omega\setminus B(x_0,\tau),\\ \frac{\eta_n}{\tau-\sigma}(\tau-dist(x,x_0)), & x\in B(x_0,\tau)\setminus B(x_0,\sigma),\\ \eta_n, & x\in B(x_0,\sigma).\end{cases} \tag{3.5}
$$

It is easy to verify that $v_n\in H_0^1(\Omega)$ and

$$
\begin{aligned}
\|v_n\|^2 &= \int_{\Omega\setminus B(x_0,\tau)}|\nabla v_n|^2dx+\int_{B(x_0,\tau)\setminus B(x_0,\sigma)}|\nabla v_n|^2dx+\int_{B(x_0,\sigma)}|\nabla v_n|^2dx\\
&= \int_{B(x_0,\tau)\setminus B(x_0,\sigma)}\frac{\eta_n^2}{(\tau-\sigma)^2}dx\\
&= \frac{\eta_n^2}{(\tau-\sigma)^2}(|B(x_0,\tau)|-|B(x_0,\sigma)|).
\end{aligned}
$$

Then it follows from (A2) and (3.2) that

$$
\begin{aligned}
\Phi(v_n) &\le \frac{m_1}{2}\|v_n\|^2+\frac{m_2}{2}\\
&\le \frac{m_1\eta_n^2(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^2}+\frac{m_2}{2}.
\end{aligned} \tag{3.6}
$$

On the other hand, using (A4), from the definition of Ψ, we infer that

$$
\Psi(v_n)\ge\int_{B(x_0,\sigma)}F(x,\eta_n)dx. \tag{3.7}
$$

Thus, by (3.6), (3.7) and (A4), for every $n\in\mathbb{N}$ large enough, one has

$$
\begin{aligned}
I_\lambda(v_n) &\le \frac{m_1\eta_n^2(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^2}+\frac{m_2}{2}-\lambda\int_{B(x_0,\sigma)}F(x,\eta_n)dx\\
&< \frac{m_1\eta_n^2(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^2}+\frac{m_2}{2}-\lambda\int_{B(x_0,\sigma)}\gamma\eta_n^2dx\\
&= \frac{m_1\eta_n^2(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^2}+\frac{m_2}{2}-\lambda\gamma|B(x_0,\sigma)|\eta_n^2\\
&= \frac{m_2}{2}+\Big(\frac{m_1(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^2}-\lambda\gamma|B(x_0,\sigma)|\Big)\eta_n^2.
\end{aligned}
$$

Since

$$
\lambda>\frac{m_1(|B(x_0,\tau)|-|B(x_0,\sigma|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|}
$$

and $\lim_{n\to+\infty}\eta_n=+\infty$, we have

$$
\lim_{n\to+\infty}I_\lambda(v_n)=-\infty.
$$

This shows that the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum. Therefore, by Theorem 2.1 (b), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty}\|u_n\| = +\infty,$$

since $\overline{K}(\|u_n\|^2) \le \frac{m_1}{2}\|u_n\|^2 + \frac{m_2}{2}$ by (A2). Therefore, the conclusion is achieved. □

Remark 3.2. Indeed, from condition (A2), we can see that the potential $\overline{K}$ has a sublinear growth. Moreover, it is not difficult to find such continuous function K satisfying (A1) and (A2), for example

$$K(t) = 1 + \frac{1}{1+t^2}, \quad t \ge 0.$$

Now, we state our second result.

Theorem 3.3. *Let $K : [0, +\infty) \to \mathbb{R}$ be a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that (A1) and the following conditions hold:*

(A2') *there exist constants $l_1 \ge 0$, $q \ge 1$ and $l_2 > 0$ such that*

$$K(t) \le l_1 t^{q-1} + l_2, \quad \text{for every } t \in [0, +\infty);$$

(A3') *there exist $b_1 > 0$, $b_2 \ge 0$ and $2 < \beta < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \ge 3$, $2^* = \infty$ if $N = 1, 2$) such that*

$$|f(x, u)| \le b_1|u| + b_2|u|^{\beta-1}, \quad \text{for all } x \in \Omega \text{ and } u \in \mathbb{R};$$

(A4') *there exist $x_0 \in \Omega$ and three constants $\tau > \sigma > 0$ and*

$$\gamma > \frac{b_1 l_2(|B(x_0,\tau)| - |B(x_0,\sigma)|)}{mS_2^2(\tau-\sigma)^2|B(x_0,\sigma)|} > 0, \tag{3.8}$$

such that

$$\begin{gathered} B(x_0,\sigma) \subset B(x_0,\tau) \subseteq \Omega, \\ F(x,u) \ge 0, \quad \text{for all } (x,u) \in (\Omega\setminus \\ B(x_0,\sigma)) \times \mathbb{R}, \\ F(x,u) \ge \gamma u^2, \quad \text{for all } (x,u) \in B(x_0,\sigma) \times [0,1], \end{gathered}$$

where $B(x_0, \sigma)$ denotes the open ball with center at x_0 and radius σ.

Then for every

$$\lambda \in \Big(\frac{l_2(|B(x_0,\tau)| - |B(x_0,\sigma)|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|}, \frac{mS_2^2}{2b_1}\Big),$$

Equation (1.1) has a sequence of solutions, which converges strongly to zero in $H_0^1(\Omega)$.

Proof. It follows from (3.8) that

$$\frac{l_2(|B(x_0,\tau)| - |B(x_0,\sigma)|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|} < \frac{mS_2^2}{2b_1}.$$

Now we fix

$$\lambda \in \Big(\frac{l_2(|B(x_0,\tau)| - |B(x_0,\sigma)|)}{2\gamma(\tau-\sigma)^2|B(x_0,\sigma)|}, \frac{mS_2^2}{2b_1}\Big).$$

Using the condition (A3') and (3.4), one has

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) &\le \sup_{u\in\Phi^{-1}(-\infty,r)}\int_\Omega |F(x,u(x))|dx \\
&\le \sup_{u\in\Phi^{-1}(-\infty,r)}\Big[\frac{b_1}{2}\int_\Omega |u(x)|^2dx + \frac{b_2}{\beta}\int_\Omega |u(x)|^\beta dx\Big] \\
&\le \sup_{u\in\Phi^{-1}(-\infty,r)}\big(\frac{b_1}{2S_2^2}\|u\|^2 + \frac{b_2}{\beta S_\beta^\beta}\|u\|^\beta\big) \\
&< \frac{b_1 r}{mS_2^2} + \frac{b_2}{\beta S_\beta^\beta}(\frac{2r}{m})^{\beta/2}.
\end{aligned}$$

Thus, there holds

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)}\frac{(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)) - \Psi(u)}{r-\Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r} \\
&< \frac{b_1}{mS_2^2} + \frac{b_2}{\beta S_\beta^\beta}(\frac{2}{m})^{\beta/2}\cdot r^{\frac{\beta-2}{2}},
\end{aligned} \tag{3.9}$$

which implies

$$\delta := \liminf_{r\to 0^+}\varphi(r) \le \frac{b_1}{mS_2^2} < +\infty.$$

Now we choose a sequence $\{\eta_n\}$ of positive numbers satisfying $\lim_{n\to+\infty}\eta_n = 0$. For all $n\in\mathbb{N}$, let v_n defined by (3.5) with the above η_n. Then, using (A2') and (A4'), for every $n\in\mathbb{N}$ large enough one has

$$\begin{aligned}
I_\lambda(v_n) &\le \frac{l_1\eta_n^{2q}(|B(x_0,\tau)|-|B(x_0,\sigma)|)^q}{2q(\tau-\sigma)^{2q}} + \frac{l_2\eta_n^{2}(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^{2}} \\
&\quad - \lambda\int_{B(x_0,\sigma)} F(x,\eta_n)dx \\
&< \frac{l_1\eta_n^{2q}(|B(x_0,\tau)|-|B(x_0,\sigma)|)^q}{2q(\tau-\sigma)^{2q}} + \frac{l_2\eta_n^{2}(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^{2}} \\
&\quad - \lambda\int_{B(x_0,\sigma)} \gamma\eta_n^2 dx \\
&= \frac{l_1\eta_n^{2q}(|B(x_0,\tau)|-|B(x_0,\sigma)|)^q}{2q(\tau-\sigma)^{2q}} + \frac{l_2\eta_n^{2}(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^{2}} \\
&\quad - \lambda\gamma|B(x_0,\sigma)|\eta_n^2 \\
&= \frac{l_1\eta_n^{2q}(|B(x_0,\tau)|-|B(x_0,\sigma)|)^q}{2q(\tau-\sigma)^{2q}} \\
&\quad + \big(\frac{l_2(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2(\tau-\sigma)^{2}} - \lambda\gamma|B(x_0,\sigma)|\big)\eta_n^2 < 0.
\end{aligned}$$

Thus,

$$\lim_{n\to+\infty} I_\lambda(v_n) = I_\lambda(0) = 0,$$

which shows that zero is not a local minimum of I_λ. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional

I_λ does not have a local minimum at the unique global minimum of Φ. Therefore, by Theorem 2.1 (c), there exists a sequence $\{u_n\}$ of critical points of I_λ, which converges weakly to zero. The proof is complete. □

Remark 3.4. Now we give an example to illustrate Theorem 3.3. Consider the Kirchhoff equation

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u(x)|^2dx\Big)\Delta u = \lambda f(x,u), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.10}$$

where $a>0, b\geq 0$ and $\Omega\subset\mathbb{R}^N$ is bounded domain. Set $K(t)=a+bt$. Obviously, (A1) and (A2') are satisfied. Let

$$f(x,u)=\alpha(x)u,$$

and $\alpha\in L^\infty(\Omega)$ with

$$\inf_{x\in\Omega}\alpha(x) > \frac{\sup_{x\in\Omega}\alpha(x)(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2S_2^2(\tau-\sigma)^2|B(x_0,\sigma)|},$$

where x_0 is a point of Ω and two constants $\tau>\sigma>0$ satisfying

$$B(x_0,\sigma)\subset B(x_0,\tau)\subseteq\Omega.$$

Thus, (A3') and (A4') hold. Therefore, by Theorem 3.3, Equation (3.10) has infinitely many nontrivial solutions for every

$$\lambda\in\Big(\frac{a(|B(x_0,\tau)|-|B(x_0,\sigma)|)}{2\inf_{x\in\Omega}\alpha(x)(\tau-\sigma)^2|B(x_0,\sigma)|}, \frac{aS_2^2}{\sup_{x\in\Omega}\alpha(x)}\Big).$$

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4. Addendum posted on September 29, 2016

The authors would like to correct Theorems 3.1, 3.3 and their proofs, since the assumptions of Theorems 3.1 and 3.3 can not be verified. Also a new example is given to illustrate Theorem 3.3 instead of the one in Remark 3.4. However, only the case of $N = 1$ is done and the case of $N \geq 2$ has not been solved yet.

In Theorem 3.1, we assume that $N = 1$ and $\Omega = (a, b)$. The assumption (A3) is removed and the assumption (A4) is replaced by the following.

(A5) there exist $x_0 \in (a, b)$ and two constants $\tau > \sigma > 0$ with $B(x_0, \sigma) \subset B(x_0, \tau) \subseteq (a, b)$ such that

$$F(x, u) \geq 0, \quad \text{for all } (x, u) \in (a, b) \times [0, +\infty),$$

$$\alpha_\infty < \frac{2m(\tau-\sigma)}{m_1(b-a)}\beta_\infty,$$

where

$$F(x,u) = \int_0^u f(x,s)ds, \quad \text{for all } (x,u) \in (a,b)\times\mathbb{R},$$

$$\alpha_\infty := \liminf_{t\to+\infty} \frac{\int_a^b \max_{|\xi|\le t} F(x,\xi)dx}{t^2}, \quad \beta_\infty := \limsup_{t\to+\infty} \frac{\int_{B(x_0,\sigma)} F(x,t)dx}{t^2}.$$

Moreover, the range of the parameter λ becomes $\big(\frac{m_1}{(\tau-\sigma)\beta_\infty}, \frac{2m}{(b-a)\alpha_\infty}\big)$. Based on these changes, we restate Theorem 3.1 as follows.

Theorem 4.1. *Let $N = 1$ and $\Omega = (a,b)$. Assume that conditions* (A1), (A2), (A5) *hold. Then for every interval $(\frac{m_1}{(\tau-\sigma)\beta_\infty}, \frac{2m}{(b-a)\alpha_\infty})$, Equation* (1.1) *has a sequence of solutions $\{u_n\}$ in $H_0^1(a,b)$ satisfying*

$$\lim_{n\to+\infty}\|u_n\| = +\infty.$$

Proof. Now we fix $\lambda \in \big(\frac{m_1}{(\tau-\sigma)\beta_\infty}, \frac{2m}{(b-a)\alpha_\infty}\big)$. Then for any $r > 0$, inequality (3.3) leads to

$$\begin{aligned}
\Phi^{-1}(-\infty,r) &= \{u \in H_0^1(a,b) : \Phi(u) < r\}\\
&\subseteq \Big\{u \in H_0^1(a,b) : \|u\| < \sqrt{\frac{2r}{m}}\Big\}\\
&\subseteq \Big\{u \in H_0^1(a,b) : |u(x)| \le \frac{1}{2}\sqrt{\frac{2r(b-a)}{m}} \quad \text{for all } x \in (a,b)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r)} \int_a^b F(x,u)dx\\
&\le \int_a^b \sup_{|u|\le\frac{1}{2}\sqrt{\frac{2r(b-a)}{m}}} F(x,u)dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\big(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)\big) - \Psi(u)}{r-\Phi(u)}\\
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}\\
&< \frac{\int_a^b \sup_{|u|\le\kappa\sqrt{\frac{2r}{m}}} F(x,u)dx}{r}.
\end{aligned} \tag{4.1}$$

Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_a^b \sup_{|u|\le t_n} F(x,u)dx}{t_n^2} = \alpha_\infty. \tag{4.2}$$

We now choose another positive numbers sequence $\{r_n\}$ with $r_n = \frac{2m}{(b-a)}t_n^2$ for every $n \in \mathbb{N}$. It follows from (4.1)) and (4.2) that

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{(b-a)}{2m}\alpha_\infty < +\infty.$$

Since $\frac{1}{\lambda} < (\tau-\sigma)\beta_\infty/m_1$, there exists a sequence $\{\eta_n\}$ of positive numbers and $\mu > 0$ such that $\eta_n \to \infty$ and

$$\frac{1}{\lambda} < \mu < \frac{(\tau-\sigma)}{m_1}\frac{\int_{B(x_0,\sigma)} F(x,\eta_n)dx}{\eta_n^2}.$$

Define $v_n(x)$ as in (3.5) with $\Omega = (a,b)$ and it is easy to verify that $\|v_n\|^2 = \frac{2\eta_n^2}{\tau-\sigma}$ when $N = 1$. Moreover, by $(A5)$ one has

$$\Psi(v_n) = \int_a^b F(x,v_n)dx \geq \int_{B(x_0,\sigma)} F(x,\eta_n)dx. \tag{4.3}$$

Thus, it follows from (4.3) and (A2) that for every $n \in \mathbb{N}$ large enough,

$$\begin{aligned} I_\lambda(v_n) &= \frac{1}{2}\overline{K}\left(\|v_n\|^2\right) - \lambda\int_a^b F(x,v_n)dx \\ &\leq \frac{m_1}{2}\|v_n\|^2 + \frac{m_2}{2} - \lambda\int_{B(x_0,\sigma)} F(x,\eta_n)dx \\ &= \frac{m_1\eta_n^2}{\tau-\sigma} + \frac{m_2}{2} - \lambda\int_{B(x_0,\sigma)} F(x,\eta_n)dx \\ &< \frac{m_1\eta_n^2}{(\tau-\sigma)}(1-\lambda\mu) + \frac{m_2}{2}, \end{aligned}$$

which implies that the functional I_λ is unbounded from below, since $\eta_n \to +\infty$ and $1 - \lambda\mu < 0$. It follows that I_λ has no global minimum. Therefore, by Theorem 2.1 (b), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty} \|u_n\|^2 = +\infty.$$

The conclusion is achieved. □

In Theorem 3.3, we likewise assume that $N = 1$ and $\Omega = (a,b)$. The assumption (A3') is removed and the assumption (A4') is replaced by the following.

(A6) there exist $x_0 \in (a,b)$ and three constants $\varepsilon > 0$ and $\tau > \sigma > 0$ with $B(x_0,\sigma) \subset B(x_0,\tau) \subseteq (a,b)$ such that

$$F(x,u) \geq 0, \quad \text{for all } (x,u) \in (a,b)\times[0,\varepsilon),$$
$$\alpha_0 < \frac{2m(\tau-\sigma)}{(b-a)l_2}\beta_0,$$

where

$$\alpha_0 := \liminf_{t\to0^+} \frac{\int_a^b \max_{|\xi|\leq t} F(x,\xi)dx}{t^2}, \quad \beta_0 := \limsup_{t\to0^+} \frac{\int_{B(x_0,\sigma)} F(x,t)dx}{t^2}.$$

In addition, the range of the parameter λ becomes $\left(\frac{l_2}{(\tau-\sigma)\beta_0}, \frac{2m}{(b-a)\alpha_0}\right)$. In view of these, we restate Theorem 3.3 as follows.

Theorem 4.2. *Let $N = 1$ and $\Omega = (a,b)$. Assume that conditions* (A1), (A2'), (A6) *hold. Then for every interval* $\left(\frac{l_2}{(\tau-\sigma)\beta_\infty}, \frac{2m}{(b-a)\alpha_\infty}\right)$*, Equation* (1.1) *has a sequence of solutions, which converges strongly to zero in $H_0^1(a,b)$.*

The proof is omitted here, since it is similar to that of Theorem 3.1.

Finally, we give a new example to replace the one in Remark 3.4.

Example 4.3. Let $K(t) = 1 + t$. Then it is easy to verify that (A1) and (A2') hold if we choose $m = l_1 = l_2 = 1$ and $q = 2$. Let $(a, b) = (0, 1)$, $x_0 = \frac{1}{2}, \tau = \frac{1}{3}$ and $\sigma = \frac{1}{4}$. Then $B(x_0, \sigma) \subset B(x_0, \tau) \subseteq (0, 1)$. We take

$$f(x,u) = f(u) = \begin{cases} u(2a - 2\sin(\ln |u|) - \cos(\ln |u|)), & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}$$

where $1 < a < \frac{7}{5}$. A direct calculation shows that

$$F(u) = \int_0^u f(t)dt = \begin{cases} u^2(a - \sin(\ln |u|), & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

It is clear that $F(u) \geq 0$ for all $u \in \mathbb{R}$,

$$\alpha_0 = \liminf_{t \to 0^+} \frac{\max_{|u| \leq t} F(u)}{t^2} = a - 1$$

and

$$\beta_0 = \limsup_{t \to 0^+} \frac{F(t)}{t^2} = a + 1.$$

Moreover, $\alpha_0 < \frac{2m(\tau - \sigma)}{(b-a)l_2} \beta_0$, which implies that (A6) holds. Therefore, for every $\lambda \in \left(\frac{12}{a+1}, \frac{2}{a-1}\right)$, the following problem

$$-\Big(1 + \int_0^1 |u'|^2 dx\Big) u'' = \lambda f(u), \quad \text{in } (0,1),$$
$$u(0) = u(1) = 0,$$

admits infinitely many nontrivial solutions strongly converging at 0 in $H_0^1(0, 1)$.

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End of addendum.

Juntao Sun (corresponding author)
College of Science, Hohai University, Nanjing 210098, China,
School of Science, Shandong University of Technology, Zibo 255049, China
E-mail address: sunjuntao2008@163.com

Yongbao Ji
Institute of Finance and Economics, Shanghai University of Finance and Economics, Shanghai 200433, China
E-mail address: jiyongbao@126.com

Tsung-fang Wu
Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan
E-mail address: tfwu@nuk.edu.tw