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GROWTH OF LOCAL SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS AROUND AN ISOLATED ESSENTIAL SINGULARITY

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Abstract. In this article we study the growth of solutions to a class of linear differential equations around an isolated essential singularity point. By using conformal mapping we apply some results from the complex plane to a neighborhood of a singular point. We point out that there are several similarities between the results for complex plane and results in this article.

1. Introduction and statement of results

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [14, 23]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [3, 18, 19, 20, 22]. In this paper, we concentrate our investigation near an isolated essential singular point. We start to give the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} - \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function of f by

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r, \tag{1.1}$$

where $n(t, f)$ counts the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ each pole according to its multiplicity; and the proximity function by

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^{+} |f(z_0 - re^{i\varphi})| d\varphi. \tag{1.2}$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f). \tag{1.3}$$

In addition, the order of meromorphic function $f(z)$ near z_0 is defined by

$$\sigma_T(f, z_0) = \limsup_{r \to 0} \frac{\log^{+} T_{z_0}(r, f)}{-\log r}. \tag{1.4}$$

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For an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$, we have also the definition

$$\sigma_M(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{-\log r}, \tag{1.5}$$

where $M_{z_0}(r, f) = \max\{|f(z)| : |z - z_0| = r\}$.

For example, the function $f(z) = \exp\{\frac{1}{(z_0 - z)^n}\}$, where $n \in \mathbb{N}\backslash\{0\}$, we have $M_{z_0}(r, f) = \exp\{\frac{1}{r^n}\}$, and then $\sigma_M(f, z_0) = n$. We have also $T_{z_0}(r, f) = m_{z_0}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(z_0 - re^{i\varphi})| d\varphi = \frac{1}{r^n}$, and so $\sigma_T(f, z_0) = n$.

For the function $f(z) = \exp\{\frac{-1}{(1-z)}\}$, we have $\sigma(f, 1) = 1$ while in the unit disc we have $\sigma_T(f) = \sigma_M(f) = 0$.

We see that in the unit disc we have $\sigma_T(f) \le \sigma_M(f) \le \sigma_T(f) + 1$ and in the complex plane we have $\sigma_T(f) = \sigma_M(f)$. Now, how about the relation between $\sigma_T(f, z_0)$ and $\sigma_M(f, z_0)$? Below, in Lemma 2.2, we will prove that if $f(z)$ is meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is a meromorphic function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$; where $R = \frac{1}{r}$; which implies that $\sigma_T(f, z_0) = \sigma_M(f, z_0)$. So, we can use the notation $\sigma(f, z_0)$ without any ambiguity.

In the usual manner, we define the hyper order near z_0 as

$$\sigma_{2,T}(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ T_{z_0}(r, f)}{-\log r}, \tag{1.6}$$

$$\sigma_{2,M}(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ \log^+ M_{z_0}(r, f)}{-\log r}. \tag{1.7}$$

The linear differential equation

$$f'' + A(z)e^{az} f' + B(z)e^{bz} f = 0, \tag{1.8}$$

where $A(z)$ and $B(z)$ are entire functions, is investigated by many authors; see for example [1, 4, 5, 10]. In [4], Chen proved that if $ab \ne 0$ and $\arg a \ne \arg b$ or $a = cb$ ($0 < c < 1$ or $c > 1$), then every solution $f(z) \not\equiv 0$ of (1.8) is of infinite order. Recently, the second author proved results similar to (1.8) in the unit disc for the differential equation

$$f'' + A(z)e^{\frac{a}{(z_0 - z)^\mu}} f' + B(z)e^{\frac{b}{(z_0 - z)^\mu}} f = 0, \tag{1.9}$$

where $A(z)$ and $B(z)$ are analytic in the unit disc, $\mu > 0$ and $\arg a \ne \arg b$ or $a = cb$ ($0 < c < 1$), see [12]. However, the method of [12] does not work in general for the case $0 < \mu \le 1$: see the discussion in [12, Remark 3.1]. The case $\mu = 1$ will be investigated in the following theorem with certain modifications on $A(z)$ and $B(z)$.

Theorem 1.1. *Let z_0, a, b be complex constants such that $\arg a \ne \arg b$ or $a = cb$ ($0 < c < 1$) and n be a positive integer. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)e^{\frac{a}{(z_0 - z)^n}} f' + B(z)e^{\frac{b}{(z_0 - z)^n}} f = 0. \tag{1.10}$$

satisfies $\sigma(f, z_0) = \infty$ with $\sigma_2(f, z_0) = n$.

In [7], Frei proved the following result in the complex plane.

Theorem 1.2 ([7]). *If the differential equation*

$$g'' + e^{-w} g' + cg = 0 \tag{1.11}$$

where $c \neq 0$ is a complex constant, possesses a solution $g \not\equiv 0$ of finite order, then $c = -k^2$ where k is a positive integer. Conversely, for each positive integer k, the equation (1.11) *with $c = -k^2$ possesses a solution g which is a polynomial in e^w of degree k.*

The analogous of this result, near a singular point z_0, is as follows.

Theorem 1.3. *Let $c \neq 0, z_0$ be complex numbers. If the differential equation*

$$f'' + \Big(\frac{1}{(z_0 - z)^2} e^{\frac{-1}{(z_0 - z)}} - \frac{2}{(z_0 - z)}\Big) f' + \frac{c}{(z_0 - z)^4} f = 0 \tag{1.12}$$

possesses a solution $f(z) \not\equiv 0$ of finite order $\sigma(f, z_0) < \infty$ then $c = -k^2$, where k is an integer. Conversely, for each positive integer k, the equation (1.12) *with $c = -k^2$, possesses a solution f which is a polynomial in $e^{\frac{1}{(z_0 - z)}}$ of degree k.*

Example 1.4. $f_1(z) = 1 + e^{\frac{1}{(z_0 - z)}}$ is a solution of the differential equation

$$f'' + (\frac{1}{(z_0 - z)^2} e^{\frac{-1}{(z_0 - z)}} - \frac{2}{(z_0 - z)}) f' - \frac{1}{(z_0 - z)^4} f = 0.$$

Example 1.5. $f_2(z) = 1 + 4e^{\frac{1}{(z_0 - z)}} + 6e^{\frac{2}{(z_0 - z)}}$ is a solution of the differential equation

$$f'' + \Big(\frac{1}{(z_0 - z)^2} e^{\frac{-1}{(z_0 - z)}} - \frac{2}{(z_0 - z)}\Big) f' - \frac{4}{(z_0 - z)^4} f = 0.$$

Theorem 1.6. *Let $A_0(z) \not\equiv 0, A_1(z), \dots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying*

$$|A_0(z)| \geq \exp\{\frac{\alpha}{r^\mu}\}, \tag{1.13}$$

$$|A_j(z)| \leq \exp\{\frac{\beta}{r^\mu}\},\ j \neq 0, \tag{1.14}$$

where $\alpha > \beta \geq 0$, $\mu > 0$, $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2) \subset [0, 2\pi)$ and $|z_0 - z| = r \to 0$. Then, every solution $f(z) \not\equiv 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.15}$$

satisfies $\sigma_2(f, z_0) \geq \mu$.

Similar results to Theorem 1.6 in the complex plane are given in [2, 11].

Theorem 1.7. *Let $A_0(z) \not\equiv 0, A_1(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$. Then, every solution $f(z) \not\equiv 0$ of* (1.15) *satisfies $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.*

2. Preliminaries lemmas

Throughout this paper, we use the following symbols that do not have necessarily the same at each occurrence: $r_0 > 0$, $\varepsilon > 0$, $\gamma > 1$, $\lambda > 0$ are real constants. The set $E_1^* \subset (0, r_0]$ has finite logarithmic measure $\int_0^{r_0} \frac{\chi_{E_1^*}}{t} dt < \infty$. The set $E_2^* \subset [0, 2\pi)$ has a linear measure zero, $\int_0^{2\pi} \chi_{E_2^*} dt = 0$.

Lemma 2.1 ([9]). *Let g be a transcendental meromorphic function in $\mathbb{C}$, and let $\gamma > 1$, $\varepsilon > 0$ be given real constants; then* *(i) there exists a set $E_1 \subset (1,\infty)$ that has a finite logarithmic measure and a constant $\lambda > 0$ that depends only on γ such that for all $R = |w|$ satisfying $R \notin E_1$, we have*

$$\Big|\frac{g^{(k)}(w)}{g(w)}\Big| \leq \lambda[T(\gamma R, g)\log T(\gamma R, g)]^k; \tag{2.1}$$

(ii) there exists a set $E_2 \subset [0,2\pi)$ that has a linear measure zero and a constant $\lambda > 0$ that depends only on γ such that for all $\theta \in [0,2\pi)\backslash E_2$ there exists a constant $R_0 = R_0(\theta) > 0$ such that for all z satisfying $\arg z \in [0,2\pi)\backslash E_2$ *and $r = |z| > R_0$, we have*

$$\Big|\frac{g^{(k)}(w)}{g(w)}\Big| \leq \lambda[T(\gamma R, g)R^{\varepsilon}\log T(\gamma R, g)]^k. \tag{2.2}$$

Lemma 2.2. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and set $g(w) = f(z_0 - \frac{1}{w})$. Then, $g(w)$ is meromorphic in $\mathbb{C}$ and we have*

$$T(R, g) = T_{z_0}\big(\frac{1}{R}, f\big).$$

Proof. It is easy to prove the following statements:

(i) $w_0 \neq 0$ is a pole of g of order n if and only if $\frac{1}{w_0} - z_0$ is a pole of f of order n.

(ii) 0 is a pole of g of order n if and only if ∞ is a pole of f of order n.

(iii) The change of variable $w = \frac{1}{z_0 - z}$ maps the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ on the region $\{w \in \mathbb{C} : |w| \leq \frac{1}{t}\}$.

From these statements, $g(w)$ is meromorphic in $\mathbb{C}$ and by using the change of variable $T = \frac{1}{t}$, we obtain

$$\begin{aligned} N(R, g) &= \int_0^R \frac{n(T, g) - n(0, g)}{T} dT + n(0, g)\ln R \\ &= -\int_{\infty}^{\frac{1}{R}} \frac{n(t, f) - n(\infty, f)}{t} dt + n(\infty, f)\ln R \\ &= -\int_{\infty}^{\frac{1}{R}} \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f)\ln\frac{1}{R} = N_{z_0}(\frac{1}{R}, f). \end{aligned}$$

Which means that $N(R, g) = N_{z_0}(\frac{1}{R}, f)$. We have

$$\begin{aligned} m(R, g) &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |g(Re^{i\varphi})| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(z_0 - \frac{1}{R}e^{-i\varphi})| d\varphi \\ &= \frac{-1}{2\pi}\int_0^{-2\pi} \ln^+ |f(z_0 - \frac{1}{R}e^{i\varphi})| d\varphi \\ &= \frac{1}{2\pi}\int_{-2\pi}^{0} \ln^+ |f(z_0 - \frac{1}{R}e^{i\varphi})| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(z_0 - \frac{1}{R}e^{i\varphi})| d\varphi = m_{z_0}(\frac{1}{R}, f). \end{aligned} \tag{2.3}$$

So, we conclude that $T(R, g) = T_{z_0}(\frac{1}{R}, f)$. □

Remark 2.3. By Lemma 2.2, if f is a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$ then $\sigma(f, z_0) = \sigma(g)$.

Lemma 2.4. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ and let $\gamma>1$, $\varepsilon>0$ be given constants; then*

(i) there exists a set $E_1^*\subset(0,r_0]$ that has finite logarithmic measure $\int_0^{r_0}\frac{\chi_{E_1^*}}{t}dt<\infty$ and a constant $\lambda>0$ that depends only on γ such that for all $r=|z-z_0|$ satisfying $r\in(0,r_0]\backslash E_1^*$, we have

$$|\frac{f^{(k)}(z)}{f(z)}|\le\lambda[\frac{1}{r^2}T_{z_0}(\frac{r}{\gamma},f)\log T_{z_0}(\frac{r}{\gamma},f)]^k\quad(k\in\mathbb{N});\tag{2.4}$$

(ii) there exists a set $E_2^*\subset[0,2\pi)$ that has a linear measure zero and a constant $\lambda>0$ that depends only on γ such that for all $\theta\in[0,2\pi)\backslash E_2^*$ there exists a constant $r_0=r_0(\theta)>0$ such that for all z satisfying $\arg(z-z_0)=\theta$ and $r=|z-z_0|<r_0$, we have

$$|\frac{f^{(k)}(z)}{f(z)}|\le\lambda[\frac{1}{r^{2+\varepsilon}}T_{z_0}(\frac{r}{\gamma},f)\log T_{z_0}(\frac{r}{\gamma},f)]^k\ (k\in\mathbb{N}).\tag{2.5}$$

Proof. Set $g(w)=f(z_0-\frac{1}{w})$. $g(w)$ is meromorphic in $\mathbb{C}$ and by Lemma 2.1, we have (2.1) and (2.2). We have $f(z)=g(w)$ such that $w=\frac{1}{z_0-z}$; then $f'(z)=\frac{1}{(z_0-z)^2}g'(w)$ and then

$$\frac{f'(z)}{f(z)}=\frac{1}{(z_0-z)^2}\frac{g'(w)}{g(w)}.\tag{2.6}$$

By Lemma 2.1, we have

$$|\frac{g'(w)}{g(w)}|\le\lambda[T(\gamma R,g)\log T(\gamma R,g)],\quad R\notin E_1;$$

and by Lemma 2.2 and (2.6), we obtain

$$\begin{aligned}|\frac{f'(z)}{f(z)}|&\le\lambda[\frac{1}{r^2}T_{z_0}(\frac{1}{\gamma R},f)\log T_{z_0}(\frac{1}{\gamma R},f)]\\&\le\lambda[\frac{1}{r^2}T_{z_0}(\frac{r}{\gamma},f)\log T_{z_0}(\frac{r}{\gamma},f)],\ r\notin E_1^*.\end{aligned}$$

where $\frac{1}{r}=R\notin E_1\Leftrightarrow r\notin E_1^*$ and $\int_0^{r_0}\frac{\chi_{E_1^*}}{t}dt=\int_{1/r_0}^{\infty}\frac{\chi_{E_1}}{T}dT<\infty$.

We have $f''(z)=\frac{1}{(z_0-z)^4}g''(w)+\frac{2}{(z_0-z)^3}g'(w)$; and so

$$\frac{f''(z)}{f(z)}=\frac{1}{(z_0-z)^4}\frac{g''(w)}{g(w)}+\frac{2}{(z_0-z)^3}\frac{g'(w)}{g(w)}.$$

and by Lemma 2.1 and Lemma 2.2, we obtain

$$|\frac{f''(z)}{f(z)}|\le\lambda\big[\frac{1}{r^2}T_{z_0}(\frac{r}{\gamma},f)\log T_{z_0}(\frac{r}{\gamma},f)\big]^2\quad r\notin E_1^*.$$

In general, we can obtain

$$f^{(k)}(z)=\frac{1}{(z_0-z)^{2k}}g^{(k)}(w)+\frac{a_{k-1}}{(z_0-z)^{2k-1}}g^{(k-1)}(w)+\dots+\frac{a_1}{(z_0-z)^{k+1}}g'(w),$$

where a_j $(j=1,2,\dots,k-1)$ are integers; and thus

$$\begin{aligned}\frac{f^{(k)}(z)}{f(z)}&=\frac{1}{(z_0-z)^{2k}}\frac{g^{(k)}(w)}{g(w)}+\frac{a_{k-1}}{(z_0-z)^{2k-1}}\frac{g^{(k-1)}(w)}{g(w)}+\dots\\&\quad+\frac{a_1}{(z_0-z)^{k+1}}\frac{g'(w)}{g(w)}.\end{aligned}\tag{2.7}$$

Also by suing Lemma 2.1 and Lemma 2.2 with (2.7), for $r = |z - z_0| < r_0$, we obtain

$$|\frac{f^{(k)}(z)}{f(z)}| \le \lambda\big[\frac{1}{r^2} T_{z_0}(\frac{r}{\gamma}, f) \log T_{z_0}(\frac{r}{\gamma}, f)\big]^k \quad r \notin E_1^*.$$

Now for (2.5) we can use the same method as above and by using Lemma 2.1 and Lemma 2.2 with (2.7), we obtain, for $r = |z - z_0| < r_0$ and $\arg(z - z_0) \in [0, 2\pi)\backslash E_2^*$,

$$|\frac{f^{(k)}(z)}{f(z)}| \le \lambda\big[\frac{1}{r^{2+\varepsilon}} T_{z_0}(\frac{r}{\gamma}, f) \log T_{z_0}(\frac{r}{\gamma}, f)\big]^k,$$

where $\theta \in E_2 \Leftrightarrow 2\pi - \theta \in E_2^*$ ($E_2^* \subset [0, 2\pi)$ has a linear measure zero). $\square$

The following lemma is a particular case of Lemma 2.4.

Lemma 2.5. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ of finite order $\sigma(f, z_0) < \infty$; let $\varepsilon > 0$ be a given constant. Then the following two statements hold.*

(i) There exists a set $E_1^* \subset (0, r_0]$ that has finite logarithmic measure $\int_0^{r_0} \frac{\chi_{E_1^*}}{t} dt < \infty$ such that for all $r = |z - z_0| \in (0, r_0]\backslash E_1^*$, we have

$$|\frac{f^{(k)}(z)}{f(z)}| \le \frac{1}{r^{k(\sigma+2+\varepsilon)}}, \quad (k \in \mathbb{N}). \tag{2.8}$$

(ii) There exists a set $E_2^* \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi)\backslash E_2^*$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z - z_0) = \theta$ and $r = |z - z_0| < r_0$, the inequality (2.8) holds.

The question which arises here is the following: can we get similar estimations on $|\frac{f^{(k)}(z)}{f(z)}|$ in (2.4), (2.5) and (2.8) for a non constant function that is meromorphic only on a bounded region of the form $\{z \in \mathbb{C} : 0 < |z - z_0| \le r_0\}$?

Lemma 2.6. *Let h be a non constant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ of order $\sigma(f, z_0) > \alpha > 0$. Then, there exists a set $F \subset (0, r_0]$ of infinite logarirhmic measure $\int_0^{r_0} \frac{\chi_F}{t} dt = \infty$ such that for all $r \in F$ and $|h(z)| = M_{z_0}(r, h)$, we have*

$$\log |h(z)| > \frac{1}{r^\alpha}.$$

Proof. By the definition of $\sigma(f, z_0)$, there exists a decreasing sequence $\{r_m\} \to 0$ satisfying $\frac{m}{m+1} r_m > r_{m+1}$ and

$$\lim_{m\to\infty} \frac{\log \log M_{z_0}(r_m, f)}{-\log r_m} > \alpha.$$

Then, there exists m_0 such that for all $m > m_0$ and for a given $\varepsilon > 0$ small enough, we have

$$\log M_{z_0}(r_m, f) > \frac{1}{r_m^{\alpha+\varepsilon}}. \tag{2.9}$$

There exists m_1 such that for all $m > m_1$, and for any $r \in [\frac{m}{m+1} r_m, r_m]$ and for a given $\varepsilon > 0$, we have

$$(\frac{m}{m+1})^{\alpha+\varepsilon} > r^\varepsilon. \tag{2.10}$$

By (2.9) and (2.10), for all $m > m_2 = \max\{m_0, m_1\}$ and for any $r \in [\frac{m}{m+1} r_m, r_m]$, we have

$$\log M_{z_0}(r, f) > \log M_{z_0}(r_m, f) > \frac{1}{r_m^{\alpha+\varepsilon}} > \frac{1}{r^{\alpha+\varepsilon}}(\frac{m}{m+1})^{\alpha+\varepsilon} > \frac{1}{r^\alpha}.$$

Set $F = \cup_{m=m_2}^{\infty}[\frac{m}{m+1}r_m, r_m]$; then we have

$$\sum_{m=m_2}^{\infty}\int_{\frac{m}{m+1}r_m}^{r_m}\frac{dt}{t} = \sum_{m>m_2}\log\frac{m+1}{m} = \infty.$$

□

We recall a particular case of an important result due to Chiang and Hayman in [6].

Lemma 2.7 ([6]). *Let A_j be meromorphic functions in $\mathbb{C}$ and f be a meromorphic solution of (1.15), assuming that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T(R) := \sum_{j=0}^{k-1} T(R, A_j)$, we have*

$$\log m(R, f) < T(R)\{\log R \log T(R)\}^{\gamma}.$$

We can transform this result near a singular point as follows.

Lemma 2.8. *Let A_j be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ and f be a meromorphic solution of (1.15) in $\overline{\mathbb{C}} - \{z_0\}$, assuming that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T_{z_0}(r) := T_{z_0}(r, A_0) + \sum_{j=1}^{k-1}\sum_{i=j}^{k-1} T_{z_0}(r, A_i) + O(\log\frac{1}{r})$, we have*

$$\log m_{z_0}(r, f) < T_{z_0}(r)\{\log\frac{1}{r}\log(T_{z_0}(r))\}^{\gamma}.$$

Proof. Set $g(w) = f(z_0 - \frac{1}{w})$; $g(w)$ is meromorphic in $\mathbb{C}$. We have $f(z) = g(w)$ such that $w = \frac{1}{z_0 - z}$; then $f'(z) = \frac{1}{(z_0-z)^2}g'(w) = w^2 g'(w)$, $f''(z) = w^4 g''(w) + 2w^3 g'(w)$. In general, we can obtain that

$$f^{(k)}(z) = w^{2k}g^{(k)}(w) + a_{k-1}w^{2k-1}g^{(k-1)}(w) + \cdots + a_1 w^{k+1}g'(w), \tag{2.11}$$

where a_j $(j = 1, 2, \dots, k-1)$ are integers. Substituting (2.11) in (1.15), we obtain

$$g^{(k)}(w) + B_{k-1}(w)g^{(k-1)}(w) + \cdots + B_1(w)g'(w) + B_0(w)g(w) = 0,$$

such that $B_0(w) = \frac{1}{w^{2k}}A_0(z_0 - \frac{1}{w})$ and for $j \neq 0$, $B_j(w) = \sum_{i=j}^{k-1}\frac{c_{ij}}{w^{n_{ij}}}A_i(z_0 - \frac{1}{w})$ where c_{ij} and n_{ij} are integers with $0 < n_{ij} \leq 2k$. Since $A_j(z_0 - \frac{1}{w})$ $(j = 0, 1, \dots, k-1)$ are meroporphic functions in $\mathbb{C}$, then $B_j(w)$ are meromorphic functions in $\mathbb{C}$ and by Lemma 2.7, we have

$$\log m(R, g) < T(R, B)\{\log R \log T(R, B)\}^{\gamma}, \tag{2.12}$$

where $T(R, B) = \sum_{j=0}^{k-1} T(r, B_j)$. By (2.3), we have

$$m(R, g) = m(\frac{1}{R}, f) = m_{z_0}(r, f) \tag{2.13}$$

and by Lemma 2.2, we obtain that

$$\begin{aligned} T(R, B) &\leq T\big(R, A_0(z_0 + \frac{1}{w})\big) + \sum_{j=1}^{k-1}\sum_{i=j}^{k-1} T\big(R, A_i(z_0 + \frac{1}{w})\big) + O(\log R) \\ &\leq T_{z_0}(r, A_0) + \sum_{j=1}^{k-1}\sum_{i=j}^{k-1} T_{z_0}(r, A_i) + O(\log\frac{1}{r}). \end{aligned} \tag{2.14}$$

From (2.12)–(2.14), we conclude that

$$\log m_{z_0}(r,f) < T_{z_0}(r)\big\{\log\frac{1}{r}\log(T_{z_0}(r))\big\}^{\gamma}.$$

□

Lemma 2.9. *Let $A(z)$ be analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\sigma(A,z_0)<n$. Set $g(z)=A(z)\exp\{\frac{a}{(z_0-z)^n}\}$ ($n\geq 1$ is an integer), $a=\alpha+i\beta\neq 0$, $z_0-z=re^{i\varphi}$, $\delta_a(\varphi)=\alpha\cos(n\varphi)+\beta\sin(n\varphi)$, and $H=\{\varphi\in[0,2\pi):\delta_a(\varphi)=0\}$ (obviously, H is of linear measure zero).*

Then for any given $\varepsilon>0$ and any $\varphi\in[0,2\pi)\backslash H$, there exists $r_0>0$ such that for $0<r<r_0$, we have

(i) if $\delta_a(\varphi)>0$, then

$$\exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\}\leq|g(z)|\leq\exp\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\},\tag{2.15}$$

(ii) if $\delta_a(\varphi)<0$, then

$$\exp\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\}\leq|g(z)|\leq\exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\}.\tag{2.16}$$

Proof. Set $h(w):=g(z_0-\frac{1}{w})=A(z_0-\frac{1}{w})\exp\{aw^n\}$, where $A(z_0-\frac{1}{w})$ is a analytic function in $\mathbb{C}$ of order $\sigma=\sigma(A,z_0)<n$. We have

$$|\exp\{aw^n\}|=|\exp\{\frac{a}{(z_0-z)^n}\}|=\exp\{\frac{\delta_a(\varphi)}{r^n}\}.$$

Using the analogous lemma in $\mathbb{C}$ (see [5, 22]), we get (2.15) and (2.16). □

3. Proof of main results

Proof of Theorem 1.1. From (1.10), we can write

$$|B(z)||e^{\frac{b}{(z_0-z)^n}}|\leq|\frac{f''}{f}|+|A(z)||e^{\frac{a}{(z_0-z)^n}}||\frac{f'}{f}|.\tag{3.1}$$

Case 1. $\arg a\neq\arg b$: then there exist $(\varphi_1,\varphi_2)\subset[0,2\pi)$ such that for $\arg(z_0-z)=\varphi\in(\varphi_1,\varphi_2)$ we have $\delta_b(\varphi)>0$ and $\delta_a(\varphi)<0$. Since $\max\{\sigma(A,z_0),\sigma(B,z_0)\}<n$, then by Lemma 2.9, (2.5) and (3.1), we obtain

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{r^n}\}\leq\frac{\lambda}{r^{2(2+\varepsilon)}}[T_{z_0}(\frac{r}{\gamma},f)]^4\exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\}.\tag{3.2}$$

From (3.2), it is easy to obtain that $\sigma_2(f,z_0)\geq n$. In the other side, by Lemma 2.8, we can get $\sigma_2(f,z_0)\leq n$. Thus we conclude that $\sigma_2(f,z_0)=n$.

Case 2. $a=cb$ $(0<c<1)$: then there exist $(\varphi_1,\varphi_2)\subset[0,2\pi)$ such that for $\arg(z_0-z)=\varphi\in(\varphi_1,\varphi_2)$ we have $\delta_a(\varphi)=c\delta_b(\varphi)>0$.

Since $\max\{\sigma(A,z_0),\sigma(B,z_0)\}<n$, by Lemma 2.9, (2.5) and (3.1), we obtain

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{r^n}\}\leq\frac{\lambda}{r^{2(2+\varepsilon)}}\big[T_{z_0}(\frac{r}{\gamma},f)\big]^4\exp\{(1+\varepsilon)c\delta_b(\varphi)\frac{1}{r^n}\}.\tag{3.3}$$

From (3.3) and by taking $0<\varepsilon<\frac{1-c}{1+c}$, we obtain that $\sigma_2(f,z_0)\geq n$. By Lemma 2.8, we have $\sigma_2(f,z_0)\leq n$. Thus we conclude that $\sigma_2(f,z_0)=n$. □

Proof of Theorem 1.3. Using the change of variable $w = \frac{1}{z_0 - z}$ and setting $g(w) = f(z)$, we get $f'(z) = \frac{1}{(z_0-z)^2} g'(w) = w^2 g'(w)$, $f''(z) = w^4 g''(w) + 2w^3 g'(w)$. Then the differential equation (1.12) becomes

$$g''(w) + e^{-w} g'(w) + cg(w) = 0. \tag{3.4}$$

By Theorem 1.2, if (3.4) possesses a solution $g \not\equiv 0$ of finite order, then $c = -k^2$ where k is a positive integer. Conversely, for each positive integer k, the equation (1.11) with $c = -k^2$, possesses a solution g which is a polynomial in e^w of degree k. By Remark 2.3, we have $\sigma(f, z_0) = \sigma(g)$. So, if the differential equation (1.12) possesses a solution $f(z) \not\equiv 0$ of finite order $\sigma(f, z_0) < \infty$ then $c = -k^2$. Conversely, for each positive integer k, the equation (1.12) with $c = -k^2$, possesses a solution f which is a polynomial in $e^{\frac{1}{(z_0-z)}}$ of degree k. □

Proof of Theorem 1.6. From (1.15), we can write

$$|A_0(z)| \le |\frac{f^{(k)}}{f}| + |A_{k-1}(z)||\frac{f^{(k-1)}}{f}| + \dots + |A_1(z)||\frac{f'}{f}|. \tag{3.5}$$

Using (1.13), (1.14) and (2.5) in (3.5), for $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2) \subset [0, 2\pi)$ and $|z_0 - z| = r$ near enough to 0, we obtain

$$\exp\{\frac{\alpha}{r^\mu}\} \le \frac{\lambda}{r^{k(2+\varepsilon)}} \big[T_{z_0}(\frac{r}{\gamma}, f)\big]^{2k} \exp\{\frac{\beta}{r^\mu}\}. \tag{3.6}$$

From (3.6), we obtain that $\sigma(f, z_0) \ge \mu$. □

Proof of Theorem 1.7. From (1.15), we can write

$$|A_0(z)| \le |\frac{f^{(k)}}{f}| + |A_{k-1}(z)||\frac{f^{(k-1)}}{f}| + \dots + |A_1(z)||\frac{f'}{f}|. \tag{3.7}$$

Set $\max\{\sigma(A_j, z_0) : j \ne 0\} < \beta < \alpha < \sigma(A_0, z_0)$. For any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for all r satisfying $r_0 \ge r > 0$, we have

$$|A_j(z)| \le \exp\{\frac{1}{r^{\beta+\varepsilon}}\}, \quad j = 1, 2, \dots, k-1. \tag{3.8}$$

By taking $\beta + \varepsilon < \alpha < \sigma(A_0, z_0)$, and by Lemma 2.6, there exists a set $F \subset (0, r_0]$ of infinite logarirhmic measure such that for all $r \in F$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we have

$$|A_0(z)| > \exp\{\frac{1}{r^\alpha}\}. \tag{3.9}$$

Using (3.8)–(3.9) with (2.4) in (3.7), we obtain

$$\exp\{\frac{1}{r^\alpha}\} \le \frac{\lambda}{r^{2k}} \big[T_{z_0}(\frac{r}{\gamma}, f)\big]^{2k} \exp\{\frac{1}{r^{\beta+\varepsilon}}\}. \tag{3.10}$$

From (3.10), we obtain that $\sigma(f, z_0) \ge \alpha$.

On the other hand, applying Lemma 2.8 with (1.15), we obtain that $\sigma(f, z_0) \le \sigma(A_0, z_0)$. Since $\alpha \le \sigma(f, z_0) \le \sigma(A_0, z_0)$ holds for every $\alpha < \sigma(A_0, z_0)$, then we conclude that $\sigma(f, z_0) = \sigma(A_0, z_0)$. □

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