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EXISTENCE OF SOLUTIONS FOR KIRCHHOFF EQUATIONS WITH A SMALL PERTURBATIONS

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Abstract. In this article, we consider the Kirchhoff equation

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u)+\mu g(x,u), \quad x\in\Omega,$$
$$u=0, \quad x\in\partial\Omega,$$

and under suitable assumptions on the main term f in the equation, some existence results are obtained by the variational methods and some analysis techniques.

1. Introduction and main results

The problem

$$\begin{gathered}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u), \quad x\in\Omega,\\ u=0, \quad x\in\partial\Omega,\end{gathered} \tag{1.1}$$

is related to the stationary case of the nonlinear equation

$$\frac{\partial^2 u}{\partial t^2}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u)$$

which was proposed by Kirchhoff in 1883 [11] as an extension of the well-known d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. Kirchhoff's model takes into account the transversal oscillations of a stretched string, where a is related to the intrinsic properties of the string, b represents the initial tension, f is the external force, and u describes a process which depends on the average of itself. Such problems are often referred to as being nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2\,dx\Delta u$ which implies that the equation in (1.1) is no longer pointwise. Nonlocal effect finds its applications in biological systems. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problem particularly interesting. There are abundant results about Kirchhoff type equation. Some interesting studies by variational methods can be found in [1]–[8], [12]–[14], [17], [19]–[21], [24]–[27] and the references therein. For example, by using the variational method, Alves et al. [1] obtained positive solutions of (1.1) provided that the nonlinear function f

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satisfies suitable conditions. Perera and Zhang [17] obtained a nontrivial solution of (1.1) by using Yang index and critical group. By minimax techniques and the construction of suitable invariant sets, Mao and Zhang [14] obtained sign changing solutions of (1.1). He and Zou [7] and [8], by using the local minimum methods and the fountain theorems, obtain the existence and multiplicity of nontrivial solutions of (1.1). In [2], the authors considered (1.1) with concave and convex nonlinearities by using Nehari manifold and fibering map methods, and obtained the existence of multiple positive solutions. Sun and Liu [19] obtained a nontrivial solution via Morse theory by computing the relevant critical groups for problem (1.1) with the nonlinearity which is superlinear near zero but asymptotically 4-linear at infinity and asymptotically near zero but 4-linear at infinity. There are some other sufficient conditions such as the monotonicity condition (see [4] and [20]). For more results about (1.1) and its variants on bounded domains, we refer the interested readers to [13, 21, 23, 25, 27] and the references therein. Kirchhoff-type problems setting on an unbounded domain also attract a lot of attention, see [3, 5, 6, 12, 26] and the references therein. In this paper, we shall study problem (1.1) with a small perturbation and prove the existence of solutions by variational method, critical point theory and some analysis techniques.

Let us consider the Dirichlet boundary value problem

$$
\begin{gathered}
-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u)+\mu g(x,u), \quad x\in\Omega,\\
u=0, \quad x\in\partial\Omega,
\end{gathered} \tag{1.2}
$$

where $a\geq 0, b>0$ are real constants, Ω is an open bounded domain in $\mathbb{R}^3$ with a C^2-boundary, $f(x,t)$ and $g(x,t)$ are continuous on $\overline{\Omega}\times\mathbb{R}$ and $f(x,t)=0$ for $t<0$, μ is a real parameter whose absolute value is small. In the case $\mu=0$, problem (1.2) has appeared in wide variety of topics, extensive results related to this problem are available in the literature, see [1]–[8], [12]–[14], [17], [19]–[20], [24]–[27] for references.

In the literature there are some works where the authors showed multiplicity of solutions for some problems related to (1.2) with $b=0$. In [9, 10], the author considered the semilinear elliptic equation with a small perturbation

$$
\begin{gathered}
-\Delta u = f(x,u)+\mu g(x,u), \quad x\in\Omega,\\
u=0, \quad x\in\partial\Omega,
\end{gathered} \tag{1.3}
$$

where μ is a real parameter whose absolute value is small. Using variational techniques, the author [10] show the existence of a positive solution. Moreover, the author prove the existence of at least two positive solutions if the perturbation term is nonnegative. In [9], they study the sublinear elliptic equation having two nonlinear terms, where the main term $f(x,u)$ is sublinear and odd with respect to u and the perturbation term is any continuous function with a small coefficient. Then they prove the existence of multiple small solutions.

In view of the results of Kajikiya [9, 10], it is natural to ask if the same kind of result holds for the Kirchhoff type problem (1.2). To the best of our knowledge, the existence and multiplicity of positive solutions for Kirchhoff (1.2) has not been studied by variational methods. The main goal of this paper is to present a positive answer to this question. We assume the following conditions:

(A1) For any $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that
$$|f(x,t)t| \le \varepsilon |t|^{2^*} + a(\varepsilon) \quad \text{for } t \in \mathbb{R},\ x \in \Omega,$$
where $2^* = \frac{2N}{N-2} = 6$ is the Sobolev critical exponent.

(A2) There exist constants $\alpha > 4$, $\theta \in [0, 4)$, $c > 0$ such that
$$\alpha F(x,t) - tf(x,t) \le c|t|^\theta + c \quad \text{for } t \in \mathbb{R},\ x \in \Omega,$$
where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$.

(A3) There exist $x_0 \in \Omega$, $\delta_0 > 0$ such that
$$\lim_{t\to\infty} \Big(\min_{|x-x_0|\le\delta_0} \frac{F(x,t)}{t^4} \Big) = \infty.$$

(A4)
$$\limsup_{t\to 0} \Big(\max_{x\in\overline{\Omega}} \frac{f(x,t)}{t^3} \Big) < \lambda_1,$$
where
$$\lambda_1 = \inf\Big\{ \Big(\int_\Omega |\nabla u|^2 \mathrm{d}x \Big)^2 : u \in H_0^1(\Omega), \int_\Omega |u|^4\,\mathrm{d}x = 1 \Big\}. \tag{1.4}$$
As shown in [27], $\lambda_1 > 0$ is the principal eigenvalue of
$$\begin{gathered} -\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x \Big) \Delta u = \lambda u^3, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.5}$$
and there is a corresponding eigenfunction $\phi_1 > 0$ in Ω.

(A5)
$$\liminf_{t\to 0} \Big(\min_{x\in\overline{\Omega}} \frac{f(x,t)}{t^3} \Big) > -\infty.$$

Hypotheses (A1)–(A4) guarantee that $f(x,u)$ has mountain pass geometry as well as satisfaction of the Palais-Smale condition, and (A5) ensures that a mountain pass solution is strictly positive. For any continuous function $g(x,u)$ and $|\mu|$ small enough, we obtain the existence of a positive solution. Moreover, if $g(x,0) \ge 0$, we prove the existence of another small positive solution. The main result of this article reads as follows.

Theorem 1.1. *Let $f(x,t)$ and $g(x,t)$ be continuous on $\overline{\Omega} \times \mathbb{R}$. Suppose that conditions* (A1)–(A5) *are satisfied. Then*

(i) There exists a $\mu_0 > 0$ such that problem (1.2) *has a positive solution u_μ when $|\mu| \le \mu_0$. Furthermore, for any sequence μ_j converging to zero, along a subsequence u_{μ_j}, converges to u_0 in $W^{2,q}(\Omega)$ for all $q \in [1,\infty)$, where u_0 is a mountain pass solution of problem* (1.2) *with $\mu = 0$ and where $W^{2,q}(\Omega)$ denotes the Sobolev space.*

(ii) If $g(x,0) \ge 0 \not\equiv 0$ in Ω, then problem (1.2) *has another nonnegative solution v_μ for $\mu > 0$ small enough such that $0 \le v_\mu < u_\mu$ and $v_\mu \to 0$ in $W^{2,q}(\Omega)$ as $\mu \to 0$ for all $q \in [1,\infty)$. Moreover, if*
$$\liminf_{t\to 0} \Big(\min_{x\in\overline{\Omega}} \frac{g(x,t) - g(x,0)}{t^3} \Big) > -\infty, \tag{1.6}$$
then each v_μ is strictly positive.

Remark 1.2. By assumption (A1), we are dealing with functionals satisfying the so-called non-standard growth conditions. The theory of functionals with non-standard growth conditions started with a series of the well known papers by Marcellini [15, 16] and has been developed in many different aspects. The situation is more delicate. We need to face more difficulties than the case the nonlinearity $f(x,t)$ satisfies the subcritical growth: there exists a constant $C_0 > 0$ such that

$$|f(x,t)| \le C_0(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R},\ x \in \Omega,$$

where $1 < p < 2^*$. One is to prove the global compactness, since in this situation with non-standard growth conditions, the standard method of getting the global compactness is not applicable. We have to analyze the (PS) sequence carefully and prove the global compactness indirectly.

The following corollary of Theorem 1.1 is valid under the assumption

(A1')

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t|t|^{2^*-2}} = 0 \quad \text{uniformly a.e. } x \in \Omega.$$

Corollary 1.3. *Let $f(x,t)$ and $g(x,t)$ be continuous on $\overline{\Omega} \times \mathbb{R}$. Suppose that conditions* (A1') *and* (A2)–(A5) *hold. Then the same conclusion as in Theorem 1.1 holds.*

This article is organized as follows. In Section 2 we study the unperturbed problem $-\big(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\big)\Delta u = f(x,u)$; In section 3, we give the proof of the main theorem.

Hereafter we use the following notation:

- For any $1 \le s < +\infty$, $L^s(\Omega)$ is the usual Lebesgue space endowed with the norm
$$\|u\|_s^s := \int_\Omega |u|^s \mathrm{d}x.$$
- $H_0^1(\Omega)$ is the usual Sobolev space endowed with the standard scalar product and norm
$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \|u\|^2 := \int_\Omega |\nabla u|^2 \mathrm{d}x;$$
- C, C', C_i are various positive constants which can change from line to line.
- $o_n(1)$ is a quantity that approaches zero as $n \to \infty$.

2. Unperturbed problem

In this section we recall some standard definitions and collect several lemmas needed to establish our main result. First, let us recall the basic definition which we shall need later. We always assume (A1)–(A5) hold.

Definition 2.1 ([18, 23])**.** A functional I is said to satisfy the (PS) condition, if every sequence $\{u_n\} \subset X$ with

$$I(u_n) \text{ bounded, and } I'(u_n) \to 0 \text{ as } n \to \infty \tag{2.1}$$

possesses a convergent subsequence.

Lemma 2.2 (Mountain pass lemma [18, 23])**.** *If X is a Banach space, $I \in C^1(X)$ satisfies the (PS) condition, there exist $x_0, x_1 \in X$ and $r > 0$, such that $\|x_0 - x_1\| > r$,*

$$\max\{I(x_0), I(x_1)\} < \inf\{I(x) : \|x_0 - x_1\| = r\} = \eta_r,$$
$$\Gamma := \{\gamma : C[0,1] \to X | \gamma(0) = x_0,\ \gamma(1) = x_1\}, \quad c := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t))$$

then $c \ge \eta_r$ and c is a critical value of I.

For (1.2) with $\mu = 0$, we define the Lagrangian functional
$$I_0(u) = \frac{a}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^2 - \int_\Omega F(x,u)\,\mathrm{d}x,$$
where $F(x,u)$ is defined in (A2).

Lemma 2.3. *The functional I_0 satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be any sequence in $H_0^1(\Omega)$ such that $I_0(u_n)$ is bounded and $\|I_0'(u_n)\|$ converges to zero; that is,
$$I_0(u_n) \to c, \quad \|I_0'(u_n)\| \to 0$$
which shows that
$$c = I_0(u_n) + o(1), \quad \langle I_0'(u_n), u_n\rangle = o(1) \tag{2.2}$$
where $o(1) \to 0$ as $n \to \infty$. So, for n is large enough, using Sobolev embedding, we have
$$\begin{aligned}
1 + c &\ge I_0(u_n) - \frac{1}{\alpha}\langle I_0'(u_n), u_n\rangle\\
&= (\frac{a}{2} - \frac{a}{\alpha})\|u_n\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u_n\|^4 + \int_\Omega \Big(\frac{1}{\alpha} f(x,u_n)u_n - F(x,u_n)\Big)\,\mathrm{d}x\\
&\ge (\frac{a}{2} - \frac{a}{\alpha})\|u_n\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u_n\|^4 - c_1|u_n|_\theta^\theta - c_2|\Omega|\\
&\ge (\frac{a}{2} - \frac{a}{\alpha})\|u_n\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u_n\|^4 - c_3\|u_n\|^\theta.
\end{aligned}$$
Since $\alpha > 4$ and $\theta < 4$, $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

By the continuity of embedding, we have $\|u_n\|_{2^*}^{2^*} \le C_2 < \infty$ for all n. Going if necessary to a subsequence, one we have
$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \text{ and } u_n \to u \text{ in } L^r(\Omega), \text{ where } 2 \le r < 2^*.$$
Using (A1), for every $\varepsilon > 0$, there exists $a(\varepsilon) > 0$, such that
$$|f(x,t)t| \le \frac{1}{2C_2}\varepsilon|t|^{2^*} + a(\varepsilon), \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{2.3}$$
Let $\delta = \varepsilon/(2a(\varepsilon)) > 0$, $E \subseteq \Omega$ $\operatorname{meas} E < \delta$, it follows from (2.3) that
$$\begin{aligned}
\Big|\int_E f(x,u_n)u_n\,\mathrm{d}x\Big| &\le \int_E |f(x,u_n)u_n|\,\mathrm{d}x\\
&\le \int_E a(\varepsilon)\,\mathrm{d}x + \frac{1}{2C_2}\varepsilon\int_E |u_n|^{2^*}\,\mathrm{d}x\\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $\{\int_\Omega f(x,u_n)u_n\,\mathrm{d}x,\ n\in N\}$ is equi-absolutely-continuous. From Vitali Convergence Theorem it follows that

$$\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \to \int_\Omega f(x,u)u\,\mathrm{d}x. \tag{2.4}$$

Using (A1) again, for any $\varepsilon>0$ there exists $a(\varepsilon)>0$ such that

$$|f(x,t)|\le \frac{1}{2c_1c_2}\varepsilon|t|^{2^*-1}+a(\varepsilon)\quad \text{for } t\in\mathbb{R},\ x\in\Omega.$$

where $c_1\ge \big(\int_\Omega |u_n|^{2^*}\,\mathrm{d}x\big)^{(2^*-1)/2^*}$ for all n, $c_2:=\big(\int_\Omega |u|^{2^*}\,\mathrm{d}x\big)^{1/2^*}$ are positive constants with the assumption of $u\not\equiv 0$. From Hölder's inequality, for every $E\subseteq\Omega$, we have

$$\int_E a(\varepsilon)|u|\,\mathrm{d}x \le a(\varepsilon)(\operatorname{meas}E)^{\frac{2^*-1}{2^*}}\Big(\int_E |u|^{2^*}\,\mathrm{d}x\Big)^{1/2^*}\le a(\varepsilon)(\operatorname{meas}E)^{\frac{2^*-1}{2^*}}c_1,$$

$$\int_E |u_n|^{2^*-1}|u|\,\mathrm{d}x \le \Big(\int_E |u_n|^{2^*}\,\mathrm{d}x\Big)^{\frac{2^*-1}{2^*}}\Big(\int_E |u|^{2^*}\,\mathrm{d}x\Big)^{1/2^*}\le c_1c_2.$$

Let $\delta=\Big(\frac{\varepsilon}{2c_1a(\varepsilon)}\Big)^{\frac{2^*}{2^*-1}}>0$, $E\subseteq\Omega$, $\operatorname{meas}E<\delta$, using (2.3) again, we have

$$\begin{aligned}
\Big|\int_E f(x,u_n)u\,\mathrm{d}x\Big| &\le \int_E |f(x,u_n)u|\,\mathrm{d}x\\
&\le \int_E a(\varepsilon)|u|\,\mathrm{d}x+\frac{1}{2c_1c_2}\varepsilon\int_E |u_n|^{2^*-1}|u|\,\mathrm{d}x\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

hence $\{\int_\Omega f(x,u_n)u\,\mathrm{d}x,\ n\in N\}$ is also equi-absolutely-continuous. From Vitali Convergence Theorem it follows that

$$\int_\Omega f(x,u_n)u\,\mathrm{d}x \to \int_\Omega f(x,u)u\,\mathrm{d}x. \tag{2.5}$$

Since

$$\langle I_0'(u_n),u\rangle=(a+b\|u_n\|^2)\int_\Omega \nabla u_n\cdot\nabla u\,\mathrm{d}x-\int_\Omega f(x,u_n)u\,\mathrm{d}x\to 0, \tag{2.6}$$

it follows that

$$\langle I_0'(u_n),u_n\rangle=(a+b\|u_n\|^2)\int_\Omega \nabla u_n\cdot\nabla u_n\,\mathrm{d}x-\int_\Omega f(x,u_n)u_n\,\mathrm{d}x\to 0. \tag{2.7}$$

Now, combining (2.4)–(2.7) we have $\|u_n\|\to\|u\|$. □

Next we consider the case that $u\equiv 0$. It follows from (2.6) and (2.7) that $\|u_n\|\to 0$. By Kadec-Klee property, we have $u_n\to u$ in $H_0^1(\Omega)$.

Lemma 2.4. *The functional I_0 has a mountain pass geometry; i.e., there exist $u_1\in H_0^1(\Omega)$ and constants $r,\rho>0$ such that $I_0(u_1)<0$, $\|u_1\|>r$ and*

$$I_0(u)\ge\rho,\quad \text{when } \|u\|=r. \tag{2.8}$$

Proof. Indeed, By (A4), we have $t_0>0$ and $\lambda\in(0,\lambda_1)$ such that

$$\frac{f(x,t)}{t^3}<b\lambda,\quad \text{for } |t|<t_0, \tag{2.9}$$

where

$$\lambda_1 = \inf \Big\{ \|u\|^4 : u \in H_0^1(\Omega), \int_\Omega |u|^4 \,\mathrm{d}x = 1 \Big\}. \tag{2.10}$$

As shown in [27], $\lambda_1 > 0$ is the principal eigenvalue of

$$\begin{gathered} -\Big(\int_\Omega |\nabla u|^2 \,\mathrm{d}x \Big) \Delta u = \lambda u^3, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.11}$$

and there is a corresponding eigenfunction $\phi_1 > 0$ in Ω. Hence, by (2.9), we have

$$F(x,t) \le \frac{b\lambda}{4} t^4, \quad \text{for } |t| \le t_0.$$

This inequality with (A1) shows that

$$F(x,t) \le \frac{b\lambda}{4} t^4 + C|t|^{2^*}, \quad \text{for } t \in \mathbb{R}$$

with some positive constant C.

It follows from (2.10) that $\|u\|^4 \ge \lambda_1 \|u\|_4^4$ for $u \in H_0^1(\Omega)$. Then I_0 is estimated as

$$\begin{aligned} I_0(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b\lambda}{4}\int_\Omega u^4\,\mathrm{d}x - C\int_\Omega |u|^6\,\mathrm{d}x \\ &\le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b\lambda_1}{4}\int_\Omega u^4\,\mathrm{d}x - C\int_\Omega |u|^6\,\mathrm{d}x \\ &\le \frac{a}{2}\|u\|^2 - c\|u\|^6. \end{aligned}$$

This shows the existence of $r > 0$ and $\rho > 0$ satisfying

$$I_0(u) \ge \rho, \quad \text{when } \|u\| = r.$$

Let x_0, δ_0 be as in (A3). Let ϕ be a function such that $\phi \in C_0^1(\Omega)$, $\phi \ge 0$, $\phi \not\equiv 0$ and the support of ϕ is in $B(x_0,\delta_0)$. Here $B(x_0,\delta_0)$ is a ball centered at x_0 with radius δ_0. By (A3),

$$\lim_{t\to\infty} \Big(\min_{x\in B(x_0,\delta_0)} \frac{F(x,t)}{t^4} \Big) = \infty.$$

Put $\beta := \|\phi\|_\infty/2$ and

$$D := \{x \in B(x_0,\delta_0) : \phi(x) \ge \beta\}.$$

For $t \ge 0$, we compute

$$\begin{aligned} I_0(t\phi) &= \frac{at^2}{2}\|\phi\|^2 + \frac{bt^4}{4}\|\phi\|^4 - \int_\Omega F(x,t\phi)\,\mathrm{d}x \\ &\le \frac{at^2}{2}\|\phi\|^2 + \frac{bt^4}{4}\|\phi\|^4 - t^4\int_D \frac{F(x,t\phi)}{t^4\phi^4}\phi^4\,\mathrm{d}x \to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

We fix $t > 0$ so large that $I_0(t\phi) < 0$ and $t\|\phi\| > r$. Then $u_1 := t\phi$ satisfies the assertion of the lemma, i.e. I_0 has a mountain pass geometry. □

Lemma 2.5. *c_0 is a critical value of I_0.*

Proof. For u_1 as in Lemma 2.4, we define

$$\Gamma := \{\gamma : C[0,1] \to H_0^1(\Omega) | \gamma(0) = 0,\ \gamma(1) = u_1\},$$
$$c_0 := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I_0(\gamma(t)).$$

It turns out that the Mountain Pass Theorem holds. Then c_0 is a critical value of I_0. □

We call u a mountain pass solution of I_0 if $I_0'(u) = 0$ and $I_0(u) = c_0$. In general, a mountain pass solution is not necessarily unique but we have an a priori estimate for all mountain pass solutions in the next lemma.

Lemma 2.6. *There exists a constant $C > 0$ such that $\|u\|_{C^1(\overline{\Omega})} \le C$ for any mountain pass solution u of I_0.*

Proof. Let u be any mountain pass solution of I_0. Since $I_0'(u) = 0$ and $I_0(u) = c_0$, we use (2.2) with (A2) to obtain

$$\begin{aligned}
1 + c &\le I_0(u) - \frac{1}{\alpha}\langle I_0'(u), u\rangle \\
&= (\frac{a}{2} - \frac{a}{\alpha})\|u\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u\|^4 + \int_\Omega \Big(\frac{1}{\alpha} f(x,u)u - F(x,u)\Big)\,\mathrm{d}x \\
&\le (\frac{a}{2} - \frac{a}{\alpha})\|u\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u\|^4 - c_1|u|_\theta^\theta - c_2|\Omega| \\
&\le (\frac{a}{2} - \frac{a}{\alpha})\|u\|^2 + (\frac{b}{4} - \frac{b}{\alpha})\|u\|^4 - c_3\|u\|^\theta
\end{aligned}$$

This gives an a priori bound of the $H_0^1(\Omega)$ norm of u; i.e., $\|u\| \le C$ with a $C > 0$ independent of u. By the bootstrap argument with (A1) and the elliptic regularity theorem, we get the upper bound of the $W^{2,q}(\Omega)$ norm of u for all $q \in [1,\infty)$. Especially, an a priori $C^1(\Omega)$ estimate of u follows. □

By Lemma 2.6, we have an $M > 0$ such that

$$\|u\|_\infty \le M \quad \text{for any mountain pass solution } u \text{ of } I_0. \tag{2.12}$$

3. Proof of main result

Now, we define

$$\widetilde{g}(x,t) = \begin{cases} g(x,0), & \text{if } t \le 0 \\ g(x,t), & \text{if } 0 \le t \le 2M \\ g(x,2M), & \text{if } t \ge 2M. \end{cases}$$

Then $\widetilde{g}(x,t)$ is continuous and bounded on $\overline{\Omega}\times\mathbb{R}$. We choose a function $h \in C_0^\infty(\mathbb{R})$ such that $0 \le h \le 1$ in $\mathbb{R}$, $h(t) = 1$ for $|t| \le 2M$ and $h(t) = 0$ for $|t| \ge 4M$. We define

$$I_\mu(u) := \frac{a}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^2 - \int_\Omega \Big(F(x,u) - \mu h(u)\widetilde{G}(x,u)\Big)\,\mathrm{d}x,$$

where $\widetilde{G}(x,t) = \int_0^t \widetilde{g}(x,s)\,\mathrm{d}s$. A critical point of I_μ is a solution of

$$\begin{gathered}
-\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) + \mu h(u)\widetilde{g}(x,u) + \mu h'(u)\widetilde{G}(x,u), \quad x \in \Omega, \\
u = 0 \quad x \in \partial\Omega.
\end{gathered} \tag{3.1}$$

Our plan for proving Theorem 1.1 is as follows. First, we find a mountain pass solution u_μ of I_μ. Next, we prove that $0 < u_\mu(x) \le 2M$ for $|\mu|$ small enough. Then $\mu h'(u_\mu) = 0$, $h(u_\mu) = 1$, $\widetilde{g}(x, u_\mu) = g(x, u_\mu)$ and therefore u_μ becomes a solution of (1.2).

Using the same argument as in Lemma 2.3 with the fact that $h(t)\widetilde{g}(x,t)$ and its partial derivative on t are bounded, we obtain the next lemma.

Lemma 3.1. *For each $\mu \in \mathbb{R}$, I_μ satisfies the (PS) condition.*

Lemma 3.2. *There exists a μ_0 such that I_μ has a mountain pass geometry when $|\mu| \le \mu_0$.*

Proof. Since $h(t)\widetilde{G}(x,t)$ is bounded on $\overline{\Omega} \times \mathbb{R}$, we have

$$I_0(u) - |\mu|C \le I_\mu(u) \le I_0(u) + |\mu|C \quad \text{for } u \in H_0^1(\Omega). \tag{3.2}$$

where $C > 0$ is independent of μ and u. Let r, ρ and u_1 be as in Lemma 2.4. For $|\mu|$ small enough, it follows that

$$I_\mu(u_1) \le I_0(u_1) + |\mu|C < 0, \tag{3.3}$$

$$I_\mu(u) \ge \rho - |\mu|C \ge \frac{\rho}{2} \quad \text{when } \|u\| = r. \tag{3.4}$$

The proof is complete. $\square$

We define the mountain pass value c_μ of I_μ by

$$c_\mu := \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I_\mu(\gamma(t)).$$

Then $c_\mu \to c_0$ as $\mu \to 0$ by (3.2).

Lemma 3.3. *Let $\mu_n \in \mathbb{R}$ be a sequence converging to zero and u_n a mountain pass solution of I_{μ_n}. Then a subsequence of u_n converges to a limit u_0 in $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$, where u_0 is a mountain pass solution of I_0.*

Proof. By definition, $I_{\mu_n}(u_n) = c_{\mu_n}$, $I'_{\mu_n}(u_n) = 0$ and hence u_n satisfies (3.1) with μ replaced by μ_n. Using the same argument as in Lemma 2.6 with the boundedness of c_{μ_n}, we can prove that the $W^{2,q}(\Omega)$ norm of u_n is bounded for any $q \in [1, \infty)$. By the compact embedding, a subsequence of u_n converges to a limit u_0 in $C^1(\overline{\Omega})$. Then u_0 satisfies that $I_0(u_0) = c_0$ and $I'_0(u_0) = 0$; i.e., that u_0 is a mountain pass solution of I_0. The right-hand side of (3.1) with $u = u_n$ and $\mu = \mu_n$ converges to that with $u = u_0$ and $\mu = 0$ uniformly on $x \in \overline{\Omega}$. The elliptic regularity theorem again ensures that u_n converges to u_0 strongly in $W^{2,q}(\Omega)$ for any $q \in [1, \infty)$.

We shall prove the positivity and a priori estimate of mountain pass solutions for I_μ. To this end, for $\delta > 0$, we put

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\},$$

where $\operatorname{dist}(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. $\square$

Lemma 3.4. *There exist constants $\mu_0, \delta, \alpha, \beta > 0$ such that any mountain pass solution u of I_μ with $|\mu| \le \mu_0$ satisfies:*

(i) *$0 < u(x) \le 2M$ in Ω, where M has been defined by* (2.12)*, and*

(ii) *$\frac{\partial u}{\partial \nu} < -\alpha$ in Ω_δ and $u(x) > \beta$ in $\Omega \setminus \Omega_\delta$. Here $\frac{\partial}{\partial \nu}$ is well defined at each point in Ω_δ for $\delta > 0$ small because $\partial\Omega$ is smooth.*

Proof. First, we shall prove $|u(x)| \le 2M$ for $|\mu| > 0$ small enough. Suppose that our claim is false. Then there exist sequences $\mu_n \in \mathbb{R}$ and u_n such that μ_n converges to zero, u_n is a mountain pass solution of I_{μ_n} and $\|u_n\|_\infty > 2M$. By Lemma 3.3, a subsequence of u_n converges to a mountain pass solution u_0 of I_0 in $C^1(\overline{\Omega})$. Since $\|u_0\|_\infty \le M$ by (2.12), it follows that $\|u_n\|_\infty \le 2M$ for n large enough. A contradiction occurs. Thus we have $\|u_n\|_\infty \le 2M$. The positivity of u in (i) follows from (ii). $\square$

Next, we shall prove that $\frac{\partial u}{\partial \nu} < -\alpha$ in Ω_δ with some α, $\delta > 0$ independent of u. Suppose on the contrary that there exist μ_n, x_n, u_n such that $\mu_n \to 0$, $\operatorname{dist}(x_n, \partial\Omega) \to 0$, u_n is a mountain pass solution of I_{μ_n} and

$$\liminf_{t\to 0} \frac{\partial u_n(x_n)}{\partial \nu} \ge 0.$$

We choose a subsequence of x_n which converges to a limit $x_0 \in \partial\Omega$. By Lemma 3.3, a subsequence of u_n converges to a mountain pass solution u_0 of I_0 in $C^1(\overline{\Omega})$. Then $\frac{\partial u_0(x_0)}{\partial \nu} \ge 0$, a contradiction to the strong maximum principle [22]. Indeed, Let u be a nontrivial solution of (1.2) with $\mu = 0$. Put

$$D := \{x \in \Omega : u(x) < 0\}.$$

Assume that $D \ne \emptyset$. We have

$$-\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Thus $u \equiv 0$ in D, a contradiction. Therefore D must be empty; i.e., $u \ge 0$ in Ω. Put $A := \|u\|_\infty$. By (A5), there exists a $C > 0$ such that $f(x,t) \ge -Cs^3$ for $0 \le t \le A$ and $x \in \Omega$. This inequality gives us

$$Cu^3 - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = Cu^3 + f(x,u) \ge 0 \quad \text{in } \Omega.$$

By the strong maximum principle [22], u is strictly positive and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.

Thus $\frac{\partial u}{\partial \nu} \le -\alpha$ in Ω_δ with some $\alpha, \delta > 0$. Fix such α, $\delta > 0$. Then by the same method as above, we can prove that $u(x) > \beta$ in $\Omega \setminus \Omega_\delta$ with some $\beta > 0$.

In Lemma 2.4, we replace r by any positive constant smaller than r. Then (2.8) is still valid after ρ is replaced by a smaller positive constant. Hence (3.2) still holds if $|\mu|$ is replaced by a small one. Thus the next lemma follows.

Lemma 3.5. *There exists an $r_0 > 0$ such that for any $r \in (0, r_0)$, there exist constants ρ, $\mu' > 0$ which satisfy*

$$I_\mu(u) \ge \rho, \quad \text{when } \|u\| = r,\ |\mu| < \mu'.$$

The lemma above will be used to find a small positive solution of (1.2). We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Choose $\mu_0 > 0$ which satisfies Lemma 3.2 and Lemma 3.4. Let u_μ be a mountain pass solution of I_μ with $|\mu| < \mu_0$. Then $0 < u_\mu(x) \le 2M$ by Lemma 3.4. Thus $h'(u_\mu) = 0$, $h(u_\mu) = 1$, $\widetilde{g}(x, u_\mu) = g(x, u_\mu)$ and therefore u_μ becomes a solution of (1.2). Let μ_j be any sequence converging to zero. By Lemma 3.3, a subsequence $u_{\mu'_j}$ converges to a mountain pass solution u_0 of I_0 in $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$.

We now suppose that $g(x,0) \geq 0$, $g(x,0) \not\equiv 0$ in Ω. By (3.3), we have

$$\inf_{\|u\|=r} I_\mu(u) \geq \frac{\rho}{2} > 0 = I_\mu(0).$$

Let B be the set of $u \in H_0^1(\Omega)$ such that $\|u\| \leq r$. Then the minimum of I_μ in B is achieved at an interior point v_μ. Indeed, choose a sequence u_n in B such that $I_\mu(u_n)$ converges to the infimum of I_μ in B. A subsequence of u_n converges weakly in $H_0^1(\Omega)$ to a point v_μ in B. By the weakly lower semicontinuity of I_μ, we have

$$\liminf_{n\to\infty} I_\mu(u_n) \geq I_\mu(v_\mu),$$

which means that v_μ is a minimum point of I_μ in B. Since $I_\mu(0) = 0$, we have $I_\mu(v_\mu) \leq 0 < I_\mu(u_\mu)$, where u_μ is a mountain pass solution of I_μ. Therefore $v_\mu \neq u_\mu$. In the same way as in Lemma 3.2 with $|\mu|$ and $r > 0$ small enough, we can prove that$\|v_\mu\|_\infty \leq M$. Hence $\widetilde{g}(x, v_\mu) = g(x, v_\mu)$ and v_μ is a solution of problem (1.2). Moreover, $v_\mu \not\equiv 0$ because $g(x,0) \not\equiv 0$. Thus v_μ is a nontrivial solution. We shall show that $v_\mu(x) \geq 0$ for $\mu > 0$. Let D be the set of $x \in \Omega$ such that $v_\mu(x) < 0$. Assumptions (A4) and (A5) imply $f(x,0) = 0$. Since $f(x,t) = f(x,0) = 0$ and $g(x,t) = g(x,0) \geq 0$ for $t < 0$, we see that for $\mu > 0$,

$$\begin{gathered} -\Big(a + b\int_\Omega |\nabla v_\mu|^2 dx\Big)\Delta v_\mu = f(x, v_\mu) + \mu g(x, v_\mu) \geq 0, \quad x \in D, \\ v_\mu = 0 \quad x \in \partial D. \end{gathered}$$

which shows that $v_\mu \geq 0$ in D, a contradiction to the definition of D. Thus D must be empty, and $v_\mu(x) \geq 0$ in Ω. By Lemma 3.5, $\|v_\mu\|_2 \to 0$ as $\mu \to 0$. By the bootstrap argument, the $W^{2,q}(\Omega)$ norm of v_μ converges to zero for all $q \in [1,\infty)$, and hence $v_\mu \to 0$ in $C^1(\overline{\Omega})$. Since u_μ is a mountain pass solution of I_μ with $|\mu| < \mu_0$. Then Lemma 3.4(ii) shows that $v_\mu < u_\mu$ in Ω for $\mu > 0$ small enough.

We suppose that (1.6) holds. Put $A := \|v_\mu\|_\infty$. By (1.6), there is a $C > 0$ such that

$$g(x,t) - g(x,0) \geq -Ct^3, \quad \text{for } 0 \leq t \leq A,\ x \in \Omega.$$

Moreover, $f(x,t) \geq -Ct^3$ for $0 \leq t \leq A$ in the proof of Lemma 3.4. Then we have

$$\begin{aligned} &(1+\mu)Cv_\mu^3 - (a + b\|v_\mu\|^2)\Delta v_\mu \\ &= Cv_\mu^3 + f(x, v_\mu) + \mu[g(x, v_\mu) - g(x,0) + Cv_\mu^3] + \mu g(x,0) \geq 0. \end{aligned}$$

By the strong maximum principle [22], v_μ is strictly positive. The proof is complete. □

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