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EXISTENCE OF SOLUTIONS FOR ELLIPTIC NONLINEAR PROBLEMS ON THE UNIT BALL OF $\mathbb{R}^3$

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Abstract. We consider an elliptic PDE with critical nonlinearity involving the Laplacian operator with zero Dirichlet boundary condition on the unit ball of $\mathbb{R}^3$. We assume some perturbation conditions and obtain what seems to be the first existence result for this problem.

1. Introduction and statement of the main result

Let $\mathbb{B}^3$ be the unit ball of $\mathbb{R}^3$, and let $K : \mathbb{B}^3 \to \mathbb{R}$ be a given function. We are looking for a map $u : \mathbb{B}^3 \to \mathbb{R}$ satisfying the nonlinear PDE with zero Dirichlet boundary condition

$$\begin{gathered} -\Delta u = K(x)u^5 \\ u > 0 \quad \text{in } \mathbb{B}^3 \\ u = 0 \quad \text{on } \partial\mathbb{B}^3 \end{gathered} \tag{1.1}$$

This problem is so-called critical in the sense that lack of compactness occurs. It is easy to see that a necessary condition for solving (1.1), is that K be positive somewhere. In addition, when $K = 1$, Pohozaev [7] proved that the problem has no solution. While many existence results have been established for the equivalent problem of (1.1) in dimensions $n \geq 4$, see (e.g [4, 5]), as far as we know, there is no existence result for (1.1). The objective of this paper is to state conditions on $K(x)$ to provide the existence of solutions to (1.1). In this article we use the assumption

(A1) $K(x)$ is a positive Morse function on $\overline{\mathbb{B}^3}$ with $\frac{\partial K}{\partial \nu}(x) < 0$ for all $x \in \partial\mathbb{B}^3$, where ν denotes the unit outward normal vector on $\partial\mathbb{B}^3$.

Let $\mathcal{K}$ denote the set of all critical points of $K(x)$. For any $y \in \mathcal{K}$, we denote by $\operatorname{ind}(K, y)$ the Morse index of K at y. Our main result reads as follows:

Theorem 1.1. *Under assumption* (A1)*, if there exists $\ell_0 \in \mathbb{N}$ such that*

(i) $3 - \operatorname{ind}(K, y) \neq \ell_0 + 1$ *for all $y \in \mathcal{K}$, and*
(ii) $\sum_{y\in\mathcal{K}, 3-\operatorname{ind}(K,y)\leq\ell_0} (-1)^{3-\operatorname{ind}(K,y)} \neq 1$,

then (1.1) *has a solution, provided K is close to 1.*

Remark 1.2. Observe that for any $\ell_0 \geq 3$, condition (i) of Theorem 1.1 is satisfied. In this case, the above sum will be over all critical points of K.

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Remark 1.3. Unlike the existence result in [4] for dimension 4, and in [5] for dimension ≥ 4, our result for dimension 3 does not use any condition about $-\Delta K(y)$, for $y \in \mathcal{K}$.

2. Lack of compactness

Define

$$\Sigma = \big\{u \in H_0^1(\Omega) : \|u\| = \Big(\int_\Omega |\nabla u|^2\Big)^{1/2} = 1\big\}, \quad \text{and} \quad \Sigma^+ = \big\{u \in \Sigma,\ u \geq 0\big\}.$$

Let

$$J(u) = \frac{\int_\Omega |\nabla u|^2}{\big(\int_\Omega K(x) u^6\big)^{1/3}}, \quad u \in H_0^1(\Omega) \setminus \{0\}.$$

It is well known that if u is a critical point of J in Σ^+, then $J(u)^{3/4}u$ is a solution of (1.1). Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is not compact, the functional J does not satisfy the Palais-Smale condition on Σ^+, and there are sequences which do not satisfy the Palais-Smale condition. To describe those sequences, we introduce some notation. For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = c_0 \Big(\frac{\lambda}{1 + \lambda^2 |x-a|^2}\Big)^{1/2},$$

where $c_0 > 0$ chosen such that $\delta_{a,\lambda}$ is the family of solutions of the problem

$$-\Delta u = u^5, \qquad u > 0 \text{ in } \mathbb{R}^3.$$

Let P be the projection from $H^1(\Omega)$ on $H_0^1(\Omega)$; that is for any $f \in H^1(\Omega)$, $P(f)$ is the unique solution of

$$\begin{gathered} -\Delta u = \Delta f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let

$$\begin{aligned} V(p,\varepsilon) = \Big\{ & u \in \Sigma^+, : \exists a_1, \dots, a_p \in \Omega,\ \exists \lambda_1, \dots, \lambda_p > \varepsilon^{-1} \text{ and} \\ & \alpha_1, \dots, \alpha_p > 0 \text{ with } \|u - \sum_{i=1}^p \alpha_i P\delta_{a_i,\lambda_i}\| < \varepsilon,\ \varepsilon_{ij} < \varepsilon, \forall i \neq j, \\ & \lambda_i d_i > \varepsilon^{-1} \text{ and } |J(u)^3 \alpha_i^4 K(a_i) - 1| < \varepsilon, \forall i = 1, \dots, p \Big\}. \end{aligned}$$

Here, $d_i = d(a_i, \partial\Omega)$ and

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2\Big)^{-1/2}.$$

The failure to satisfy the Palais-Smale condition can be described as follows.

Proposition 2.1 ([2]). *Assume that* (1.1) *has no solutions. Let* $(u_k)_k$ *be a sequence in* Σ^+ *such that* $J_{\varepsilon_0}(u_k)$ *is bounded and* $\partial J(u_k)$ *approaches zero. Then there exists a positive integer* p*, a sequence* (ε_k) *with* $\varepsilon_k \to 0$ *as* $k \to +\infty$ *and an extracted subsequence of* $(u_k)_k$*, again denoted* $(u_k)_k$*, such that* $u_k \in V(p, \varepsilon_k)$ *for all* k.

The following proposition gives a parametrization of $V(p,\varepsilon)$.

Proposition 2.2 ([2]). *For all $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that for any $\varepsilon \leq \varepsilon_p$ and any u in $V(p,\varepsilon)$, the problem*

$$\min \big\{ \|u - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i}\| : \alpha_i > 0, \lambda_i > 0, a_i \in \Omega \big\}.$$

has a unique solution (up to a permutation). Thus, we can uniquely write u as

$$u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} + v,$$

where $v \in H_0^1(\Omega)$ and satisfies

$$\langle v, \psi \rangle = 0 \quad \textit{for } \psi \in \big\{ P\delta_i, \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_i}{\partial a_i}, i = 1, \dots, p \big\}. \tag{2.2}$$

Here, $P\delta_i = P\delta_{a_i,\lambda_i}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product on $H_0^1(\Omega)$ associated to the norm $\| \cdot \|$.

The following proposition deals with the v-part of u and shows that is negligible with respect to the concentration phenomenon.

Proposition 2.3 ([1, 2]). *There is a $\mathcal{C}^1$-map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i}$ belonging to $V(p,\varepsilon)$ associates $\overline{v} = \overline{v}(\alpha_i, a_i, \lambda_i)$, where $\overline{v}$ is the unique solution of the minimization problem*

$$\min \Big\{ J\Big(\sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} + v \Big) : v \in H_0^1(\Omega) \textit{ and satisfies } (2.2) \Big\}.$$

Moreover, there exists a change of variables $v - \overline{v} \to V$ such that

$$J\Big(\sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} + v \Big) = J\Big(\sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} + \overline{v} \Big) + \|V\|^2.$$

Definition 2.4 ([1]). A critical point of J at infinity is a limit of a non-compact flow line $u(s)$ of the gradient vector field $(-\partial J)$. By Propositions 2.1 and 2.2, $u(s)$ can be written as

$$u(s) = \textstyle\sum_{i=1}^{p} \alpha_i(s) P\delta_{a_i(s),\lambda_i(s)} + v(s).$$

Denoting $y_i = \lim_{s\to+\infty} a_i(s)$ and $\alpha_i = \lim_{s\to+\infty} \alpha_i(s)$, we denote such critical point at infinity by

$$\sum_{i=1}^{p} \alpha_i P\delta_{y_i,\infty} \quad \text{or} \quad (y_1, \dots, y_p)_\infty .$$

We point out that the topological argument that we will use in the proof of Theorem 1.1 avoid all critical points at infinity which are in $V(p,\varepsilon), p \geq 2$. For this, we need the next proposition to characterize the critical points at infinity in $V(1,\varepsilon)$.

Proposition 2.5. *Under assumption* (A1)*, the critical points of J at infinity, in $V(1,\varepsilon)$, are*

$$(y)_\infty = \frac{1}{K(y)^{1/2}} P\delta_{(y,\infty)}, \quad y \in \mathcal{K},$$

where $\mathcal{K}$ is the set of all critical points of $K(x)$. Furthermore, the Morse index of each $(y)_\infty$ is $3 - \operatorname{ind}(K,y)$.

Proof. Let $u = \alpha P\delta_{(a,\lambda)} \in V(1,\varepsilon)$. It is proved in [4, Propositions 2.3 and 2.4] and [5, Propositions 3.4 and 3.5] that for any $n \geq 4$, we have

$$\langle \partial J(u), \lambda \frac{\partial P\delta_{(y,\infty)}}{\partial \lambda} \rangle = J(u)\Big(c_1 \frac{\Delta K(a)}{\lambda^2} - c_2 \frac{H(a,a)}{\lambda^{n-2}}\Big) + o\Big(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-1}}\Big), \tag{2.3}$$

$$\begin{aligned} \langle \partial J(u), \frac{1}{\lambda} \frac{\partial P\delta_{(y,\infty)}}{\partial a} \rangle &= -J(u)^{\frac{2(n-1)}{n-2}} c_3 \frac{\nabla K(a)}{\lambda} + c_4 \frac{1}{\lambda^{n-1}} \frac{\partial H(a,a)}{\partial a} \\ &\quad + o\Big(\frac{1}{(\lambda d)^{n-1}}\Big), \end{aligned} \tag{2.4}$$

where $d = d(a, \partial\mathbb{B}^3)$ and H is the regular part of the Green's function of the Laplacian with Dirichlet boundary condition on $\mathbb{B}^3$. Their proof can be extended in dimension 3. Indeed, it is known that

$$\partial J(u) = 2J(u)\big[u + J(u)^3 \Delta^{-1}(Ku^5)\big].$$

Thus, for $u = \alpha P\delta_{(a,\lambda)}$, we obtain

$$\begin{aligned} &\langle \partial J(u), \alpha\lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} \rangle \\ &= 2J(u)\alpha^2 \Big[\langle P\delta_{(a,\lambda)}, \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} \rangle - J(u)^3 \alpha^4 \int_{\mathbb{B}^3} K P\delta^5_{(a,\lambda)} \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} dx\Big]. \end{aligned}$$

Using that

$$\begin{gathered} P\delta_{(a,\lambda)} = \delta_{(a,\lambda)} - \frac{1}{\lambda^{1/2}} H(a,\cdot) + O(\frac{1}{\lambda^{\frac{3}{2}} d}), \\ \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} = \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} + \frac{1}{2\lambda^{1/2}} H(a,\cdot) + O(\frac{1}{\lambda^{\frac{3}{2}} d}), \end{gathered}$$

where $d = d(a, \partial\mathbb{B}^3)$, we obtain

$$\langle P\delta_{(a,\lambda)}, \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} \rangle = c \frac{H(a,a)}{\lambda} + o(\frac{1}{\lambda}).$$

Here $c = \int_{\mathbb{B}^3} \frac{dz}{(1+|z|^2)^{\frac{5}{2}}}$. Moreover, by expanding $K(x)$ about a, we obtain

$$\int_{\mathbb{B}^3} K P\delta^5_{(a,\lambda)} \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} dx = K(a) \langle P\delta_{(a,\lambda)}, \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda} \rangle - \tilde{c} \frac{\Delta K(a)}{\lambda^2} + o(\frac{1}{\lambda^2}),$$

where

$$\tilde{c} = \int_{\mathbb{B}^3} |z|^2 \frac{1 - |z|^2}{(1+|z|^2)^4} dz.$$

Using now that $J(u)^3 \alpha^4 = \frac{1}{K(a)} + o(1)$, estimate (2.3) follows.

Concerning (2.4), it follows from the same argument and the fact that

$$\frac{1}{\lambda} \frac{\partial P\delta_{(a,\lambda)}}{\partial a} = \frac{1}{\lambda} \frac{\partial \delta_{(a,\lambda)}}{\partial a} + \frac{1}{\lambda^{\frac{3}{2}}} \frac{H(a,.)}{\partial a} + O(\frac{1}{\lambda^4 d^2}).$$

In [4, Theorem 3.1] and [5, Propositions 4.2], the authors showed that under condition (A1), the boundary of a domain Ω does not have any effect in the existence of critical points at infinity. Therefore, to establish our proof, it remains only to focus on the existence of critical points at infinity in

$$\widetilde{V}(1,\varepsilon) = \{\alpha P\delta_{(a,\lambda)} + \bar{v} \in V(1,\varepsilon), d(a, \partial\mathbb{B}^3) \geq d_0\},$$

where $d_0 > 0$ is small. The following Lemma studies the concentration phenomenon of J in $\widetilde{V}(1,\varepsilon)$. Its proof will be given later.

Lemma 2.6. *There exists a pseudo-gradient W in $\widetilde{V}(1,\varepsilon)$ such that for any $u = \alpha P\delta_{(a,\lambda)} \in \widetilde{V}(1,\varepsilon)$ we have:*

(i) $\langle \partial J(u), W(u)\rangle \le -c\Big(\frac{1}{\lambda} + \frac{|\nabla K(a)|}{\lambda}\Big)$,

(ii)
$$\langle \partial J(u+\bar v), W(u) + \frac{\partial \bar v}{\partial(\alpha,a,\lambda)}(W(u))\rangle \le -c\Big(\frac{1}{\lambda} + \frac{|\nabla K(a)|}{\lambda}\Big).$$

Moreover, the concentration $\lambda(s)$ of the flow line of W increases and approaches $+\infty$, as $a(s)$ approaches y, $y \in \mathcal{K}$.

In the above Lemma, we observe that if the concentration point $a(s)$ of the flow line of the pseudo-gradient W enter in some neighborhood of any critical point y of $K(x)$, $\lambda(s)$ increases on the flow line and approaches $+\infty$. Thus, we obtain a critical point at infinity. In this statement, the functional J can be expended after a suitable change of variables as
$$J(\alpha P\delta_{a,\lambda} + \bar v) = J(\widetilde{\alpha} P\delta_{\widetilde{a},\widetilde{\lambda}}) = \frac{S_3}{\widetilde{\alpha}^4 (K(x))^{1/2}}\Big(1 + \frac{1}{\widetilde{\lambda}}\Big).$$
Thus, the index of such critical point at infinity is $3 - \operatorname{ind}(K,y)$. Since J behaves in this region as $\frac{1}{(K(x))^{1/2}}$. This completes the proof. $\square$

Proof of Lemma 2.6. Let $\delta > 0$ be small enough, and set the cut-off function $\theta : \mathbb{R} \to \mathbb{R}$ by
$$\theta(t) = \begin{cases} 1 & \text{if } |t| \le \delta/2 \\ 0 & \text{if } |t| \ge \delta. \end{cases}$$
For $u = \alpha P\delta_{(a,\lambda)} \in \widetilde{V}(1,\varepsilon)$, define
$$\dot\lambda = \lambda \quad\text{and}\quad \dot a = \frac{1}{\lambda}\frac{\nabla K(a)}{|\nabla K(a)|}, \quad a \in \mathbb{B}^3 \setminus \mathcal{K}.$$
We set
$$W(u) = \theta(|\nabla K(a)|)\alpha \frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\dot\lambda + \Big(1 - \theta(|\nabla K(a)|)\Big)\alpha\frac{\partial P\delta_{a,\lambda}}{\partial a}\dot a.$$
We claim that
$$\langle \partial J(u), W(u)\rangle \le -c\Big(\frac{1}{\lambda} + \frac{|\nabla K(a)|}{\lambda}\Big). \tag{2.5}$$
Indeed, $|\nabla K(a)| \le \delta$, by expansion (2.3) we have
$$\langle \partial J(u), \alpha\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\rangle \le \frac{-c}{\lambda}, \tag{2.6}$$
since $n = 3$ and $H(x,x)$ is smooth and positive on $\mathbb{B}^3$. Observe that in our case
$$\frac{|\nabla K(a)|}{\lambda} \le \frac{\delta}{\lambda},$$
so we can include $-\frac{|\nabla K(a)|}{\lambda}$ in the upper bound of (2.6) and therefore we obtain
$$\langle \partial J(u), \alpha\lambda\frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\rangle \le -c\Big(\frac{1}{\lambda} + \frac{|\nabla K(a)|}{\lambda}\Big). \tag{2.7}$$
Now if $|\nabla K(a)| \ge \frac{\delta}{2}$, by expansion (2.4) we obtain
$$\langle \partial J(u), \alpha\frac{1}{\lambda}\frac{\nabla K(a)}{|\nabla K(a)|}\frac{\partial P\delta_{a,\lambda}}{\partial a}\rangle \le -c\frac{|\nabla K(a)|}{\lambda} + O\Big(\frac{1}{\lambda^2}\Big). \tag{2.8}$$

Observe that in our statement we have $\frac{1}{\lambda^2} = o\Big(\frac{|\nabla K(a)|}{\lambda}\Big)$ as λ approaches $+\infty$. Indeed,

$$\frac{1}{\lambda^2}\frac{\lambda}{|\nabla K(a)|} \leq \frac{2}{\delta}\frac{1}{\lambda}.$$

Moreover, $\frac{1}{\lambda} \leq \frac{2}{\delta}\frac{|\nabla K(a)|}{\lambda}$. Therefore, we can include $-\frac{1}{\lambda}$ in the upper bound of (2.8) and obtain

$$\langle \partial J(u), \alpha\frac{1}{\lambda}\frac{\nabla K(a)}{|\nabla K(a)|}\frac{\partial P\delta_{a,\lambda}}{\partial a}\rangle \leq -c\Big(\frac{1}{\lambda} + \frac{|\nabla K(a)|}{\lambda}\Big). \tag{2.9}$$

Hence claim (2.5) is valid. This completes the proof of part (i) in Lemma 2.6.

Part (ii) follows as in [3, Appendix 2] from (i) and the following Lemma which shows that $\|\bar{v}\|^2$ is small with respect to the absolute value of the upper bound of (i).

Lemma 2.7 ([5]). *There exists $c > 0$ such that*

$$\|\bar{v}\| \leq c\Big(\frac{1}{\lambda^{\frac{3}{2}}} + \frac{|\nabla K(a)|}{\lambda} + \frac{\ln \lambda^{5/6}}{\lambda^{\frac{5}{6}}}\Big).$$

This completes the proof of Lemma 2.6. □

3. Proof of Theorem 1.1

Let

$$J_1(u) = \frac{1}{\big(\int_{\mathbb{B}^3} u^6 dx\big)^3}, \quad u \in \Sigma,$$

be the Euler Lagrange functional associated to Yamabe's problem on $\mathbb{B}^3$. Let

$$S = \frac{1}{\big(\int_{\mathbb{B}^3} \delta_{a,\lambda}^6 dx\big)^3},$$

be the best Sobolev constant. S does not depend on a and λ. It is known that

$$S = \inf_{u\in\Sigma} J_1(u).$$

For $c \in \mathbb{R}$ and for any function f on Σ, we define $f^c = \{u \in \Sigma : f(u) \leq c\}$. It is easy to see that if $\|K-1\|_{L^\infty(\mathbb{B}^3)}$ is small enough, we have

$$J^{S+\frac{S}{4}} \subset J_1^{S+\frac{S}{2}} \subset J^{S+\frac{3S}{4}}. \tag{3.1}$$

This is due to the fact that $J(u) = J_1(u)\Big(1 + O\big(\|K-1\|_{L^\infty(\mathbb{B}^3)}\big)\Big)$, with $O\big(\|K-1\|_{L^\infty(\mathbb{B}^3)}\big)$ independent of u. Indeed,

$$\begin{aligned}
J(u) &= \frac{1}{\big(\int_{\mathbb{B}^3} u^6 dx + \int_{\mathbb{B}^3}(K-1)u^6 dx\big)^3} \\
&= J_1(u)\frac{1}{\big[1 + \big(\int_{\mathbb{B}^3} u^6 dx\big)^{-1}\int_{\mathbb{B}^3}(K-1)u^6 dx\big]^3} \\
&= J_1(u)\big[1 + O\big(\|K-1\|_{L^\infty(\mathbb{B}^3)}\big)\big].
\end{aligned}$$

Now let $(y_1, \dots, y_q)_\infty$ be a critical point at infinity of q masses. It is known that the level of J at $(y_1, \dots, y_q)_\infty$ is given by $S\big(\sum_{k=1}^q \frac{1}{K(y)^{1/2}}\big)^{2/3}$, see [5]. Hence it approaches qS as K is close to 1. Therefore, for $\|K-1\|_{L^\infty(\mathbb{B}^3)}$ small, we have:

$$\text{All critical points at infinity of } q\text{-masses, } q \geq 2 \text{ are above } S + \frac{3}{4}S, \tag{3.2}$$

$$\text{all critical points at infinity of } J \text{ of one masse are below } S + \frac{S}{4}. \tag{3.3}$$

Arguing by contradiction and assume that (1.1) has no solution. Let $\ell_0 \in \mathbb{N}$ be the integer defined in Theorem 1.1. Define

$$X^\infty_{\ell_0} = \cup_{y\in\mathcal{K}, 3-\widetilde{i}(y)\leq \ell_0} \overline{W^\infty_u(y)}_\infty,$$

where $\overline{W^\infty_u(y)}_\infty$ is the closure of the unstable manifold of $(-\partial J)$ at the critical point $(y)_\infty$, defined by adding to $W^\infty_u(y)_\infty$ the unstable manifolds of critical points or critical points at infinity dominated by $(y)_\infty$. These manifolds are then of dimension less or equal to $\ell_0 - 1$. Therefore $X^\infty_{\ell_0}$ define a stratified set of top dimension $\leq \ell_0$. Without loss of generality, we may assume that it is equals to ℓ_0. From (3.3), we can see that $X^\infty_{\ell_0}$ lies in $J^{S+\frac{S}{4}}$. We claim that

$$X^\infty_{\ell_0} \text{ is contractible in } J^{S+\frac{S}{4}}. \tag{3.4}$$

Indeed, using the flow lines of $(-\partial J)$, from (3.2) and (3.3) it follows that

$$J^{S+\frac{3S}{4}} \simeq J^{S+\frac{S}{4}},$$

where $\simeq$ denotes retract by deformation. Hence by (2.3), we obtain

$$J_1^{S+\frac{S}{2}} \simeq J^{S+\frac{S}{4}}.$$

It is known that $J_1^{S+\frac{S}{2}}$ and $\mathbb{B}^3$ have the same homotopy type. See [2, Remark 5] and [6, Remark 3]. Thus, $J_1^{S+\frac{S}{2}}$ is contractible. This leads to the controllability of $J^{S+\frac{S}{4}}$. Hence claim (2.4) follows. Let

$$H : [0,1] \times X^\infty_{\ell_0} \to J^{S+\frac{S}{4}}$$

be a contraction of $X^\infty_{\ell_0}$ in $J^{S+\frac{S}{4}}$ and let $\Theta(X^\infty_{\ell_0}) = H([0,1] \times X^\infty_{\ell_0})$. $\Theta(X^\infty_{\ell_0})$ is a contractible stratified set of dimension $\ell_0 + 1$. Deform $\Theta(X^\infty_{\ell_0})$. By dimension argument and under assumption (i) of Theorem 1.1, we obtain

$$\Theta(X^\infty_{\ell_0}) \simeq X^\infty_{\ell_0}. \tag{3.5}$$

Apply now the Euler-Poincaré characteristic of both sides of (3.5), we obtain

$$1 = \sum_{y\in\mathcal{K}, 3-\widetilde{i}(y)\leq \ell_0} (-1)^{3-\widetilde{i}(y)}.$$

Such equality contradicts assumption (ii) of Theorem 1.1. This completes the proof. of Theorem 1.1.

Remark 3.1. Any function K of the form $K = 1 + \varepsilon K_0$, where $K_0 \in C^2(\overline{\mathbb{B}^3})$, having more than one local maximum on $\mathbb{B}^3$, no critical points of Morse index 2 and satisfying $\frac{\partial K_0}{\partial \nu} < 0$ on $\partial \mathbb{B}^3$, satisfies the assumptions of Theorem 1.1 for $\varepsilon > 0$ small enough.

Next, we provide an explicit example of function $K = 1 + \varepsilon K_0$ satisfying the conditions of Theorem 1.1. For this, we construct a C^2-function K_0 on $\overline{\mathbb{B}^3}$ having only three critical points y_1, y_2 and y_3 which are nondegenerate with $\operatorname{ind}(K, y_1) = \operatorname{ind}(K, y_2) = 3$ and $\operatorname{ind}(K, y_3) = 0$. Moreover, it satisfies $\frac{\partial K}{\partial \nu} < 0$ on $\partial\mathbb{B}^3$. In that case, K satisfies the hypothesis of Theorem 1.1, for ε positive small enough and fore $\ell_0 = 1$. To define K_0, let $y_1 = (1/2, 0, 0), y_2 = (-1/2, 0, 0)$ and $y_3 = 0_{\mathbb{R}^3}$. For $\rho = 1/8$, we define the cut-off function $\phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\phi(t) = \begin{cases} 1 & \text{if } t \le \rho \\ 0 & \text{if } t \ge 2\rho \\ \phi'(t) < 0 & \text{if } \rho < t < 2\rho. \end{cases}$$

For $x \in \overline{\mathbb{B}^3}$, we define

$$\begin{aligned} &K_0(x) \\ &= -\phi(\|x - y_1\|^2)\|x - y_1\|^2 - \phi(\|x - y_2\|^2)\|x - y_2\|^2 + \phi(\|x - y_3\|^2)\|x - y_3\|^2 \\ &\quad - \Big[1 - \phi(\|x - y_1\|^2) - \phi(\|x - y_2\|^2) - \phi(\|x - y_3\|^2)\Big]\|x\|^2. \end{aligned}$$

Observe that inside the balls $B(y_i, \rho)$, $i = 1, 2, 3$, we have

$$K_0(x) = \begin{cases} -\|x - y_i\|^2 & \text{if } i = 1, 2; \\ \|x - y_i\|^2 & \text{if } i = 3. \end{cases}$$

Therefore, y_i, $i = 1, 2$ are nondegenerate critical points of Morse index 3 and y_3 is a nondegenrate critical points of Morse index 0. Recall that a critical point y of a function f is said nondegenerate if the Hessian matrix of f at y, $\operatorname{Hess}_y f$ is nondegenrate, in that case the Morse index of f at y, $\operatorname{ind}(f, y)$ is defined as the number of negative eigenvalues of $\operatorname{Hess}_y f$. Observe also that outside of $B(y_i, \rho)$, $i = 1, 2, 3$, we have $K_0(x) = -\|x\|^2$. Therefore, for any $x \in \partial\mathbb{B}^3$ we have

$$\frac{\partial K}{\partial \nu}(x) = -2\langle x, x\rangle = -2, \tag{3.6}$$

since $\nu(x) = x$ for all $x \in \partial\mathbb{B}^3$.

Remark 3.2. Theorem 1.1 can be extended in dimension $n \ge 4$ as follows. We assume that for every $y \in \mathcal{K}$, we have

$$\begin{gathered} \Delta K(y) \neq 0, \quad \text{if } n \ge 5, \\ \frac{1}{3}\Delta K(y) - 8H(y, y) \neq 0 \quad \text{if } n = 4. \end{gathered}$$

Let $\mathcal{K}^+ = \{y \in \mathcal{K} : -\Delta K(y) > 0\}$ if $n \ge 5$ and $\mathcal{K}^+ = \{y \in \mathcal{K} : -\frac{1}{3}\Delta K(y) + 8H(y, y) > 0\}$ if $n = 4$. Then, under the assumption (3.6), Theorem 1.1 is valid in dimension $n \ge 4$ by replacing $\mathcal{K}$ by $\mathcal{K}^+$.

Remark 3.3. We point out that the results of this note hold if we replace $\mathbb{B}^n$ by any smooth bounded contractible domain Ω of $\mathbb{R}^n$, $n \ge 3$. therefore, the question related to the existence of solution under the assumptions of Theorem 1.1 (or the assumptions in Remark 3.2) on a non contractible domain remains open.

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