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DYNAMICAL BIFURCATION IN A SYSTEM OF COUPLED OSCILLATORS WITH SLOWLY VARYING PARAMETERS

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Abstract. This paper deals with a fast-slow system representing n nonlinearly coupled oscillators with slowly varying parameters. We find conditions which guarantee that all ω-limit sets near the slow surface of the system are equilibria and invariant tori of all dimensions not exceeding n, the tori of dimensions less then n being hyperbolic. We show that a typical trajectory demonstrates the following transient process: while its slow component is far from the stationary points of the slow vector field, the fast component exhibits damping oscillations; afterwards, the former component enters and stays in a small neighborhood of some stationary point, and the oscillation amplitude of the latter begins to increase; eventually the trajectory is attracted by an n-dimesional invariant torus and a multi-frequency oscillatory regime is established.

1. Introduction

The coupled oscillators theory plays a significant role in understanding various patterns of collective behavior occurring in physical, chemical, biological and social systems (see, e. g., [22, 4] and references therein). The variety of behaviors exhibited by systems of coupled oscillators (SCO) ranges from synchronization to complex chaotic motions. In many cases, transient processes in SCO eventually turn into self-excited multi-frequency (quasiperiodic) oscillations on toric attractors. Such a type of behavior in non-conservative systems was observed as early as in the 20s–30s of the XX century and since that time was intensively studied (see, e. g. [9, 10, 13, 17, 19, 30, 31, 39, 43, 45]). In the middle of the XX century, there was discovered a phenomenon of a 2-dimensional torus bifurcation accompanying the stability loss of a limit cycle [27, 32, 41]. Later, studies on bifurcations of invariant tori and quasiperiodic oscillations were conducted by many authors (see, e. g., [6, 7, 8, 14, 20, 24, 29] and references therein) and the actual toolkit for qualitative investigation of such bifurcations was developed in [10, 15, 19, 31, 42, 43].

The aforementioned results concern static bifurcation theory which deals with systems dependent on time-constant parameters. Within the framework of this theory, the birth of a stable k-dimensional invariant torus from an equilibrium of

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the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^d, \quad (\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}), \tag{1.1}$$

dependent on the m-dimensional parameter u can be ensured by the following conditions: there exists a sufficiently smooth curve (equilibrium curve) $x = \xi(s), u = \upsilon(s)$, $s \in (-1, 1)$, such that $f(\xi(s), \upsilon(s)) = 0$ for all $s \in (-1, 1)$; the eigenvalues of the Jacobi matrix $f'_x(\xi(s), \upsilon(s))$ have negative real parts for all $s \in (-1, 0)$ and positive real parts for all $s \in (0, 1)$; there exists a sufficiently smooth mapping $\Xi(\cdot, \cdot)\colon \mathbb{T}^k \times [0, 1) \to \mathbb{R}^n$ (here $\mathbb{T}^k := \mathbb{R}^k/2\pi\mathbb{Z}^k$ denotes the standard torus), such that $\Xi(\varphi, \upsilon(0)) = \xi(0)$ for all $\varphi \in \mathbb{T}^k$ and $\operatorname{rank} \Xi'_\varphi(\varphi, \upsilon(s)) = k$ for all $\varphi \in \mathbb{T}^k$, $s \in (0, 1)$; finally, for any $s \in (0, 1)$ the toroidal surface $x = \Xi(\varphi, \upsilon(s))$ is a local attractor of the flow generated by the system $\dot{x} = f(x, \upsilon(s))$. Under such conditions, when the parameter u, restricted to the curve $u = \upsilon(s)$, passes through the point $u = \upsilon(0)$, we observe the stability loss of equilibrium and the birth of a stable invariant torus. It should be stressed that here the verb "passes" does not have any relation to a parameter motion over time.

On the contrary to the theory of static bifurcations, dynamical bifurcation theory deals with systems which depend on slowly varying in time parameters (fast-slow systems). Dynamical bifurcation theory focuses on qualitative behavioral transformations which happen in fast-slow systems due to the slow passage of parameters through certain critical points in the parameter space. The origin of this theory can be found in papers on relaxation oscillations (see the review [1]), although the term "dynamical bifurcation" appeared later, in the 80s of the XX century. The papers [47, 33, 34, 35, 5] gave start to studies of actually dynamic bifurcations in fast-slow systems. During the last several decades many important results concerning the considered area were obtained [12, 23, 11, 46, 40, 3, 26, 18, 28]. Some of the most peculiar features of fast-slow systems, such as the delayed loss of stability, the synchronization, the existence of the canard solutions and the blue-sky catastrophe can be of great importance in the real-world applications. Nevertheless, some phenomena have not yet been fully understood. In particular, as it was noted in [2], this can be said about the emergence of multi-frequency oscillations as a result of parameters evolution in fast-slow systems.

The present paper grounds on our previous results [37, 38] and aims to fill the gap above. Here we consider the SCO governed by the n-dimensional second order system

$$\ddot{w} + \Omega_0^2(u)w = 2\varepsilon\Lambda(u)\dot{w} + F(w, \dot{w}, u, \mu), \tag{1.2}$$

dependent on external (environmental) parameters $u = (u_1, \dots, u_m)$ and small positive parameters $\varepsilon, \mu \ll 1$. Here

$$\begin{gathered}
\Lambda(u) := \operatorname{diag}(\lambda_1(u), \dots, \lambda_n(u)), \quad \lambda_j(\cdot) \in \mathrm{C}^\infty(\mathbb{R}^m; \mathbb{R}), \\
\Omega_0^2(u) := \operatorname{diag}(\omega_{01}^2(u), \dots, \omega_{0n}^2(u)), \quad \omega_{0j}(\cdot) \in \mathrm{C}^\infty(\mathbb{R}^m; \mathbb{R}_{++}), \\
F(\cdot) \in \mathrm{C}^\infty(\mathbb{R}^{2n+m+1}; \mathbb{R}^n)
\end{gathered}$$

and

$$F(w, p, u, 0) = O(\|w\|^2 + \|p\|^2), \quad \|w\|^2 + \|p\|^2 \to 0, \tag{1.3}$$

where $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ stands for the Euclidean norm associated with the standard dot product in the coordinates vector space. Hereafter, we will also use a norm $|\cdot|$

defined as the sum of the absolute values of vector components. We study the case where the slow evolution of parameters u is governed by the system

$$\dot{u} = \mu G(w, \dot{w}, u, \mu) \tag{1.4}$$

where $G(\cdot) \in \mathrm{C}^\infty(\mathbb{R}^{2n+m+1}; \mathbb{R}^m)$. Thus, the parameter μ plays the role of the so-called slowness parameter. When $\mu = 0$, the system

$$\ddot{w} + \Omega_0^2(u)w = 2\varepsilon\Lambda(u)\dot{w} + F(w, \dot{w}, u, 0), \quad \dot{u} = 0 \tag{1.5}$$

has an invariant surface given by the equations $w = \dot{w} = 0$ and the parameter ε is responsible for the oscillations damping rate (oscillations growth rate) of variables $w, \dot{w}$ near this surface for those u that belong to the stability zone where all $\lambda_j(u) < 0$ (complete instability zone where all $\lambda_j(u) > 0$). The presence of two small parameters in fast-slow systems is a rather usual case. Initially these small parameters are completely independent, however, later we will impose a restriction that $\mu \propto \varepsilon$. All the functions involved in systems (1.2) and (1.4) may also continuously depend on ε, but we will not show this explicitly.

By the terminology of [1], the equations $w = \dot{w} = 0$ define the so-called slow surface in the phase space $\mathbb{R}^{2n+m}$, the vector $g(u) := G(0, 0, u, 0)$ is called the slow velocity vector, and, in this way, we obtain the slow system on the slow surface

$$\dot{u} = g(u).$$

In [23, 37, 38], there was considered the case when the slow system has a unique equilibrium attracting all its other trajectories. Here we study a more general situation allowing multiple equilibria, among which are stable, hyperbolic and completely unstable ones, but require the slow vector field to be gradient-like. This means that there exists a Morse function $V(\cdot) \in \mathrm{C}^2(\mathbb{R}^m; \mathbb{R}_+)$ such that $\dot{V}_g(u) := \langle \nabla V(u), g(u) \rangle < 0$ for any non-stationary point u of $V(\cdot)$. We find additional conditions under which a neighborhood of the slow surface is forward invariant under the semi-flow of system (1.2)–(1.4) and the set of all ω-limit points contained in this neighborhood consists of equilibria and invariant tori of all dimensions less than or equal to n. We show that a typical forward semi-trajectory starting at $(w_0, \dot{w}_0, u_0)$, where u_0 belongs to the instability zone of the fast system (1.5), demonstrates the following transient process: while the slow component $u(t)$ is far from the stationary points of the Morse function V, the fast component $(w(t), \dot{w}(t))$ exhibits damping oscillations; afterwards, this component eventually enters and stays in a small neighborhood of some stationary point, and the oscillation amplitude of the fast component begins to increase. Since the trajectory is attracted by an invariant torus, eventually a multi-frequency oscillatory regime is established. Such behavior can be naturally interpreted as the dynamical bifurcation of multi-frequency oscillations.

In fact, we will also be able to categorize the solutions by their ultimate behavior near the slow surface. It will be shown that in a small neighborhood of the slow surface most of the system's trajectories, in terms of the Lebesgue measure, are attracted to trajectories on the stable n-dimensional invariant torus, while the rest ones lie on the stable manifolds of hyperbolic tori of dimension less than n.

The current article is organized as follows. Section 2 provides the key hypotheses regarding system (1.2)–(1.4) and the statement of the main theorem. Then, in section 3 we introduce auxiliary lemmas, which enable us to state in section 4 certain preliminary results on the system's dynamics near its slow surface, and, consequently, describe the solutions behavior and classify them in sections 5, 6. After that, we provide an example depicting oscillation excitation in a circuit of two coupled oscillators which have components with temperature dependent properties. Finally, the paper ends with an addendum containing information on the normal forms method for systems with slowly varying parameters.

2. Main theorem

Let us describe the conditions imposed on system (1.2)–(1.4)

We will assume that the slow vector field $u \mapsto g(u) := G(0,0,u,0)$ satisfy the following conditions

(H1) there exists a Morse function $V(\cdot) \in \mathrm{C}^2(\mathbb{R}^m; \mathbb{R}_+)$ with the properties:
  1. $V(\cdot)$ has a non-empty finite set of stationary points;
  2. $V(u) \to +\infty$ when $\|u\| \to \infty$;
  3. $\dot{V}_g(u) := \langle \nabla V(u), g(u) \rangle < 0$ for any non-stationary point u of $V(\cdot)$;
  4. the Hesse matrix $\frac{\partial^2}{\partial u^2}\dot{V}_g(u)$ is negative definite at any stationary point of $V(\cdot)$.

Then, according to this hypothesis, any level set of $V(\cdot)$ is compact and if $V^* > 0$ is sufficiently large, then the sub-level set

$$\mathcal{V} := V^{-1}([0, V^*)) = \{u \in \mathbb{R}^n : V(u) < V^*\}$$

contains the set

$$\mathcal{W} := \{u \in \mathbb{R}^m : \nabla V(u) = 0\}$$

of all stationary points of $V(\cdot)$. Moreover, there exist such $\nu^* \geq \nu_* > 0$ and $\delta > 0$, that for any stationary point u_* the following inequalities hold

$$-\nu^* \|u - u_*\|^2 < \dot{V}_g(u) \leq -\nu_* \|u - u_*\|^2, \quad \|\nabla V(u)\| \leq \nu^* \|u - u_*\| \tag{2.1}$$

for all $u \colon \|u - u_*\| \leq \delta$, and δ-neighborhoods of any two points of $\mathcal{W}$ do not intersect. Obviously, $\mathcal{W}$ thereby coincides with the set of all singular points of the vector field g.

Now, for such a number V^*, let us adopt certain non-resonant hypotheses which are necessary for construction of the system's normal form in the whole domain $\mathcal{V}$, as well as in a vicinity of the set $\mathcal{W}$.

(H2) if $u \in \mathrm{cl}(\mathcal{V})$, then

$$\omega_{0i}(u) \neq \omega_{0j}(u), \quad \omega_{0i}(u) \neq \omega_{0j}(u) + \omega_{0k}(u), \quad \omega_{0i}(u) \neq \omega_{0j}(u) + \omega_{0k}(u) \pm \omega_{0l}(u)$$

for all $i, j, k, l \in \{1, 2, \dots, n\}$.

(H3) there exists such a number $N \geq 5$, that for any $u_* \in \mathcal{W}$ the equality

$$\sum_{j=1}^{n} (q_j - q_{j+n}) \omega_{0j}(u_*) = \sigma \omega_{0i}(u_*),$$

where $\sigma \in \{0, 1\}$, $i \in \{1, \dots, n\}$, $\mathbf{q} = (q_1, \dots, q_{2n}) \in \mathbb{Z}_+^{2n}$, $4 \leq \sum_{j=1}^{2n} q_j \leq N$, is valid iff $q_i = q_{i+n} + \sigma$ and $q_j = q_{j+n}$ for all $j \in \{1, \dots, n\} \setminus \{i\}$.

Furthermore, we may assume that for all $\varepsilon \in [0, \varepsilon_0]$, with $\varepsilon_0 > 0$ being small enough, and for all $u \in \mathrm{cl}(\mathcal{V})$ the frequencies

$$\omega_j(u, \varepsilon) := \sqrt{\omega_{0j}^2(u) - (\varepsilon \lambda_j(u))^2} = \omega_{0j}(u) + O(\varepsilon^2), \quad j \in \{1, \dots, n\}.$$

are correctly defined and satisfy hypotheses (H2), (H3) where each $\omega_{0j}(u)$ is replaced with $\omega_j(u,\varepsilon)$. Hereafter, to simplify our notations, we will omit explicit dependencies of functions on ε, as long as it does not lead to confusion. Thus, we will use the notation $\omega_j(u)$ instead of $\omega_j(u,\varepsilon)$ and so on.

Proceeding to the new variables $w,\dot w\mapsto x=(x_1,\dots,x_{2n})$ by means of a substitution

$$w_j=x_{2j-1},\quad \dot w_j=\varepsilon\lambda_j(u)x_{2j-1}+\omega_j(u)x_{2j},\quad j\in\{1,\dots,n\},\tag{2.2}$$

we come to an equivalent system

$$\dot x=J(u)x+\hat F(x,u,\mu),\quad \dot u=\mu\hat G(x,u,\mu),\tag{2.3}$$

where

$$\begin{gathered}
J(u):=\operatorname{diag}\Big[\begin{pmatrix}\varepsilon\lambda_1(u) & \omega_1(u)\\ -\omega_1(u) & \varepsilon\lambda_1(u)\end{pmatrix},\dots,\begin{pmatrix}\varepsilon\lambda_n(u) & \omega_n(u)\\ -\omega_n(u) & \varepsilon\lambda_n(u)\end{pmatrix}\Big],\\
\hat G(x,u,\mu):=G(w,\dot w,u,\mu)\big|_{w,\dot w\mapsto x},\\
\hat F(x,u,\mu)=(\hat F_1(x,u,\mu),\dots,\hat F_{2n}(x,u,\mu)),\\
\hat F_{2j-1}(x,u,\mu)\equiv 0,
\end{gathered}$$

$$\begin{aligned}
\hat F_{2j}(x,u,\mu):=&\frac{1}{\omega_j(u)}F(w,\dot w,u,\mu)\big|_{w,\dot w\mapsto x}-\mu\varepsilon\sum_{i=1}^m\frac{\partial\lambda_j(u)}{\partial u_i}\hat G_i(x,u,\mu)x_{2j-1}\\
&-\mu\sum_{i=1}^m\frac{\partial\omega_j(u)}{\partial u_i}\hat G_i(x,u,\mu)x_{2j}.
\end{aligned}$$

In view of (1.3), when $\mu=0$, system (2.3) has a slow invariant manifold of equilibria $\mathcal{M}_0$ defined by the equation $x=0$. Alike static bifurcation theory [7, 8], we will study the behavior of system (2.3) in a neighborhood of this manifold. And to do so, our first step will be finding conditions that guarantee the forward semi-invariance of such a neighborhood. This can be achieved by transforming the N-jet of system (2.3) to the normal form with respect to variables x.

Let $\mathcal{D}_N\subset\mathbb{R}^m$ denote a domain (or a collection of domains), such that for a fixed natural N and for any $k\in\{1,\dots,n\}$ and $\sigma\in\{0,1\}$ the equality

$$\min_{u\in\operatorname{cl}\mathcal{D}_N}\big|\sum_{l=1}^n(q_l-q_{l+n}-\sigma\delta_{kl})\omega_{0l}(u)\big|=0,$$

where $\mathbf{q}\in\mathbb{Z}_+^{2n}, 1\le|\mathbf{q}|:=\sum_{k=1}^{2n}q_k\le N$, fulfills iff

$$q_l-q_{l+n}-\sigma\delta_{kl}=0\quad\forall l\in\{1,\dots,n\}.$$

Remark 2.1. Hypothesis (H2) guarantees that $\mathcal{D}_3=\mathcal{V}$, and under hypothesis (H3) the domain $\mathcal{D}_N$ is non-empty and contains some neighborhood of the set $\mathcal{W}$.

If we introduce a vector of complex coordinates $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ and notations

$$\overrightarrow{|z|}:=(|z_1|,\dots,|z|_n),\quad \overrightarrow{|z|}^2:=(|z_1|^2,\dots,|z|_n^2),\quad \overrightarrow{|z|}^{\mathbf{p}}:=\prod_{k=1}^n|z_k|^{p_k},$$

where $\mathbf{p}:=(p_1,\dots,p_n)\in\mathbb{Z}_+^n$, then, as it is shown in the Addendum, for all sufficiently small μ and ε there exists a smoothly diffeomorphic change of variables

$(x,u)\mapsto(\operatorname{Re}z,\operatorname{Im}z,v)$ which for $v\in\mathcal{D}_N$ transforms system (2.3) into

$$\begin{aligned}\dot z_k &= \Big[\varepsilon\lambda_k(v)+\mathrm{i}\omega_k(v)+\sum_{3\le 2|\mathbf{p}|+1\le N}h_{0,k,\mathbf{p}}(v)\overrightarrow{|z|}^{2\mathbf{p}}\Big]z_k+\sum_{j=1}^{P}\mu^j\eta_{j,k}(v)z_k\\ &\quad+\sum_{j=1}^{P}\mu^j\Big[\sum_{3\le 2|\mathbf{p}|+1\le N}h_{j,k,\mathbf{p}}(v)\overrightarrow{|z|}^{2\mathbf{p}}\Big]z_k+O(\|z\|^{N+1}+\mu^{P+1}),\\ &\qquad k\in\{1,\dots,n\},\\ \dot v &= \mu\Big[g(v)+\sum_{2\le 2|\mathbf{p}|\le N}g_{0,\mathbf{p}}(v)\overrightarrow{|z|}^{2\mathbf{p}}+\sum_{j=1}^{P}\mu^j\sum_{0\le 2|\mathbf{p}|\le N}g_{j,\mathbf{p}}(v)\overrightarrow{|z|}^{2\mathbf{p}}\Big]\\ &\quad+\mu O(\|z\|^{N+1}+\mu^{P+1}).\end{aligned}\tag{2.4}$$

Here $P\ge N/2$ is an arbitrary fixed natural number, $\eta_{j,k}(\cdot),h_{j,k,\mathbf{p}}(\cdot)$ are smooth complex-valued functions in $\mathcal{D}_N$ and $g_{j,\mathbf{p}}(\cdot)$ are smooth $\mathbb{C}^n$-valued functions in $\mathcal{D}_N$. Besides that, all these functions smoothly depend on the parameter ε. The remainder terms are smooth in the sense of real calculus on the set

$$\|z\|<\delta,\quad v\in\mathcal{D}_N,\quad \mu\in[0,\mu_0],\quad \varepsilon\in[0,\varepsilon_0]$$

with sufficiently small positive numbers δ, μ_0 and ε_0, and are uniform with respect to $v\in\mathcal{D}_N$ and $\varepsilon\in[0,\varepsilon_0]$.

Further, we will also denote

$$\lambda(v)=(\lambda_1(v),\dots,\lambda_n(v)),\quad A(v)=\{a_{kl}(v)\}_{k,l=1}^{n},$$
$$a_{kl}(v):=-\operatorname{Re}h_{0,k,\mathbf{e}_l}(v),\quad b_{kl}(v):=-\operatorname{Im}h_{0,k,\mathbf{e}_l}(v),$$

where $\mathbf{e}_l\in\mathbb{Z}_+^n$ is a vector having its l-th component equal 1 and all other equal 0. As we will see later, the functions $\lambda(v)$ and $A(v)$ play the key role in emergence of the bifurcation phenomenon, which is why we require them to satisfy additional constraints.

- (H4) the symmetrical part of the matrix $A(v)$ is positive definite on $\operatorname{cl}(\mathcal{V})$, and all non-diagonal elements $a_{ij}(v)$, $i\ne j$, are non-positive at any stationary point $v\in\mathcal{W}$.
- (H5) the set $\mathcal{V}$ admits representation as a union of three nonempty sets

$$\mathcal{V}_+:=\{v\in\mathcal{V}\colon \lambda_j(v)>0\quad\forall j\in\{1,\dots,n\}\},$$
$$\mathcal{V}_-:=\{v\in\mathcal{V}\colon \lambda_j(v)<0\quad\forall j\in\{1,\dots,n\}\},$$
$$\mathcal{V}_*=\mathcal{V}\setminus[\mathcal{V}_+\cup\mathcal{V}_-],$$

  and each function $\lambda_j(\cdot)$, $j\in\{1,\dots,n\}$, is positive at any stationary point of $V(\cdot)$.

Note, that $\mathcal{W}\subset\mathcal{V}_+$ and for system (2.3) with $\mu=0$ the submanifold $\{(x,u)\in\mathcal{M}_0:u\in\mathcal{V}_-\}$ is a local attractor, while $\{(x,u)\in\mathcal{M}_0:u\in\mathcal{V}_-\}$ is a local repeller.

Let us fix sufficiently small $\kappa>0$ in such a way that

$$\mathcal{V}_-^{\kappa}:=\{v\in\mathcal{V}_-:\lambda_j(v)\le-\kappa\quad\forall j\in\{1,\dots,n\}\}\ne\emptyset.$$

Now we are in a position to state our main result.

Theorem 2.2. *There exist $\rho_0>0$, $\varsigma_0>0$, and for any $\varsigma_*\in(0,\varsigma_0)$ there is $\varepsilon_0>0$, such that once $\varepsilon\in(0,\varepsilon_0)$ and $\mu\in(\varsigma_*\varepsilon,\varsigma_0\varepsilon)$, the following statements are true:*

(1) *system* (2.4) *generates a forward semi-flow on the set*
$$\mathfrak{A} := \{(z,v) \in \mathbb{C}^n \times \mathbb{R}^m : \|z\| \le \rho_0, v \in \mathcal{V}\};$$
(2) *if* $\mathbb{R}_+ \ni t \mapsto (z(t), u(t))$ *is a solution of system* (2.4)*, such that* $(z(0), v(0)) \in \mathfrak{A}$ *and* $(t_0, t_1) \in \mathbb{R}_+$ *is an interval, such that* $\|z(t)\| > \mu^P$ *and* $v(t) \in \mathcal{V}_-^\kappa$ *for all* $t \in (t_0, t_1)$*, then*
$$\|z(t)\| \le \mathrm{e}^{-\varepsilon\kappa(t-t_0)/2}\|z(t_0)\| \quad \forall t \in (t_0, t_1);$$
(3) *to any point of the set* $\mathcal{W}$*, one can put into correspondence a finite collection of invariant tori belonging to* $\mathfrak{A}$*, and each such collection contains tori of all dimensions from* 1 *to* n*; in addition, any torus of a dimension less then* n *is truly hyperbolic, while any* n*-dimensional torus corresponding to a local minimum of the Morse function* $V(\cdot)$ *is a local attractor of system* (2.4)*;*
(4) *any non-equilibrium forward semi-trajectory of this system lying in* $\mathfrak{A}$ *is attracted by one of the invariant tori, and those trajectories that are attracted by* n*-dimensional tori form the set of the full Lebesgue measure in* $\mathfrak{A}$*;*
(5) *each forward semi-trajectory approaching the* n*-dimensional invariant torus is attracted by a forward semi-trajectory lying on this torus.*

The rest of this article will be devoted to proving and illustrating this theorem.

3. Auxiliary lemmas

Lemma 3.1. *For* $\delta > 0$*, set* $\bar{B}^d_\delta := \{x \in \mathbb{R}^d : \|x\| \le \delta\}$*. Suppose that there exists a Morse function* $W(\cdot) \in \mathrm{C}^2(\bar{B}^d_\delta; \mathbb{R})$ *having a unique stationary point* $x_* = 0$ *and a vector field* $f(\cdot) \in \mathrm{C}^1(\bar{B}^d_\delta; \mathbb{R}^d)$*, such that*
$$\langle \nabla W(x), f(x)\rangle \le -\theta\|x\|^2, \quad \|f'(x)\| \le \theta L \quad \forall x \in \bar{B}^d_\delta(0),$$
where L *and* θ *are some positive constants. Define*
$$K := \max_{\|x\| \le \delta} \|W''(x)\|$$
and let M *and* ϵ *be arbitrary positive numbers satisfying*
$$M \ge M_*(K,L) := \frac{1 + KL + \sqrt{1 + 2K^3L^3}}{L}, \quad M\epsilon < \delta. \tag{3.1}$$
Then for any $f_1(\cdot) \in \mathrm{C}^1([\tau_0,\infty) \times \bar{B}^d_\delta; \bar{B}^d_\theta)$ *and any* $x_0 \in B^d_\delta$ *a solution* $x(t)$ *of the initial problem*
$$\dot{x} = f(x) + \varepsilon f_1(t,x), \quad x(\tau_0) = x_0 \tag{3.2}$$
meets the following alternative: either there exists such $\tau^* > \tau_0$*, that* $\|x(\tau^*)\| = \delta$*, or there exists such* $\tau_* \ge \tau_0$*, that* $\|x(t)\| < M\epsilon$ *for all* $t > \tau_*$*.*

Additionally, if $0 < \epsilon < \delta/N(K,L,M)$*, where*
$$N(K,L,M) := \frac{1 + KL + \sqrt{[1 + (K-M)L]^2 + 2KL^3M^2}}{L} > M, \tag{3.3}$$
and if the first scenario takes place, but at some instant of time the solution belongs to $\bar{B}^d_{M\epsilon}$*, then*
$$W(x(\tau^*)) < W(0) - \frac{K(M\epsilon)^2}{2} \le \min\{W(x) : \|x\| = M\epsilon\}.$$

Furthermore, in the case when the stationary point $x_* = 0$ *is elliptic, numbers* $\beta \in (0, \delta)$ *and* $\epsilon > 0$ *are such, that*

$$\max\{W(x)\colon \|x\| = \beta\} < \min\{W(x)\colon \|x\| = \delta\}, \quad 0 < \epsilon < \beta/K, \tag{3.4}$$

and the solution of (3.2) *at some moment of time belongs to* $\bar{B}^d_\beta$, *then the second scenario fulfills.*

Proof. Since $\nabla W(0) = 0$, it is true that $\|\nabla W(x)\| \le K\|x\|$ and

$$\langle \nabla W(x), f(x) + \varepsilon f_1(t, x)\rangle \le \theta\left[-\|x\| + \epsilon K\right]\|x\| \quad \forall x \in \bar{B}^d_\delta.$$

Besides,

$$\max\{W(x)\colon \|x\| = \varrho\} \le W(0) + \frac{K\varrho^2}{2}, \quad \min\{W(x)\colon \|x\| = \varrho\} \ge W(0) - \frac{K\varrho^2}{2}$$

for any $\varrho \in (0, \delta]$. As the Hessian of $W(\cdot)$ at $x = 0$ is non-degenerate, we have $f(0) = 0$ and $\|f(x)\| \le \theta L\|x\|$. Hence,

$$\|f(x) + \varepsilon f_1(t, x)\| \le \theta(L\|x\| + \epsilon) \quad \forall x \in \bar{B}^d_\delta.$$

Now let us demonstrate how to choose M. At first, we will require only that $M \ge K$ and $M\epsilon < \delta$. Take an arbitrary $\varrho \in (K\epsilon, M\epsilon)$. If the moment τ^* does not exists, i. e. $\|x(t)\| < \delta$ for all $t \ge \tau_0$, then the function $W(x(t))$ strictly decreases until $x(t)$ reaches the sphere $\|x\| = \varrho$ at an instant of time $\tau_* \ge \tau_0$. The moment τ_* necessarily exists, since otherwise $W(x(t))$ would decrease unboundedly, which is impossible.

Suppose that $x(t)$ reaches the sphere $\|x\| = M\epsilon$ after the moment τ_*. Then there exist $\tau_2 > \tau_*$ and $\tau_1 \in [\tau_*, \tau_2)$, such that

$$\|x(\tau_1)\| = \varrho, \quad \varrho < \|x(t)\| < M\epsilon \quad \forall t \in (\tau_1, \tau_2), \quad \|x(\tau_2)\| = M\epsilon.$$

For $t \in [\tau_1, \tau_2]$, we have

$$\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} \le \frac{\langle x, f(x) + \varepsilon f_1(t, x)\rangle}{\|x\|}\Big|_{x = x(t)} \le \theta(L\|x(t)\| + \epsilon),$$

which implies

$$\frac{\mathrm{d}\|x(t)\|/\mathrm{d}t}{\theta(L\|x(t)\| + \epsilon)} \le 1$$

and

$$\begin{aligned} W(x(\tau_2)) - W(x(\tau_1)) &= \int_{\tau_1}^{\tau_2} \langle \nabla W(x), f(x) + \varepsilon f_1(t, x)\rangle\big|_{x = x(t)}\mathrm{d}t \\ &\le \int_{\tau_1}^{\tau_2} \theta\left[-\|x(t)\| + K\epsilon\right]\|x(t)\| \frac{\mathrm{d}\|x(t)\|/\mathrm{d}t}{\theta(L\|x(t)\| + \epsilon)}\mathrm{d}t \\ &= \int_{\varrho}^{M\epsilon} \frac{(-s + K\epsilon)s}{Ls + \epsilon}\mathrm{d}s. \end{aligned}$$

Taking into account that $W(x(\tau_1)) \le W(0) + K\varrho^2/2$ and making ϱ tend to $K\epsilon$, we obtain the estimate

$$\begin{aligned} W(x(\tau_2)) &\le W(0) + \frac{K^3\epsilon^2}{2} + \big[-\frac{s^2}{2L} + \frac{\epsilon(1 + KL)s}{L^2} - \frac{\epsilon^2(1 + KL)}{L^3}\ln\frac{Ls + \epsilon}{L}\big]_{K\epsilon}^{M\epsilon} \\ &< W(0) + \frac{K^3\epsilon^2}{2} - \frac{\epsilon^2}{2L}\big[M^2 - \frac{2(1 + KL)M}{L} + \frac{2(1 + KL)K}{L} - K^2\big]. \end{aligned}$$

If we introduce a quadratic polynomial of variable ξ,

$$\mathcal{P}(\xi;\epsilon,\eta) := \xi^2 - \frac{2(1+KL)\epsilon}{L}\xi + \frac{2(1+KL)\eta\epsilon^2}{L} - (1+2KL)\eta^2\epsilon^2,$$

where ϵ and η are positive parameters, one can verify that $M_*(K,L)$ is the greatest root of $\mathcal{P}(\xi;1,K)$. Thus, for any $M \geq M_*(K,L)$ and any $\epsilon < \delta/M$, we obtain $\mathcal{P}(M\varepsilon;\epsilon,K) = \varepsilon^2\mathcal{P}(M;1,K) \geq 0$. It means that

$$M^2 - \frac{2(1+KL)M}{L} + \frac{2(1+KL)K}{L} - K^2 \geq 2K^3L,$$

$$W(x(\tau_2)) < W(0) - \frac{K^3\epsilon^2}{2} \leq \min\{W(x)\colon \|x\| = K\epsilon\}.$$

Hence, the function $W(x(t))$ keeps on decreasing and satisfies for $t > \tau_2$ the inequality $W(x(t)) < W(x(\tau_2))$. This means that $x(t)$ never reaches the sphere $\|x\| = K\epsilon$, and moreover,

$$\inf_{t\geq\tau_2}[\|x(t)\| - K\epsilon] > 0.$$

As a consequence, $W(x(t)) \to -\infty$ as $t \to \infty$, and we come to a contradiction. Therefore, such a choice of M and ϵ guarantees the validity of the inequality $\|x(t)\| < M\epsilon$ for all $t > \tau_*$.

In a similar way, one can show that if $\|x(\tau')\| \leq M\epsilon$, but there exists $\tau^* > \tau'$, such that $\|x(\tau^*)\| = \delta$, then

$$\begin{aligned}
&W(x(\tau^*))\\
&\leq W(0) + \frac{KM^2\epsilon^2}{2} - \Big[\frac{s^2}{2L} - \frac{\epsilon(1+KL)s}{L^2}\frac{\epsilon^2(1+KL)}{L^3}\ln\frac{Ls+\epsilon}{L}\Big]_{M\epsilon}^{\delta}\\
&\leq W(0) + \frac{KM^2\epsilon^2}{2} - \frac{1}{2L}\Big[\delta^2 - \frac{2(1+KL)\epsilon}{L}\delta + \frac{2(1+KL)M\epsilon^2}{L} - (M\epsilon)^2\Big].
\end{aligned}$$

Since $N(K,L,M)$ is the greatest root of $\mathcal{P}(\xi;1,M)$, once $0 < N(K,L,M)\epsilon < \delta$, we have $\mathcal{P}(\delta;\epsilon,M) \geq 0$ and

$$W(x(\tau^*)) < W(0) - \frac{K(M\epsilon)^2}{2} \leq \min\{W(x)\colon \|x\| = M\epsilon\}.$$

Finally, if the point 0 is elliptic and inequalities (3.4) are fulfill, then

$$\langle\nabla W(x), f(x) + \varepsilon f_1(t,x)\rangle \leq [-\theta\|x\| + \epsilon K]\,\|x\| < 0, \tag{3.5}$$

as soon as $\beta \leq \|x\| \leq \delta$. Let the solution belong to $\bar{B}^d_\beta$ at some moment of time. If by reasoning ad absurdum we supposed that there existed such $\tau^* > \tau_0$, that $\|x(\tau^*)\| = \delta$, then there would exist $\tau'' < \tau^*$, such that

$$\|x(\tau'')\| = \beta, \quad \beta < \|x(t)\| < \delta \quad \forall t \in (\tau'', \tau^*).$$

Thereby, $W(x(\tau'')) \leq \max_{\|x\|=\beta} W(x) < \min_{\|x\|=\delta} W(x(\tau^*))$, which is impossible, since (3.5) yields that the function $W(x(t))$ is decreasing on (τ', τ^*). $\square$

Lemma 3.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain with a C^2-boundary, $\bar{D} := \mathrm{cl}(D)$ and let $W(\cdot) \in \mathrm{C}^2(\bar{D};\mathbb{R})$ be a Morse function with a finite set of stationary points $\mathfrak{W} \subset D$. Define*

$$K := \max_{x\in\bar{D}}\{\|\nabla W(x)\|, \|W''(x)\|\}$$

and choose sufficiently small $\delta > 0$ and $\beta \in (0,\delta)$ that meet the following requirements:

(1) *the δ-neighborhood of $\mathfrak{W}$ belongs to D;*
(2) *for any $x'_*, x''_* \in \mathfrak{W}$, such that $W(x'_*) > W(x''_*)$ the following inequality holds:*
$$\min\{W(x)\colon \|x - x'_*\| = \delta\} > \max\{W(x)\colon \|x - x''_*\| = \delta\};$$
(3) *for any elliptic point $x_* \in \mathfrak{W}$ there holds the inequality*
$$\max\{W(x)\colon \|x - x_*\| = \beta\} < \min\{W(x)\colon \|x - x_*\| = \delta\}.$$

Also, suppose that there is such a vector field $f(\cdot) \in \mathrm{C}^1(\bar{D}; \mathbb{R}^d)$, that for some positive constants θ, L such inequalities fulfill:
$$\langle \nabla W(x), f(x)\rangle < -\theta\delta^2 \quad \forall x \in D\colon \min_{y\in\mathfrak{W}} \|x - y\| > \delta,$$
$$\langle \nabla W(x), f(x)\rangle \le -\theta\|x - y\|^2, \quad \|f'(x)\| \le \theta L \quad \forall y \in \mathfrak{W}, \forall x\colon \|x - y\| \le \delta.$$
Then, with the corresponding functions defined by formulae (3.1) *and* (3.3)*, for any $M \ge M_*(K, L)$ and all $\epsilon \in (0, \epsilon_0(K, L, M))$, where*
$$\epsilon_0(K, L, M) := \min\{\frac{\delta^2}{K}, \frac{\beta}{N(K, L, M)}\},$$
the following assertion is correct. If $f_1(\cdot) \in \mathrm{C}^1([\tau_0, \infty) \times \bar{D}; \bar{B}^d_\theta)$, then for any such $x_0 \in D$, that the corresponding solution $x(\cdot)$ of initial problem (3.2) *is defined on $[\tau_0, \infty)$ and takes values in D, there exist $x_* \in \mathfrak{W}$ and $t_* > \tau_0$, such that $\|x(t) - x_*\| < M\epsilon$ for all $t > t_*$.*

Proof. Under the conditions of this lemma, we have
$$\langle \nabla W(x), f(x) + \varepsilon f_1(t, x)\rangle < 0 \quad \forall x\colon \min_{y\in\mathfrak{W}} \|x - y\| \ge M\epsilon. \tag{3.6}$$
Therefore, if the solution $x(\cdot)$ is defined on $[\tau_0, \infty)$ and takes values in D, then there exist $x^1_* \in \mathfrak{W}$ and $t_1 \ge \tau_0$, such that $\|x(t_1) - x^1_*\| < M\epsilon$. Indeed, otherwise, the function $W(x(t))$ would decrease unboundedly when $t \to \infty$, which is impossible.

By Lemma 3.1, the solution $x(\cdot)$ meets the following alternative: either for all $t \ge t_1$ we have $\|x(t) - x_*\| \le M\epsilon$, or there exists $t_2 > t_1$, such that $\|x(t_2) - x^1_*\| = \delta$ and
$$W(x(t_2)) < W(x^1_*) - \frac{K(M\epsilon)^2}{2} \le \min\{W(x)\colon \|x - x^1_*\| = M\epsilon\}.$$
The first case always takes place if x^1_* is elliptic. In the second one, on account of choice of δ and (3.6), there exist such $x^2_* \in \mathfrak{W}$ and $t_3 > t_2$, that $\|x(t_3) - x^2_*\| < M\epsilon$, and $W(x^2_*) < W(x^1_*)$. Now, it is clear that eventually the solution enters and then never leaves an $M\epsilon$-neighborhood of some point $x_* \in \mathfrak{W}$. □

For the sake of completeness of our exposition, let us represent the following result from the theory of non-negative invertible matrices.

Lemma 3.3. *Let a real matrix $P = \{p_{ij}\}^d_{i,j=1}$ have such properties:*
(1) *the matrix $P + P^\top$ is positive definite;*
(2) *$p_{ij} \le 0$ for any $i, j \in \{1, \dots, d\}$, $i \ne j$.*

Then for any vector $y = (y_1, \dots, y_d)$ with the positive elements each component of the vector $x := P^{-1}y$ satisfies the inequalities
$$x_i \ge \frac{y_i}{p_{ii}}, \quad i \in \{1, \dots, d\}, \tag{3.7}$$

$$|x| \le \frac{\max_{1\le i\le d} y_i}{p_+}, \tag{3.8}$$

where $p_+ := \min\{\langle P\xi,\xi\rangle : \xi \in \mathbb{R}^d_+, |\xi| = 1,\}$

Proof. The system $\dot{x} = -Px$, in which P has the property 1 is asymptotically stable. Hence, all eigenvalues of $-P$ have negative real parts. By [16, Theorem 5, XIII], all of the main minors of P are positive. Thus, according to [36, Theorem 6.3] the matrix P^{-1} has non-negative elements, and any row of this matrix contains at least one positive element. Obviously, then $x := P^{-1}y \in \mathbb{R}^d_{++}$, once $y \in \mathbb{R}^d_{++}$. Moreover, since

$$\sum_{j=1, j\ne i}^{d} p_{ij}x_j \ge 0,$$

the components x_i satisfy (3.7), and the inequalities

$$p_+|x|^2 \le \langle Px, x\rangle = \langle y, x\rangle \le \max_{1\le i\le d} y_i|x|$$

yield (3.8). □

4. Preliminary results on the behavior of the normalized system

Having introduced the aforementioned lemmas, we may proceed to the investigation of system (2.4) dynamics. This section provides the general description of the solutions behavior and suggests a way to classify them. Later we will refine this information. Define

$$\begin{gathered} a_+ := \min\{\langle A(v)r, r\rangle \colon \forall r \in \mathbb{R}^n_+, |r| = 1, v \in \mathrm{cl}(\mathcal{V})\}, \\ \lambda^+ := \max\{\langle \lambda(v), r\rangle \colon \forall r \in \mathbb{R}^n_+, |r| = 1, v \in \mathrm{cl}(\mathcal{V})\}. \end{gathered}$$

Proposition 4.1. *Assume hypotheses* (H1), (H2), (H4) *to be true and* $N = 3$. *Then there exist a sufficiently small* $\rho_0 > 0$ *and sufficiently large* $\rho_* > 0$, $R_* > 0$, *such that for any* $\rho > \rho_*$, $R > R_*$ *one can choose* $\varepsilon_0 > 0$ *in such a way, that once* $\varepsilon \in (0, \varepsilon_0]$ *and* $\mu \in [0, \varepsilon]$, *system* (2.4) *generates a forward semi-flow on the set* $\mathfrak{A} := \{(z, v) \in \mathbb{C}^n \times \mathbb{R}^m \colon \|z\| \le \rho_0, v \in \mathcal{V}\}$. *Furthermore, for any solution* $(z(t), v(t))$ *of* (2.4) *with the initial values* $(z(0), v(0)) \in \mathfrak{A}$, *there exist such a stationary point* v_* *of* $V(\cdot)$ *and an instant of time* t_*, *that*

$$\|z(t)\| < \rho\sqrt{\varepsilon} < \rho_0, \quad \|v(t) - v_*\| < R\varepsilon \quad \forall t \ge t_*.$$

Proof. There is a constant $c_1 > 0$, such that if $\|z\| \le \rho_0$, then a quadratic form $\|z\|^2 = \sum_{k=1}^n |z_k|^2$ and the function $V(\cdot)$ admit the following estimates for their directional derivatives along the vector field of system (2.4)

$$\begin{aligned} \|z\|^2\big|'_{(2.4)} &\le 2\varepsilon \sum_{k=1}^{n} \lambda_k(v)|z_k|^2 - 2\sum_{k,l=1}^{n} a_{kl}(v)|z_k|^2|z_l|^2 \\ &\quad + c_1(\mu\|z\|^2 + \|z\|^5 + \|z\|\mu^{P+1}) \\ &\le \big[(2\varepsilon\lambda^+ + c_1\mu)\|z\| - (2a_+ - c_1\rho_0)\|z\|^3 + c_1\mu^{P+1}\big]\|z\|, \\ V(v)\big|'_{(2.4)} &\le \mu\big[\dot{V}_g(v) + c_1\|\nabla V(v)\|(\|z\|^2 + \mu)\big]. \end{aligned} \tag{4.1}$$

It is easily seen that one can choose the positive numbers $\rho_0 < 2a_+/c_1$, and $\rho_* > 0$ in such a way, that for any $\rho > \rho_*$, $\mu \in [0, \varepsilon]$, $v \in \mathrm{cl}(\mathcal{V})$, $\varepsilon \in (0, \varepsilon_1]$, where

$\varepsilon_1 := \min\{1, \rho_0^2/\rho^2, \mu_0\}$, the inequality

$$\|z\|^2\big|'_{(2.4)} < 0,$$

holds as long as z satisfies $\rho\sqrt{\varepsilon} \le \|z\| \le \rho_0$. It means that $\mathfrak{A}$ is forward semi-invariant and, moreover, there exists $t_0 > 0$, such that $\|z(t)\| < \rho\sqrt{\varepsilon}$ for all $t > t_0$. Now we can regard $v(t)$ as a solution of a system $\dot{v} = \mu[g(v) + \tilde{g}(t, v, \varepsilon, \mu)]$ defined on $[t_0, \infty) \times \mathrm{cl}(\mathcal{V})$ and obtained from the corresponding sub-system of (2.4) via the substitution $z = z(t)$. Obviously, there exists such a constant $c_2 > 0$, that

$$\|\tilde{g}(t, v, \varepsilon, \mu)\| \le c_2\varepsilon \quad \forall (t, v, \varepsilon, \mu) \in [t_0, \infty) \times \mathrm{cl}(\mathcal{V}) \times (0, \varepsilon_1] \times [0, \varepsilon].$$

On account of (H1), (2.1) and (4.1), after an appropriate additional correction of δ in (2.1), the final part of the proof follows from Lemma 3.2 in the case when $f = \mu g$, $f_1 = \mu\varepsilon\tilde{g}$, $W = V$, $\epsilon = \varepsilon c_2$, $\theta \propto \mu$. In particular, if we find M_* and ϵ_0 from the lemma, we can set $R_* = M_*$, $R = M$ and $\varepsilon_0 = \min\{\epsilon_0/c_2, \varepsilon_1\}$. ☐

Corollary 4.2. *Let $0 < \mu < \varepsilon\kappa/(2c_1)$ and let $(t_0, t_1) \in \mathbb{R}_+$ be such an interval, that*

$$\mu^P < \|z(t)\| < \rho_0, \quad v(t) \in \mathcal{V}_-^\kappa \quad \forall t \in (t_0, t_1)$$

Then $\|z(t)\| \le \mathrm{e}^{-\varepsilon\kappa(t-t_0)/2}\|z(t_0)\|$ for all $t \in (t_0, t_1)$.

In fact, for $\mu^P \le \|z\| \le \rho_0$, $v \in \mathcal{V}_-^\kappa$ and $0 < \mu < \varepsilon\kappa/(2c_1)$, in the same way as in the proof of Proposition 4.1, we obtain the inequality

$$\begin{aligned}
\|z\|^2\big|'_{(2.4)} &\le \big[(-2\varepsilon\kappa + c_1\mu)\|z\| - (2a_+ - c_1\rho_0)\|z\|^3 + c_1\mu^{P+1}\big]\|z\| \\
&\le \big[-\frac{3\varepsilon\kappa}{2}\|z\| + \frac{\varepsilon\kappa}{2}\mu^P\big]\|z\| \le -\frac{\varepsilon\kappa}{2}\|z\|^2.
\end{aligned}$$

The above corollary proves the statement (2) of the main theorem.

Hypothesis (H3) and Proposition 4.1 allow us to focus on system (2.4) defined on the set

$$\{(r, v) \colon \|z\| < \rho\sqrt{\varepsilon}, \|v - v_*\| < R\varepsilon\}.$$

Hereafter, we will require the numbers ρ and R to be large enough.

Without loss of generality we may suppose that $v_* = 0$ in Proposition 4.1. Then, having applied the scaling $z \mapsto \sqrt{\varepsilon}z$, $v \mapsto \varepsilon v$ to (2.4), we obtain the system

$$\begin{aligned}
\dot{z}_k &= \Big[\mathrm{i}\omega_k(\varepsilon v) + \varepsilon\lambda_k(\varepsilon v) - \varepsilon\sum_{l=1}^n (a_{kl}(\varepsilon v) + \mathrm{i}b_{kl}(\varepsilon v))|z_l|^2\Big] z_k \\
&\quad + \sum_{5 \le 2|\mathbf{p}|+1 \le N} \varepsilon^{|\mathbf{p}|} h_{0,k,\mathbf{p}}(\varepsilon v)\overrightarrow{|z|}^{2\mathbf{p}} z_k + \sum_{j=1}^P \mu^j \eta_{j,k}(\varepsilon v) z_k \\
&\quad + \sum_{j=1}^P \mu^j \Big[\sum_{3 \le 2|\mathbf{p}|+1 \le N} \varepsilon^{|\mathbf{p}|} h_{j,k,\mathbf{p}}(\varepsilon v)\overrightarrow{|z|}^{2\mathbf{p}}\Big] z_k + O\big(\varepsilon^{N/2}\|z\|^{N+1} + \mu^P\big), \\
&\qquad k \in \{1, \dots, n\}, \\
\dot{v} &= \frac{\mu}{\varepsilon}\Big[g(\varepsilon v) + \sum_{2 \le 2|\mathbf{p}| \le N} \varepsilon^{|\mathbf{p}|} g_{0,\mathbf{p}}(\varepsilon v)\overrightarrow{|z|}^{2\mathbf{p}}\Big] \\
&\quad + \frac{\mu}{\varepsilon}\sum_{j=1}^P \mu^j \sum_{0 \le 2|\mathbf{p}| \le N} \varepsilon^{|\mathbf{p}|} g_{j,\mathbf{p}}(\varepsilon v)\overrightarrow{|z|}^{2\mathbf{p}} + O\big(\mu\varepsilon^{(N-1)/2}\|z\|^{N+1} + \mu^{P+2}/\varepsilon\big)
\end{aligned} \tag{4.2}$$

defined on the set

$$\{(z, v, \varepsilon, \mu)\colon \|z\| \le \rho, \|v\| \le R, \varepsilon \in (0, \varepsilon_0], \mu \in [0, \varepsilon]\}.$$

In this system we constrain the parameter μ to be $\mu = \varepsilon\varsigma$ with ς being an arbitrary fixed number satisfying

$$0 < \varsigma \le \varsigma_0 := \frac{1}{2} \min_{1\le j\le n} \frac{\lambda_j(0)}{|\eta_{1,j}(0)|}.$$

Note that such a condition ensures the validity of the inequality

$$\alpha_k := \lambda_k(0) + \varsigma \operatorname{Re} \eta_{1,k}(0) \ge \lambda_k(0)/2 > 0 \quad \forall k \in \{1, \dots, n\}.$$

Now, if we recall the earlier imposed condition $P \ge N/2$ and if for a vector $x = (x_1, \dots, x_d)$ we define $D[x] := \operatorname{diag}(x_1, \dots, x_d)$, then system (4.2) can be presented in the form

$$\begin{aligned} \dot z &= D[\mathrm{i}(\omega_0 + \varepsilon(\beta + \Omega' v - B\overrightarrow{|z|}^2))]z \\ &\quad + [\varepsilon(\alpha - A\overrightarrow{|z|}^2) + \varepsilon^2 \hat h(\overrightarrow{|z|}^2, v)]z + O(\varepsilon^{N/2}), \end{aligned} \tag{4.3}$$

$$\dot v = \varepsilon\varsigma[\Gamma v + \Upsilon\overrightarrow{|z|}^2 + \varepsilon \hat g(\overrightarrow{|z|}^2, v) + O(\varepsilon^{(N+1)/2})] \tag{4.4}$$

where, for the sake of notations simplicity, we assign

$$\begin{gathered} \omega_0 := (\omega_{01}(0), \dots \omega_{0n}(0)), \quad \eta_1 := (\eta_{1,1}(0), \dots, \eta_{1,n}(0)), \quad \alpha := \lambda(0) + \varsigma \operatorname{Re} \eta_1(0), \\ \beta := \zeta \operatorname{Im} \eta_1(0), \quad A := \big\{a_{kl}(0)\big\}_{k,l=1}^n, \quad B := \big\{b_{kl}(0)\big\}_{k,l=1}^n, \\ \Omega' := \Big\{\frac{\partial \omega_{0k}(0)}{\partial v_j}\Big\}_{k=1,}^n \quad \Gamma := g'(0), \quad \Upsilon x := \sum_{|\mathbf{p}|=1} g_{0,\mathbf{p}}(0) x^{\mathbf{p}}. \end{gathered}$$

The definitions of the remainder terms $\hat h(\cdot)$ and $\hat g(\cdot)$ inside the square brackets in the right-hand sides of (4.3), (4.4) is obvious. Recall that we agreed not to mention ε directly as functions arguments.

Proposition 4.3. *For all $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ being sufficiently small, system (4.3)–(4.4) has an equilibrium $(z^0, v^0) = (z^0(\varepsilon), v^0(\varepsilon))$, such that*

$$z^0(\varepsilon) = O(\varepsilon^{N/2}), \quad v^0(\varepsilon) = O(\varepsilon). \tag{4.5}$$

If $(z(t), v(t))$ is a solution of (4.3)–(4.4), such that $\|z(t)\| < \sqrt{\varepsilon}$ and $\|v(t)\| < R$ for all $t > 0$, then

$$\lim_{t\to\infty} \big[\|z(t) - z^0\| + \|v(t) - v^0\|\big] = 0, \tag{4.6}$$

and the set of all such solutions forms a manifold whose real dimension equals the number of eigenvalues of Γ with negative real parts. In the case when all eigenvalues of Γ have positive real parts the only solution with the stated property is the equilibrium (z^0, v^0).

Proof. The existence of the equilibrium $(z^0(\varepsilon), v^0(\varepsilon))$ satisfying (4.5) directly follows from the implicit function theorem. If $(z(t), v(t))$ is a solution of (4.3)–(4.4), such that $\|z(t)\| < \sqrt{\varepsilon}$ and $\|v(t)\| < R$ for all $t > 0$, then the functions

$$w(t) := \exp\Big[-\mathrm{i}\int_0^t D\Big[\Big(\omega_0 + \varepsilon(\Omega' v(s) - B\overrightarrow{|z(s)|}^2)\Big)\Big]\mathrm{d}s\Big] z(t)$$

and $v(t)$ satisfy a system of the form

$$\dot{w} = D\left[\varepsilon(\alpha - A\overrightarrow{|w|}^2) + \varepsilon^2\hat{h}(\overrightarrow{|w|}^2, v)\right]w + O(\varepsilon^{N/2}), \tag{4.7}$$

$$\dot{v} = \varepsilon\varsigma\left[\Gamma v + \Upsilon\overrightarrow{|w|}^2 + \varepsilon\hat{g}(\overrightarrow{|w|}^2, v) + O(\varepsilon^{(N+1)/2})\right], \tag{4.8}$$

which also has a solution

$$w = w^0(t) := \exp\Big[-\mathrm{i}\int_0^t D\big[(\omega_0 + \varepsilon(\Omega' v(s) - B\overrightarrow{|z(s)|}^2))\big]\mathrm{d}s\Big]z^0,$$
$$v = v^0, \quad t \geq 0.$$

Hence, the realification of system (4.7)–(4.8) has a pair of solutions

$$\xi(t) := (\operatorname{Re} w(t), \operatorname{Im} w(t), v(t)), \quad \xi^0(t) := (\operatorname{Re} w^0(t), \operatorname{Im} w^0(t), v^0).$$

One can consider the difference $\eta(t) := \xi(t) - \xi^0(t)$ to be a bounded solution for $t \geq 0$ of the linear system $\dot{\eta} = \varepsilon[\mathcal{A} + \mathcal{A}_1(t;\varepsilon)]\eta$ where

$$\mathcal{A} = \operatorname{diag}[D[\alpha], D[\alpha], \varsigma\Gamma],$$

and $\sup_{t\geq 0}\|\mathcal{A}_1(t;\varepsilon)\| = O(\sqrt{\varepsilon})$. Such a system is hyperbolic on $[0,\infty)$ if ε is small enough, and each of its bounded solutions approaches zero as $t \to \infty$. This yields (4.6).

Next, it is not hard to see that at (z^0, v^0) the Jacobi matrix of the right-hand side of system (4.3)–(4.4) realification has the form

$$\pm\mathrm{i}\omega_{0k} + \varepsilon(\alpha_k \pm \mathrm{i}\beta_k) + o(\varepsilon), \quad k \in \{1, \dots, n\}, \quad \varepsilon\varsigma\gamma_j + o(\varepsilon), \quad j \in \{1, \dots, m\}$$

where $\gamma_1, \dots, \gamma_m$ are the eigenvalues of Γ counted according to multiplicities. All of these numbers have non-zero real parts. It is well known that in this case the set of all solutions of the realification of (4.3)–(4.4) which approach the equilibrium $\xi^0(\varepsilon)$ forms the so-called stable manifold, whose dimension equals the number of eigenvalues of $\mathcal{A}(\varepsilon)$ with negative real parts (see, e. g. [21]). In the case when all of the eigenvalues of Γ have positive real parts, the equilibrium $(\operatorname{Re} z^0, \operatorname{Im} z^0, v^0)$ is completely unstable. $\square$

Proposition 4.4. *There exists a constant $c_3 > 0$, such that for sufficiently small $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0]$ the following assertion is valid. If $(z(t), v(t))$ is a solution of (4.3)–(4.4), such that $\|z(0)\| \leq \rho$, $|z_k(0)| \geq \sqrt{c_3\varepsilon^{N-2}}$ for some $k \in \{1, \dots, n\}$ and $\|v(t)\| < R$ for all $t > 0$, then*

$$|z_k(t)| > \sqrt{c_3\varepsilon^{N-2}}, \quad \|z(t)\| < \rho \quad \forall t > 0,$$

and there exists such $t_* > 0$, that

$$|z_k(t)| > \sqrt{\frac{\alpha_k}{2a_{kk}}}, \quad \|z(t)\| < \sqrt{\frac{2\max_{1\leq k\leq n}\alpha_k}{a_+^0}} \quad \forall t > t_*,$$

where

$$a_+^0 := \min\{\langle Ar, r\rangle : r \in \mathbb{R}_+^n, |r| = 1\}.$$

Proof. It is sufficient to note that there is such $c_4 > 0$, that for sufficiently small $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0]$ the following inequalities are satisfied.

$$\begin{aligned}\|z\|^2\big|'_{(4.3)} &\leq 2\varepsilon\langle\overrightarrow{|z|}^2, \alpha - A\overrightarrow{|z|}^2\rangle + c_4(\varepsilon^2\|z\|^2 + \varepsilon^{N/2}\|z\|)\\ &\leq 2\varepsilon\big[\max_{1\leq k\leq n}\alpha_k - a_+^0\|z\|^2 + c_4\varepsilon\big]\|z\|^2 + c_4\varepsilon^{N/2}\|z\| < 0\end{aligned}$$

if $2\max_{1\le k\le n}\alpha_k/a_+^0 \le \|z\|^2 \le \rho^2$, and, in view of (H4),

$$|z_k|^2\big|'_{(4.3)} \ge 2\varepsilon|z_k|^2\Big[\alpha_k - a_{kk}|z_k|^2 - c_4\varepsilon - c_4\varepsilon^{(N-2)/2}/|z_k|\Big] > 0$$

if $c_3\varepsilon^{N-2} \le |z_k|^2 \le \alpha_k/(2a_{kk})$, where c_3 is a sufficiently large positive constant. □

Definition 4.5. Let $\{i_1,\dots,i_s\}$ be an ordered collection of distinct natural numbers not exceeding n. We will say that a solution $(z(t),v(t))$ of (4.3)–(4.4) is of type $(i_1,\dots,i_s)$ if

$$\|v(t)\| < R, \quad |z_k(t)| < \varepsilon \quad \forall k \notin \{i_1,\dots,i_s\}, \quad \forall t > 0$$

and there exists such a moment of time $t^* > 0$, that

$$|z_k(t)| > \sqrt{\frac{\alpha_k}{2a_{kk}}} \quad \forall k \in \{i_1,\dots,i_s\}, \quad \|z(t)\| < \sqrt{\frac{2\max_{1\le k\le n}\alpha_k}{a_+^0}} \quad \forall t > t^*.$$

As a consequence of Propositions 4.3 and 4.4, we obtain the following result.

Proposition 4.6. *Let $(z(t),v(t))$ be a solution of (4.3)–(4.4), such that $\|z(0)\| \le \rho$ and $\|v(t)\| \le R$ for all $t \ge 0$. If ε_0 is sufficiently small and $\varepsilon \in (0,\varepsilon_0]$, then either this solution tends to the equilibrium (z^0,v^0) as $t\to\infty$, or it is of type $(i_1,\dots,i_s)$ for some $s \in \{1,\dots,n\}$.*

Proof. If $\|z(t)\| < \sqrt{\varepsilon}$ for all $t > 0$, then $(z(t),v(t))$ is a solution of (4.3)–(4.4) from Proposition 4.3, and therefore tends to (z^0,v^0) as $t\to\infty$. If $(z(t),v(t))$ does not possess the aforementioned property, then there exist $k \in \{1,\dots,n\}$ and $t_1 \ge 0$, such that $|z_k(t_1)|^2 > \varepsilon/n \ge c_3\varepsilon^{N-2}$. Hence, the solution $(z(t-t_1),v(t-t_1))$ satisfies the conditions of Proposition 4.4. Now it becomes apparent that in such case, basing on Proposition 4.4, one can decompose a set $\{1,\dots,n\}$ into two ordered subsets, $\{i_1,\dots,i_s\} \ni k$ and $\{j_1,\dots,j_{n-s}\} \subset \{1,\dots,n\}\setminus\{i_1,\dots,i_s\}$, and choose $t^* > 0$ in such a way, that $|z_i(t)|^2 > \alpha_i/(2a_{ii})$ for all $t > t^*$ if i belongs to the first subset, whereas $|z_j(t)|^2 \le c_3\varepsilon^{N-2} < \varepsilon^2$ for all $t > 0$ if j belongs to the second subset, with the latter being empty when $s = n$. Besides that, $\|z(t)\|$ meets the imposed requirements for all $t > t^*$. □

5. Ultimate behavior of solutions of type $(1,\dots,n)$

To study the final behavior of solutions of type $(1,\dots,n)$, introduce the polar-like coordinates $z_k = \sqrt{r_k}e^{i\varphi_k}$, $k = 1,\dots,n$, and set $r = (r_1,\dots,r_n)$. This transforms system (4.3)–(4.4) into

$$\dot r = 2\varepsilon D[r]\,[\alpha - Ar + \varepsilon\hat a(r,v)] + \varepsilon^{N/2}D^{1/2}[r]\tilde a(r,v,\varphi), \tag{5.1}$$

$$\dot v = \varepsilon\varsigma\Big[\Upsilon r + \Gamma v + \varepsilon\hat g(r,v) + \varepsilon^{(N+1)/2}\tilde g(r,v,\varphi)\Big], \tag{5.2}$$

$$\dot\varphi = \omega_0 + \varepsilon(\beta + \Omega' v - Br) + \varepsilon^2\hat b(r,v) + \varepsilon^{N/2}D^{-1/2}[r]\tilde b(r,v,\varphi) \tag{5.3}$$

where $\hat a(r,v) = \operatorname{Re}\hat h(r,v)$, $\hat b(r,v) = \operatorname{Im}\hat h(r,v)$, and $\tilde a(r,v,\varphi)$, $\tilde b(r,v,\varphi)$, $\tilde g(r,v,\varphi)$ are defined by the remainder terms of (4.3)–(4.4). On ground of Lemma 3.3 and Proposition 4.1, we can consider ρ and R to be so large, that

$$|A^{-1}\alpha| \le \frac{\max_{1\le k\le n}\alpha_k}{a_+^0} < \frac{\rho}{2}, \quad \|\Gamma^{-1}\Upsilon A^{-1}\alpha\| < \frac{R}{2}. \tag{5.4}$$

Proposition 5.1. *For sufficiently small $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0]$, the system*

$$\alpha - Ar + \varepsilon\hat{a}(r, v) = 0, \quad \Upsilon r + \Gamma v + \varepsilon\hat{g}(r, v) = 0$$

has a solution

$$r = r_*(\varepsilon) := A^{-1}\alpha + O(\varepsilon), \quad v = v_*(\varepsilon) := -\Gamma^{-1}\Upsilon A^{-1}\alpha + O(\varepsilon),$$

such that

$$r_{*k}(\varepsilon) > \frac{2\alpha_k}{3a_{kk}}, \quad k \in \{1, \dots, n\}, \quad |r_*(\varepsilon)| < \frac{3\max_{1\le k\le n}\alpha_k}{2a_+^0}, \quad \|v_*(\varepsilon)\| < \frac{2R}{3}.$$

Proof. Taking into account hypothesis (H4), Lemma 3.3 and inequalities (5.4), the desired result follows from the implicit function theorem. $\square$

Proposition 5.2. *There exist such positive numbers c_5, ς_0 and ε_0, that for any $\varsigma \in (0, \varsigma_0)$, $\varepsilon \in (0, \varepsilon_0)$ the following assertion is true. If $(z(t), v(t))$ is a solution of type $(1, \dots, n)$ of system (4.3)–(4.4) and $r(t) := \overrightarrow{|z(t)|}^2$ then there is such $t^* > 0$, that*

$$\sqrt{\|r(t) - r_*(\varepsilon)\|^2 + \|v(t) - v_*(\varepsilon)\|^2} < \frac{c_5\varepsilon^{(N-2)/2}}{\varsigma} \quad \forall t > t^*.$$

Proof. Let $(r(t), v(t), \varphi(t))$ represent the solution $(z(t), v(t))$ of type $(1, \dots, n)$ in the polar-like coordinates. On account of (5.1)–(5.3) and Proposition 5.1, the pair $(r(t), v(t))$ satisfies the system

$$\begin{aligned}
\dot{r} &= 2\varepsilon D[r]\big[(-A + \varepsilon\hat{A}_r(r, v))(r - r_*) + \varepsilon\hat{A}_v(r, v)(v - v_*)\big] \\
&\quad + \varepsilon^{N/2}D^{1/2}[r]\tilde{a}(r, v, \varphi(t)), \\
\dot{v} &= \varepsilon\varsigma\big[(\Upsilon + \varepsilon\hat{G}_r(r, v))(r - r_*) + (\Gamma + \varepsilon\hat{G}_v(r, v))(v - v_*) \\
&\quad + \varepsilon^{(N+1)/2}\tilde{g}(r, v, \varphi(t))\big],
\end{aligned}$$

where

$$\begin{aligned}
\hat{A}_r(r, v) &:= \int_0^1 \frac{\partial\hat{a}(r, v)}{\partial r}\Big|_{\substack{r\mapsto sr+(1-s)r_* \\ v\mapsto sv+(1-s)v_*}} \mathrm{d}s, \\
\hat{A}_v(r, v) &:= \int_0^1 \frac{\partial\hat{a}(r, v)}{\partial v}\Big|_{\substack{r\mapsto sr+(1-s)r_* \\ v\mapsto sv+(1-s)v_*}} \mathrm{d}s, \\
\hat{G}_r(r, v) &:= \int_0^1 \frac{\partial\hat{g}(r, v)}{\partial r}\Big|_{\substack{r\mapsto sr+(1-s)r_* \\ v\mapsto sv+(1-s)v_*}} \mathrm{d}s, \\
\hat{G}_v(r, v) &:= \int_0^1 \frac{\partial\hat{g}(r, v)}{\partial v}\Big|_{\substack{r\mapsto sr+(1-s)r_* \\ v\mapsto sv+(1-s)v_*}} \mathrm{d}s.
\end{aligned}$$

By Definition 4.5, there exists $t^* > 0$, such that for all $t > t^*$ a point $(r(t), v(t))$ belongs to the domain

$$\begin{aligned}
\mathcal{D} := \Big\{&(r, v) \in \mathbb{R}_+^n \times \mathbb{R}^m \colon r_k > \frac{\alpha_k}{2a_{kk}}\ \forall k \in \{1, \dots, n\}, \\
&|r| < \frac{2\max_{1\le k\le n}\alpha_k}{a_+^0}, \quad \|v\| < R\Big\},
\end{aligned}$$

which contains the unique equilibrium (r_*, v_*) of the system

$$\begin{aligned}\dot r &= 2\varepsilon D[r]\Big[\big(-A+\varepsilon\hat A_r(r,v)\big)(r-r_*)+\varepsilon\hat A_v(r,v)(v-v_*)\Big],\\ \dot v &= \varepsilon\varsigma\Big[(\Upsilon+\varepsilon\hat G_r(r,v))(r-r_*)+\big(\Gamma+\varepsilon\hat G_v(r,v)\big)(v-v_*)\Big],\end{aligned}\tag{5.5}$$

This equilibrium is the unique stationary point of the Morse function

$$W(r,v) := \sum_{i=1}^n \big(r_i + r_{*i}\ln(\frac{r_{*i}}{r_i}) - r_{*i}\big) + \frac{1}{2}\langle V''(0)(v-v_*), v-v_*\rangle$$

in $\mathrm{cl}(\mathcal{D})$.

The first inequality (2.1) yields $\langle V''(0)\Gamma v, v\rangle \le -\nu_*\|v\|^2$ for any $v\in\mathbb{R}^m$. Therefore, there is such a constant $c_6>0$, that for sufficiently small ε_0 and for all $\varepsilon\in(0,\varepsilon_0)$, $(r,v)\in\mathrm{cl}(\mathcal{D})$ the following inequality holds.

$$\begin{aligned}&W(r,v)|'_{(5.5)}\\ &\le 2\varepsilon\langle r-r_*, \big(-A+\varepsilon\hat A_r(r,v)\big)(r-r_*)+\varepsilon\hat A_v(r,v)(v-v_*)\rangle\\ &\quad+\varepsilon\varsigma\langle V''(0)(v-v_*), \big(\Upsilon+\varepsilon\hat G_r(r,v)\big)(r-r_*)+(\Gamma+\varepsilon\hat G_v(r,v))(v-v_*)\rangle\\ &\le -\varepsilon a^0\|r-r_*\|^2+2c_6(\varepsilon^2+\varepsilon\varsigma)\|r-r_*\|\|v-v_*\|-\frac{\varepsilon\varsigma\nu_*}{2}\|v-v_*\|^2,\end{aligned}$$

where

$$a^0 := \min\{\langle A\zeta,\zeta\rangle : \zeta\in\mathbb{R}^n, \|r\|=1\}.$$

Now, observe that for sufficiently small positive ς_0 and ε_0, and for any $\varsigma\in(0,\varsigma_0)$ and $\varepsilon\in(0,\varepsilon_0)$ the smallest eigenvalue of the matrix

$$\begin{pmatrix} a^0 & -c_6(\varepsilon+\varsigma)\\ -c_6(\varepsilon+\varsigma) & \varsigma\nu_*/2\end{pmatrix}$$

exceeds $\varsigma\kappa$, where

$$\kappa_0 := \frac{1}{2}\cdot\frac{a^0\nu_*}{2a^0+\varsigma\nu_*}.$$

Hence,

$$W(r,v)\big|'_{(5.5)} \le -\varepsilon\varsigma\kappa_0\big[\|r-r_*\|^2+\|v-v_*\|^2\big],$$

as soon as $(r,v)\in\mathrm{cl}(\mathcal{D})$, $\varsigma\in(0,\varsigma_0)$ and $\varepsilon\in(0,\varepsilon_0)$. It only remains to apply Lemma 3.2 in the case of unique stationary point of Morse function with $\theta\propto\varepsilon\varsigma$ and $\epsilon\propto\varepsilon^{N/2}$ to finish this proof. □

Next, we are going to utilize results on the existence of invariant tori obtained in [19]. To do so, we introduce a new vector variable $\xi\in\mathbb{R}^{n+m}$ via the formula

$$(r,v) = (r_*,v_*)+\varepsilon^\sigma\xi$$

with $0<\sigma<1/2$, and define the block matrices

$$\mathcal{B} := \begin{pmatrix}-2D[r_*]A & 0\\ \varsigma\Upsilon & \varsigma\Gamma\end{pmatrix},\quad \mathcal{B}_1(\varepsilon^\sigma\xi) := \begin{pmatrix}-2D[r-r_*]A & 0\\ 0 & 0\end{pmatrix}\Big|_{(r,v)=(r_*,v_*)+\varepsilon^\sigma\xi},$$

$$\hat{\mathcal{B}}(\varepsilon^\sigma\xi) := \begin{pmatrix}2D[r]\hat A_r(r,v) & 2D[r]\hat A_v(r,v)\\ \varsigma\hat G_r(r,v) & \varsigma\hat G_v(r,v)\end{pmatrix}\Big|_{(r,v)=(r_*,v_*)+\varepsilon^\gamma\xi},$$

the vector functions

$$\tilde f(\varepsilon^\sigma\xi,\varphi) := \big(D^{1/2}[r]\tilde a(r,v,\varphi), \varepsilon^{3/2}\varsigma\tilde g(r,v,\varphi)\big)\big|_{(r,v)=(r_*,v_*)+\varepsilon^\sigma\xi},$$

$$\hat{\Phi}(\varepsilon^\sigma\xi,\varphi) := \hat{b}(r,v)\big|_{(r,v)=(r_*,v_*)+\varepsilon^\sigma\xi},$$
$$\tilde{\Phi}(\varepsilon^\sigma\xi,\varphi) := \varepsilon^{(N-4)/2}D^{-1/2}[r]\tilde{b}(r,v,\varphi)\big|_{(r,v)=(r_*,v_*)+\varepsilon^\sigma\xi},$$

the vector $\omega_* := \omega_0 + \varepsilon(\beta + \Omega' v_* - Br_*)$ and, finally, the $(n\times(n+m))$-matrix $\Xi := [-B;\Omega']$. After that, system (5.1)–(5.3) takes the following form in the variables (ξ,φ).

$$\dot{\xi} = \varepsilon[\mathcal{B} + \mathcal{B}_1(\varepsilon^\sigma\xi) + \varepsilon\hat{\mathcal{B}}(\varepsilon^\sigma\xi)]\xi + \varepsilon^{N/2-\gamma}\tilde{f}(\varepsilon^\sigma\xi,\varphi), \tag{5.6}$$
$$\dot{\varphi} = \omega_* + \varepsilon^{1+\gamma}\Xi\xi + \varepsilon^2\hat{\Phi}(\varepsilon^\sigma\xi) + \varepsilon^{N/2}\tilde{\Phi}(\varepsilon^\sigma\xi,\varphi). \tag{5.7}$$

Assuming that ε is small enough for ω_* to be positive, we hereby arrive at the perturbation problem for the trivial invariant torus $\xi = 0$ of the system

$$\dot{\xi} = \varepsilon\mathcal{B}\xi, \quad \dot{\varphi} = \omega_*.$$

Note, that the quadratic forms $\langle D[r_*(0)]A\cdot,\cdot\rangle$ and $\langle V''(0)\Gamma\cdot,\cdot\rangle$ are positive and negative definite respectively, which means that the eigenvalues of $\mathcal{B}$ have non-zero real parts. The Lipschitz constants for the right-hand sides of (5.6)–(5.7) are $o(\varepsilon)$. Under such circumstances, one can easily verify that system (5.6)–(5.7) satisfies all conditions of [19, Lemma 2.1] (There is a misprint in condition (ii) on page 507: in $\mathrm{Lip}(\theta,y,z;\Sigma_{\sigma,\mu};\eta(\varepsilon,\sigma,\mu))$ the symbol η should be replaced by λ.) according to which, there exists sufficiently small $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0,\varepsilon_0)$ system (5.6)–(5.7) has an invariant torus given by the equation $\xi = \tilde{\xi}(\varphi;\varepsilon)$, where $\tilde{\xi}(\cdot;\varepsilon)\colon \mathbb{T}^n \to \mathbb{R}^{n+m}$ is a Lipschitzian mapping, such that $\max_{\varphi\in\mathbb{T}^n}\|\xi(\varphi;\varepsilon)\| \to 0$ as $\varepsilon \to 0$. Therefore, we have shown that system (5.1)–(5.3) has an invariant torus $\mathcal{T}^n_\varepsilon$ located in $\mathcal{D}$.

Also, if $\xi(t)$ is a solution of system (5.1)–(5.3) corresponding to $(r(t),v(t))$, then

$$\|\xi(t)\| < \frac{c_5\varepsilon^{(N-2)/2-\gamma}}{\varsigma} \quad \forall t > t^*.$$

Lemma 2.3 in [19] claims that the trajectory corresponding to the solution $\xi(t)$ belongs to the stable invariant manifold of the invariant torus. This yields the inequality

$$\|\tilde{\xi}(\varphi;\varepsilon)\| \le \frac{c_5\varepsilon^{(N-2)/2-\gamma}}{\varsigma} \quad \forall\psi \in \mathbb{T}^n.$$

Furthermore, if $N \ge 5$ and $V''(0)$ is positive definite, then it was shown in [37] that not only does the trajectory corresponding to the solution from Proposition 5.2 approach the torus $\mathcal{T}^n_\varepsilon$ as $t \to \infty$, but it is also attracted by some trajectory on $\mathcal{T}^n_\varepsilon$ when $t \to \infty$.

Hence, we have proved the part of statements (3)–(5) from the main theorem concerning n-dimensional tori.

6. Ultimate behavior of solutions of type $(i_1,\dots,i_s)$

Now, we will cover the case when solutions are of type $(1,\dots,s)$. Obviously, results that we are going to obtain will be applicable to other possible types $(i_1,\dots,i_s)$, too.

We introduce new variables $p := (p_1,\dots,p_s) \in \mathbb{R}^s_+$, $\vartheta := (\vartheta_1,\dots,\vartheta_s) \in \mathbb{T}^s$ and $\zeta = (\zeta_1,\dots,\zeta_{n-s}) \in \mathbb{C}^{n-s}$ via the formulae

$$z_k = \sqrt{p_k}\mathrm{e}^{\mathrm{i}\vartheta_k}, \quad k \in \{1,\dots,s\}, \quad z_{k+s} = \varepsilon\zeta_k, \quad k \in \{1,\dots,n-s\}.$$

If we denote

$$\begin{gathered}\alpha'=(\alpha_1,\dots,\alpha_s),\quad \alpha''=(\alpha_{s+1},\dots,\alpha_n),\\ \beta'=(\beta_1,\dots,\beta_s),\quad \beta''=(\beta_{s+1},\dots,\beta_n),\\ \omega_0'=(\omega_{01},\dots,\omega_{0s}),\quad \omega_0''=(\omega_{0,s+1},\dots,\omega_{0n})\end{gathered}$$

and decompose into blocks, the matrices

$$A:=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\quad B:=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix},\quad \Omega'=\begin{pmatrix}\Omega_1'\\ \Omega_2'\end{pmatrix},\quad \Upsilon=\begin{pmatrix}\Upsilon_1 & \Upsilon_2\end{pmatrix},$$

where $\dim A_{11}=\dim B_{11}=s\times s$, $\dim\Omega_1'=s\times m$, $\dim\Upsilon_1=m\times s$, we will get a smooth system of the form

$$\begin{aligned}\dot p&=2\varepsilon D[p]\Big[\alpha'-A_{11}p+\varepsilon\bar a\big(p,|\overrightarrow{\varepsilon\zeta}|^2,v\big)-\varepsilon^2A_{12}|\overrightarrow{\zeta}|^2\Big]\\ &\quad+\varepsilon^{N/2}D^{1/2}[p]\breve a(p,\operatorname{Re}\zeta,\operatorname{Im}\zeta,v,\vartheta),\end{aligned}\tag{6.1}$$

$$\begin{aligned}\dot v&=\varepsilon\varsigma\big[\Upsilon_1p+\Gamma v+\varepsilon\bar g(p,|\overrightarrow{\varepsilon\zeta}|^2,v)+\varepsilon^2\Upsilon_2|\overrightarrow{\zeta}|^2\\ &\quad+\varepsilon^{(N+1)/2}\breve g(p,\operatorname{Re}\zeta,\operatorname{Im}\zeta,v,\vartheta)\big],\end{aligned}\tag{6.2}$$

$$\begin{aligned}\dot\zeta&=D\big[\mathrm{i}(\omega_0''+\varepsilon(\beta''+\Omega_2'v-B_{21}p))+\varepsilon(\alpha''-A_{21}p)\big]\zeta\\ &\quad+\varepsilon^2D\big[-(\mathrm{i}B_{22}+A_{22})|\overrightarrow{\zeta}|^2+\bar h(p,|\overrightarrow{\varepsilon\zeta}|^2,v)\big]\zeta+\varepsilon^{(N-2)/2}\breve h(p,\operatorname{Re}\zeta,\operatorname{Im}\zeta,v,\vartheta),\\ \dot\vartheta&=\omega_0'+\varepsilon(\beta'+\Omega_1'v-B_{11}p)+\varepsilon^2\big[-B_{12}|\overrightarrow{\zeta}|^2+\bar b(p,|\overrightarrow{\varepsilon\zeta}|^2,v)\big]\\ &\quad+\varepsilon^{N/2}D^{-1/2}[p]\breve b(p,\operatorname{Re}\zeta,\operatorname{Im}\zeta,v,\vartheta),\end{aligned}$$

whose domain contains the set $\mathrm{cl}(\mathcal{D}_1)\times\{\zeta\in\mathbb{C}^{n-s}\colon\|\zeta\|\le 1\}\times\mathbb{T}^s$, with $\mathcal{D}_1\subset\mathbb{R}^s_+\times\mathbb{R}^m$ being defined by the inequalities

$$p_k>\frac{\alpha_k}{2a_{kk}},\quad k\in\{1,\dots,s\},\quad |p|<\frac{2\max_{1\le k\le s}\alpha_k}{a_+^0},\quad \|v\|<R.$$

Let $(p(t),v(t),\zeta(t),\vartheta(t))$ be a solution of this system which corresponds to a solution of type $(1,\dots,s)$, so that

$$(p(t),v(t),\zeta(t))\in\mathcal{D}_1\times\{\zeta\in\mathbb{C}^{n-s}\colon\|\zeta\|\le1\}\quad\forall t>t_*$$

if $t_*>0$ is large enough.

The following is an analogue of Proposition 5.1.

Proposition 6.1. *For sufficiently small $\varepsilon_0>0$ and for any $\varepsilon\in(0,\varepsilon_0]$, the system*

$$\alpha'-A_{11}p+\varepsilon\bar a(p,0,v)=0,\quad \Upsilon_1p+\Gamma v+\varepsilon\bar g(p,0,v)$$

has a solution

$$p=\bar p_*(\varepsilon):=A_{11}^{-1}\alpha'+O(\varepsilon),\quad v=\bar v_*(\varepsilon):=-\Gamma^{-1}\Upsilon_1A_{11}^{-1}\alpha'+O(\varepsilon),$$

such that

$$\bar p_{*k}(\varepsilon)>\frac{2\alpha_k}{3a_{kk}},\quad k\in\{1,\dots,s\},\quad |\bar p_*(\varepsilon)|<\frac{3\max_{1\le k\le n}\alpha_s}{2a_+^0},\quad \|\bar v_*(\varepsilon)\|<\frac{2R}{3}.$$

Note that the equilibrium $(\bar p_*,\bar v_*)$ of the system

$$\begin{gathered}\dot p=2\varepsilon D[p][\alpha'-A_{11}p+\varepsilon\bar a(p,0,v)],\\ \dot v=\varepsilon\varsigma[\Upsilon_1p+\Gamma v+\varepsilon\bar g(p,0,v)]\end{gathered}$$

is the unique stationary point in $\mathcal{D}_1$ of the Morse function

$$\bar{W}(p,v) := \sum_{i=1}^{s}\Big(p_i + \bar{p}_{*i}\ln(\frac{\bar{p}_{*i}}{p_i}) - \bar{p}_{*i}\Big) + \frac{1}{2}\langle V''(0)(v-\bar{v}_*), v-\bar{v}_*\rangle.$$

Proposition 6.2. *There are positive numbers c_7, ς_0 and ε_0, such that for any $\varsigma \in (0,\varsigma_0)$, $\varepsilon \in (0,\varepsilon_0)$ the following assertion is valid. If $(z(t), v(t))$ is a solution of type $(1,\dots,s)$ of system (4.3)–(4.4) and*

$$p(t) := \big(|z_1(t)|^2,\dots,|z_s(t)|^2\big), \quad \zeta(t) := \frac{1}{\varepsilon}(z_{s+1}(t),\dots,z_n(t)),$$

then there exists $t^* > 0$, such that

$$\sqrt{\|p(t)-\bar{p}_*(\varepsilon)\|^2 + \|v(t)-\bar{v}_*(\varepsilon)\|^2} < \frac{c_7\varepsilon^{(N-2)/2}}{\varsigma}, \quad \|\zeta(t)\| \le c_7\varepsilon^{(N-4)/2} \quad \forall t > t^*.$$

Proof. Since all elements of the matrix A_{21} are non-positive, then the minimal element of $\alpha'' - A_{21}p$ is not less than the minimal element of α''. Thus, there exists a constant $c_8 > 0$, such that for all sufficiently small $\varepsilon > 0$ there hold the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta(t)\|^2 \ge \|\zeta(t)\|\big[\varepsilon \min_{s+1\le k\le n}\alpha_k\|\zeta(t)\| - c_8\varepsilon^{(N-2)/2}\big], \quad \|\zeta(t)\| \le 1 \quad \forall t \ge t^*,$$

and consequently,

$$\|\zeta(t)\| \le \frac{c_8\varepsilon^{(N-4)/2}}{\min_{s+1\le k\le n}\alpha_k}$$

for all $t \ge t^*$.

Now observe that for

$$\|\zeta\| \le \frac{c_8\varepsilon^{(N-3)/2}}{\min_{s+1\le k\le n}\alpha_k}$$

sub-system (6.1)–(6.2) takes the form

$$\begin{aligned}\dot{p} &= 2\varepsilon D[p]\,[\alpha' - A_{11}p + \varepsilon\bar{a}(p,0,v)] + O(\varepsilon^{N/2}),\\ \dot{v} &= \varepsilon\varsigma\,[\Upsilon_1 p + \Gamma v + \varepsilon\bar{g}(p,0,v)] + O(\varepsilon^{(N+3)/2}).\end{aligned}$$

Using Lemma 3.2 for the function $\bar{W}(p,v)$, one can obtain the inequality for $p(t)$ and $v(t)$ precisely in the same way as we did in Proposition 5.2. $\square$

If we introduce new variables $\eta \in \mathbb{R}^{s+m}$ and $\xi \in \mathbb{R}^{s+m}\times\mathbb{C}^{n-s}$ via the formulae

$$(p,v) = (\bar{p}_*,\bar{v}_*) + \varepsilon^{\sigma}\eta,$$

then again we come to the perturbation problem for the trivial hyperbolic invariant torus of the system

$$\begin{aligned}\dot{\eta} &= \varepsilon\begin{pmatrix}-2D[\bar{p}_*(0)]A_{11} & 0\\ \varsigma\Upsilon_1 & \varsigma\Gamma\end{pmatrix}\eta,\\ \dot{\zeta} &= D\,[\mathrm{i}(\omega_0'' + \varepsilon(\beta'' + \Omega_2'\bar{v}_*(0) - B_{21}\bar{p}_*(0))) + \varepsilon(\alpha'' - A_{12}\bar{p}_*(0))]\,\zeta,\\ \dot{\vartheta} &= \omega_0' + \varepsilon\,[\beta' + \Omega_1'\bar{v}_*(0) - B_{11}\bar{p}_*(0)]\,.\end{aligned}$$

The real parts of the eigenvalues of this system's matrix are non-zero, since they are defined by the eigenvalues of the matrix

$$\varepsilon\,\mathrm{diag}(-2D[\bar{p}_*(0)]A_{11},\, \varsigma\Gamma,\, \alpha'' - A_{12}\bar{p}_*(0)).$$

It is not hard to see that the perturbation terms satisfy the conditions of [19, Lemma 2.1]. Using the same arguments as above for the realification of the perturbed system in the variables $\eta, \operatorname{Re}\zeta, \operatorname{Im}\zeta, \vartheta$, we can prove the existence of an s-dimensional truly hyperbolic invariant torus $\mathcal{T}_\varepsilon^s(1,\dots,s)$, which attracts the trajectories corresponding to the solutions of type $(1,\dots,s)$. In the coordinates (p, v, ζ, ϑ) this torus is given by the equations

$$(p, v) = (\bar{p}_*(\varepsilon), \bar{v}_*(\varepsilon)) + \varepsilon^\sigma \eta_*(\vartheta;\varepsilon), \quad \zeta = \zeta_*(\vartheta;\varepsilon),$$

where $\eta_*(\cdot;\varepsilon)\colon \mathbb{T}^s \to \mathbb{R}^{s+m}$ and $\zeta_*(\cdot;\varepsilon)\colon \mathbb{T}^s \to \mathbb{C}^{n-s}$ are Lipschitzian mappings satisfying the conditions

$$\max_{\vartheta\in\mathbb{T}^s} \|\eta_*(\vartheta;\varepsilon)\| \le \frac{c_8\varepsilon^{(N-2)/2-\gamma}}{\varsigma}, \quad \max_{\vartheta\in\mathbb{T}^s} \|\zeta_*(\vartheta;\varepsilon)\| \le c_8\varepsilon^{(N-4)/2}.$$

Hence, we have proved the part of statements (3)–(5) from the main theorem concerning the tori of dimensions less then n. Note, that those forward semi-trajectories in $\mathfrak{A}$ that are not attracted by stable n-dimensional tori lie on stable manifolds of truly hyperbolic tori, and thus form the set of zero Lebesgue measure.

7. Excitation of two-frequency oscillations in a system of two coupled generators due to slow cooling

Let us now provide an example of a hypothetical device where the described phenomenon can be observed. Consider a system of two coupled generators (Figure 1).

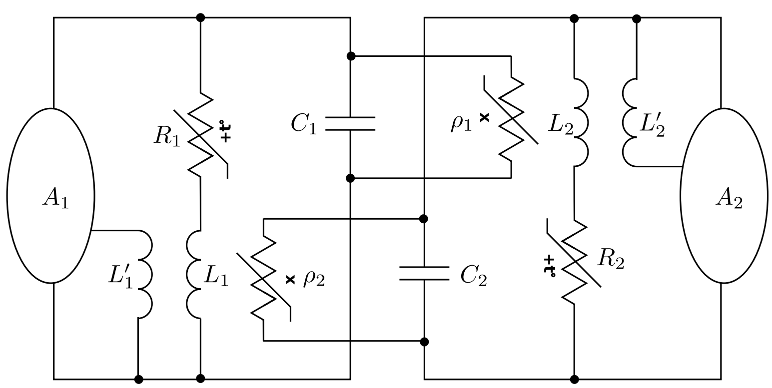

Figure 1. Coupled generators

Here, the i-th generator consists of the following elements: a thermistor R_i, a magneto resistor ρ_i, magnetically connected inductors L_i, L_i' having a negative mutual induction $M_i = \gamma_i\sqrt{L_iL_i'}$, a capacitor C_i, and an active feedback element A_i. We suppose that the I-V characteristics χ_i of A_i is a smooth function of voltage difference V_i on L_i' and admits the expansion

$$\chi_i = \chi_i(V_i) := \chi_{i1}V_i - \chi_{i3}V_i^3 + O(V_i^4), \quad \chi_{i1} > 0, \quad \chi_{i3} > 0.$$

We assume that the i-th thermistor has a positive temperature coefficient and that its resistance depends on the thermistor's temperature T_i by the relation

$$R_i = R_i(T_i) := \varepsilon R_{i0} + \mu R_{i1}T_i, \quad i \in \{1, 2\},$$

where R_{i0}, R_{i1} are positive constants and ε and μ are small parameters. We also suppose the inductances L_i, L_i' and the capacities C_i to be constant.

The resistance of the magneto resistor depends on an external magnetic field. If the latter is orthogonal to the direction of the current in the resistor, then the change of its resistance is approximately proportional to the square of the magnetic field magnitude [25].

The generators interact with each other in the following way. The current I_i in the inductor L_i produces a magnetic field of a magnitude proportional to I_i, and this magnetic field influences the resistance of the magneto resistor ρ_j. Thus, it is natural to consider the case where

$$\rho_j = \rho_j(I_i) = \rho_{j0} + \rho_{j2}(L_i I_i)^2, \quad \rho_{j0} > 0, \quad \rho_{j2} > 0, \quad i \in \{1, 2\}. \tag{7.1}$$

Taking into account the Newton law of cooling and the Joule – Lenz law of ohmic heating and assuming the environment temperature to be zero, we adopt the next equation for the resistor R_i temperature change

$$\dot{T}_i = -\mu k_i T_i + K_i I_i^2 R_i, \tag{7.2}$$

where k_i and K_i are some positive coefficients.

Our goal is to show that under an appropriate choice of the generators parameters one can observe such a phenomenon. If the described device of the coupled generators with the sufficiently high initial temperatures $T_i(0)$ of the resistances R_i is placed into the environment, then first for a long period of time this device remains in the sleep mode. However, when the resistances temperatures drop almost to zero, the device wakes up and after a transient process, in general, it starts producing two-frequency oscillations. Such a kind of behavior can be treated as a phenomenon of dynamical bifurcation.

Denote by q_i the charge of the capacitor C_i, and let I_i, I_{C_i}, I_{ρ_i} and I_{A_i} stand for the currents through the resistor R_i, the capacitor C_i, the magneto resistor ρ_i and the active element A_i respectively. The Kirchhoff laws yield that

$$I_i + I_{C_i} + I_{\rho_i} = I_{A_i} = \chi_i, \tag{7.3}$$

$$L_i \dot{I}_i + R_i I_i = \frac{q_i}{C_i} = \rho_i I_{\rho_i}. \tag{7.4}$$

Since $\dot{q}_i = I_{C_i}$, by differentiation of (7.4) we obtain

$$L_i \ddot{I}_{R_i} + R_i \dot{I}_i + \dot{R}_i I_i = \frac{I_{C_i}}{C_i}.$$

But from (7.4) and (7.3) we can find

$$I_{\rho_i} = [L_i \dot{I}_i + R_i I_i]/\rho_i,$$

$$I_{C_i} = \chi_i - I_i - \frac{1}{\rho_i}[L_i \dot{I}_i + R_i I_i].$$

Hence,

$$\ddot{I}_i + \frac{R_i}{L_i}\dot{I}_i + \frac{I_i}{L_i C_i} + \frac{1}{\rho_i L_i C_i}[L_i \dot{I}_i + R_i I_i] = \frac{\chi_i}{L_i C_i} - \frac{\dot{R}_i}{L_i} I_i,$$

and on account of (7.2)

$$\ddot{I}_i + [\frac{R_i}{L_i} + \frac{1}{\rho_i C_i}]\dot{I}_i + \frac{1}{L_i C_i}[1 + \frac{R_i}{\rho_i}]I_i = \frac{\chi_i}{L_i C_i} - \frac{\mu R_{i1}}{L_i}[-\mu k_i T_i + K_i I_i^2 R_i] I_i.$$

Introducing new variables $I_i = \sqrt{\rho_{j0}}w_i$, $T_i = R_{i0}u_i/(\varsigma R_{i1})$, imposing the constraint $\varsigma\varepsilon = \mu$ with the constant ς playing the same role as in section 4 and taking into account (7.1) and (7.2), we finally reach the system

$$
\begin{aligned}
&\ddot{w}_i + \Big[\frac{\varepsilon R_{i0}(1+u_i)}{L_i} + \frac{1}{C_i\rho_{i0}(1+\rho_{i2}L_j^2w_j^2)}\Big]\dot{w}_i + \frac{1}{L_iC_i}\Big[1 + \frac{\varepsilon R_{i0}(1+u_i)}{\rho_{i0}(1+\rho_{i2}L_j^2w_j^2)}\Big]w_i \\
&= \frac{1}{L_iC_i\sqrt{\rho_{j0}}}\chi_i(M_i\sqrt{\rho_{j0}}\dot{w}_i) - \frac{\mu\varepsilon R_{i0}}{L_i}[-k_iu_i + K_iR_{i1}(1+u_i)\rho_{j0}w_i^2]w_i, \\
&\qquad \dot{u_i} = \mu[-k_iu_i + R_{i1}K_i\rho_{j0}(1+u_i)w_i^2], \quad i \in \{1,2\}, j \in \{1,2\}, \quad i \neq j.
\end{aligned}
$$

This system can be represented in the form (2.3) if the parameters of the generators are chosen in such a way, that

$$
\frac{\chi_{i1}M_i}{L_iC_i} - \frac{1}{C_i\rho_{i0}} = \varepsilon b_i, \quad i \in \{1,2\},
$$

where b_1, b_2 are positive constants satisfying the inequalities

$$
b_i > R_{i0}/L_i, \quad i \in \{1,2\}. \tag{7.5}
$$

In fact, we have

$$
\omega_{0i}(u) = \sqrt{\frac{1}{L_iC_i}} + O(\varepsilon), \quad \lambda_i(u) = b_i - \frac{R_{i0}(1+u_i)}{L_i}, \quad i \in \{1,2\},
$$

and

$$
\begin{aligned}
&F_i(w_1, w_2, \dot{w}_1, \dot{w}_2, u, 0) \\
&= \sum_{k+l+m+n=3} f_{i,jklm}w_1^j\dot{w}_1^kw_2^l\dot{w}_2^m + O(\|w\|^4 + \|\dot{w}\|^4), \quad i \in \{1,2\},
\end{aligned}
$$

where the only non-zero coefficients are

$$
\begin{gathered}
f_{1,0300} = -\frac{\chi_{13}\rho_{20}M_1^3}{L_iC_i}, \quad f_{1,0120} = \frac{\rho_{12}L_1^2}{C_1\rho_{10}}, \\
f_{2,2001} = \frac{\rho_{22}L_2^2}{C_2\rho_{20}}, \quad f_{2,0003} = -\frac{\chi_{23}\rho_{10}M_2^3}{L_iC_i}.
\end{gathered}
$$

Therefore, performing the change of variables (2.2), we obtain

$$
\hat{F}(x, u, \varsigma\varepsilon) = \begin{pmatrix} 0 \\ \frac{1}{\omega_{01}(u)}F_1(x_1, x_3, \omega_{01}(u)x_2, \omega_{02}(u)x_4, u, 0) \\ 0 \\ \frac{1}{\omega_{02}(u)}F_2(x_1, x_3, \omega_{01}(u)x_2, \omega_{02}(u)x_4, u, 0) \end{pmatrix} + O(\varepsilon).
$$

It is easily seen the eigenvectors of the matrix $J(u)$ are

$$
s_1^+ = \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2^+ = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{i} \\ 1 \end{pmatrix}, \quad s_1^- = \begin{pmatrix} \mathrm{i} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2^- = \begin{pmatrix} 0 \\ 0 \\ \mathrm{i} \\ 1 \end{pmatrix}.
$$

If we now introduce new complex variables $z_1, z_2 \in \mathbb{C}$ via

$$
x_1 = -\mathrm{i}z_1 + \mathrm{i}\bar{z}_1, \quad x_2 = z_1 + \bar{z}_1, \quad x_3 = -\mathrm{i}z_2 + \mathrm{i}\bar{z}_2, \quad x_4 = z_2 + \bar{z}_2,
$$

we will be able to find the elements of the matrix $A(u)$ by extraction of the resonant terms from cubic nonlinearities:

$$\begin{aligned}
&\frac{1}{2\omega_{01}(u)}F_1(-\mathrm{i}z_1+\mathrm{i}\bar z_1,-\mathrm{i}z_2+\mathrm{i}\bar z_2,\omega_{01}(u)(z_1+\bar z_1),\omega_{02}(u)(z_2+\bar z_2),u,0)\\
&=-a_{11}(u)|z_1|^2z_1-a_{12}(u)|z_2|^2z_1+[\text{nonresonant terms}]\\
&\frac{1}{2\omega_{02}(u)}F_2(-\mathrm{i}z_1+\mathrm{i}\bar z_1,-\mathrm{i}z_2+\mathrm{i}\bar z_2,\omega_{01}(u)(z_1+\bar z_1),\omega_{02}(u)(z_2+\bar z_2),u,0)\\
&=-a_{21}(u)|z_1|^2z_2-a_{22}(u)|z_2|^2z_2+[\text{nonresonant terms}].
\end{aligned}$$

It turns out that

$$\begin{gathered}
a_{11}(u)=-\frac12 f_{1,2100}-\frac32\omega_{01}^2(u)f_{1,0300},\\
a_{12}(u)=-f_{1,0120}-\omega_{02}^2(u)f_{1,0102},\\
a_{21}(u)=-f_{2,2001}-\omega_{01}^2(u)f_{2,0201},\\
a_{22}(u)=-\frac12 f_{2,0021}-\frac32\omega_{02}^2(u)f_{2,0003}.
\end{gathered}$$

In our case, when $\varepsilon=0$, these elements does not depend on u:

$$\begin{gathered}
a_{11}=\frac{3\chi_{13}\rho_{20}M_1^3}{2(C_1L_1)^2},\quad a_{12}=-\frac{\rho_{12}L_2^2}{C_1\rho_{10}},\\
a_{21}=-\frac{\rho_{22}L_1^2}{C_2\rho_{20}},\quad a_{22}=\frac{3\chi_{23}\rho_{10}M_2^3}{2(C_2L_2)^2}.
\end{gathered}$$

Hence, when $\varepsilon=0$, the positive definiteness condition of the symmetric part of the matrix A takes the form

$$9\chi_{13}\chi_{23}(\rho_{10}\rho_{20}M_1M_2)^3>\big[L_1L_2(\rho_{12}\rho_{20}C_2L_2^2+\rho_{10}\rho_{22}C_1L_1^2)\big]^2. \tag{7.6}$$

Since $g(u)=(-k_1u_1,-k_2u_2)$, the Morse function can be chosen as $V(u)=u_1^2+u_2^2$. It has a unique stationary point $u_*=(0,0)$. The instability and stability sets are defined as follows

$$\begin{aligned}
\mathcal V_+&=\{(u_1,u_2)\in\mathbb R^2\colon u_1^2+u_2^2<V^*,u_i<b_iL_{i0}/R_{i0}-1,i\in\{1,2\}\},\\
\mathcal V_-&=\{(u_1,u_2)\in\mathbb R^2\colon u_1^2+u_2^2<V^*,u_i>b_iL_{i0}/R_{i0}-1,i\in\{1,2\}\},
\end{aligned}$$

where $V^*>0$ is large enough.

Thus, if the numbers $\sqrt{C_1L_1}$, $\sqrt{C_2L_2}$ are rationally independent and conditions (7.5) and (7.6) fulfill, then hypotheses (H1)–(H5) hold, and Theorem 2.2 implies that under the appropriate choice of $\varsigma_0>0$, $0<\varsigma_*<\varsigma_0$, $\varepsilon_0>0$ the aforementioned changes in behavior of the coupled generators can actually be observed, once $\varepsilon\in(0,\varepsilon_0)$, $\mu\in(\varsigma_*\varepsilon,\varsigma_0\varepsilon)$.

8. Addendum

Consider the formal system

$$\begin{aligned}
\dot x&=J(u)x+\sum_{i\ge0}\mu^iF_i(x,u),\\
\dot u&=\mu\Big[g(u)+\sum_{i\ge0}\mu^iG_i(x,u)\Big]
\end{aligned}\tag{8.1}$$

obtained from (2.3) by expanding its right-hand sides into the Taylor series expansions in powers of μ. Our goal is to simplify this system with aid of the formal change of variables

$$x = y + \sum_{i\geq 0} \mu^i X_i(y,v), \quad u = v + \mu \sum_{i\geq 0} \mu^i U_i(y,v). \tag{8.2}$$

Construction of such a transformation consists in solving homological equations of the form

$$\mathfrak{L}_{J(v)y}[\mathcal{Y}_j(v)y^j] = \mathcal{P}_j(v)y^j - \mathcal{R}_j(v)y^j, \tag{8.3}$$

$$\partial_{J(v)y}[\mathcal{Z}_j(v)y^j] = \mathcal{Q}_j(v)y^j - \mathcal{S}_j(v)y^j. \tag{8.4}$$

Here $\mathcal{P}_j(v)y^j$ and $\mathcal{Q}_j(v)y^j$ are known j-th order homogeneous forms which take values in $\mathbb{R}^{2n}$ and $\mathbb{R}^m$ respectively and smoothly depend on the parameter v, whereas $\partial_{J(v)y}$ and $\mathfrak{L}_{J(v)y}$ are, respectively, the directional and the Lie derivatives along the vector field $J(v)y$. Namely,

$$\partial_{J(v)y} Z(y) := \frac{\partial Z(y)}{\partial y} J(v)y \quad \forall Z(\cdot) \in \mathrm{C}^1(\mathbb{R}^{2n};\mathbb{R}^m),$$

$$\mathcal{L}_{J(v)y} Y(y) := \frac{\partial Y(y)}{\partial y} J(v)y - J(v)Y(y) \quad \forall Y(\cdot) \in \mathrm{C}^1(\mathbb{R}^{2n};\mathbb{R}^{2n}).$$

The forms $\mathcal{Y}_j(v), \mathcal{R}_j(v)y^j$, $\mathcal{Z}_j(v)$ and $\mathcal{S}_j(v)y^j$ are determined in such a way, that they smoothly depend on v and satisfy the corresponding equations.

If $j = 0$, equation (8.3) has obvious solutions

$$\mathcal{R}_0(v) \equiv 0, \quad \mathcal{Y}_0(v) = -J^{-1}(v)\mathcal{P}_0(v), \quad \mathcal{S}_0(v) = \mathcal{Q}_0(v), \quad \mathcal{Z}_0(v) = 0. \tag{8.5}$$

In the case when $j \geq 1$, in the same manner as in [37], we can introduce a suitable basis in the space of vector-valued polynomial forms. Note, that the matrix $J(v)$ has constant linearly independent eigenvectors $s_j^{\pm} \in \mathbb{C}^{2n}$, $j = 1,\dots,n$, such that

$$J(v)s_j^{\pm} = [\varepsilon\lambda_j(v) \pm \mathrm{i}\omega_j(v)]s_j^{\pm},$$

with vectors s_j^- and s_j^+ being complex conjugate for all $j = 1,\dots,n$. Denote by S the matrix with the columns $s_1^+,\dots,s_n^+,s_1^-,\dots,s_n^-$, and define the homogeneous forms

$$\mathfrak{s}_{\mathbf{q}}(y) := [S^{-1}y]^{\mathbf{q}}, \quad e_{j,\mathbf{q}}^{\pm}(y) = \mathfrak{s}_{\mathbf{q}}(y)s_j^{\pm},$$

where $\mathbf{q} := (q_1,\dots,q_{2n}) \in \mathbb{Z}_+^{2n}$ and $x^{\mathbf{q}} := x_1^{q_1}\cdots x_{2n}^{q_{2n}}$ for $x = (x_1,\dots,x_{2n})$. This gives us the following expansions.

$$\mathcal{P}_j(v)y^j = \sum_{k=1}^n \sum_{|\mathbf{q}|=j} [P_{k,\mathbf{q}}^+(v)e_{k,\mathbf{q}}^+(y) + P_{k,\mathbf{q}}^-(v)e_{k,\mathbf{q}}^-(y)], \quad \mathcal{Q}_j(v)y^j = \sum_{|\mathbf{q}|=j} \mathfrak{s}_{\mathbf{q}}(y)Q_{\mathbf{q}}(v),$$

$$\mathcal{R}_j(v)y^j = \sum_{k=1}^n \sum_{|\mathbf{q}|=j} [R_{k,\mathbf{q}}^+(v)e_{k,\mathbf{q}}^+(y) + R_{k,\mathbf{q}}^-(v)e_{k,\mathbf{q}}^-(y)], \quad \mathcal{S}_j(v)y^j = \sum_{|\mathbf{q}|=j} \mathfrak{s}_{\mathbf{q}}(y)S_{\mathbf{q}}(v).$$

Proposition 8.1. *Let $\mathcal{D}_N \subset \mathbb{R}^m$ be such a domain, that for any $k \in \{1,\dots,n\}$ and $\sigma \in \{0,1\}$ the equality*

$$\min_{v\in \mathrm{cl}(\mathcal{D}_N)} \Big| \sum_{l=1}^n (q_l - q_{l+n} - \sigma\delta_{kl})\omega_{0l}(v)\Big| = 0,$$

where $\mathbf{q} \in \mathbb{Z}_+^{2n}$, $1 \le |\mathbf{q}| := \sum_{k=1}^{2n} q_k \le N$ *and* δ_{kl} *is Kronecker's delta, holds if and only if*

$$q_l - q_{l+n} - \sigma\delta_{kl} = 0 \quad \forall l \in \{1, \dots, n\}.$$

Suppose also that the forms $\mathcal{P}_j(v)y^j$ *and* $\mathcal{Q}_j(v)y^j$ *smoothly depend on* $v \in \mathrm{cl}(\mathcal{D}_N)$ *and assign*

$$R^{\pm}_{k,\mathbf{q}}(v) = \begin{cases} P^{\pm}_{k,\mathbf{q}}(v) & \text{if } \sum_{l=1}^n |q_l - q_{l+n} \mp \delta_{kl}| = 0, \\ 0 & \text{if } \sum_{l=1}^n |q_l - q_{l+n} \mp \delta_{kl}| \ne 0, \end{cases}$$

$$S_{\mathbf{q}}(v) = \begin{cases} Q_{\mathbf{q}}(v) & \text{if } \sum_{l=1}^n |q_l - q_{l+n}| = 0, \\ 0 & \text{if } \sum_{l=1}^n |q_l - q_{l+n}| \ne 0. \end{cases}$$

Then, for sufficiently small $\varepsilon_0 > 0$ *and for all* $\varepsilon \in [0, \varepsilon_0]$, *there exist an* $\mathbb{R}^{2n}$*-valued form* $\mathcal{Y}_j(v)y^j$ *and an* $\mathbb{R}^m$*-valued form* $\mathcal{Z}_j(v)y^j$, *which satisfy equations* (8.3) *and* (8.4) *respectively. The coefficients of these forms are smooth functions in* $\mathrm{cl}(\mathcal{D}_N)$.

Proof. Since

$$S^{-1}J(v)S = \mathrm{diag}\,\big[\varepsilon\lambda_1(v) + \mathrm{i}\omega_1(v), \dots, \varepsilon\lambda_n(v) + \mathrm{i}\omega_n(v), \varepsilon\lambda_1(v) - \mathrm{i}\omega_1(v), \dots, \varepsilon\lambda_n(v) - \mathrm{i}\omega_n(v)\big],$$

we have

$$\begin{aligned} \partial_{J(v)y}\mathfrak{s}_{\mathbf{q}}(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}[(S^{-1}\mathrm{e}^{J(v)t}y)^{\mathbf{q}}] \\ &= \Big[\varepsilon\sum_{l=1}^n (q_l + q_{l+n})\lambda_l(v) + \mathrm{i}\sum_{l=1}^n (q_l - q_{l+n})\omega_l(v)\Big]\mathfrak{s}_{\mathbf{q}}(y) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{L}_{J(v)y}e^{\pm}_{k,\mathbf{q}}(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{e}^{-J(v)t}e^{\pm}_{k,\mathbf{q}}\big(\mathrm{e}^{J(v)t}y\big) \\ &= \Big[\varepsilon\sum_{l=1}^n (q_l + q_{l+n} - \delta_{kl})\lambda_l(v) + \mathrm{i}\sum_{l=1}^n (q_l - q_{l+n} \mp \delta_{kl})\omega_l(v)\Big]e^{\pm}_{k,\mathbf{q}}(y). \end{aligned}$$

After expanding the forms

$$\mathcal{Y}_j(v)y^j = \sum_{k=1}^n \sum_{|\mathbf{q}|=j} \big[Y^+_{k,\mathbf{q}}(v)e^+_{k,\mathbf{q}}(y) + Y^-_{k,\mathbf{q}}(v)e^-_{k,\mathbf{q}}(y)\big],$$

$$\mathcal{Z}_j(v)y^j = \sum_{|\mathbf{q}|=j} \mathfrak{s}_{\mathbf{q}}(y)Z_{\mathbf{q}}(v),$$

the homological equations are reduced to

$$\begin{aligned} &\Big[\varepsilon\sum_{l=1}^n (q_l + q_{l+n} - \delta_{kl})\lambda_l(v) + \mathrm{i}\sum_{l=1}^n (q_l - q_{l+n} \mp \delta_{kl})\omega_l(v)\Big]Y^{\pm}_{k,\mathbf{q}}(v) \\ &= P^{\pm}_{k,\mathbf{q}}(v) - R^{\pm}_{k,\mathbf{q}}(v), \end{aligned}$$

$$\Big[\varepsilon\sum_{l=1}^n (q_l + q_{l+n})\lambda_l(v) + \mathrm{i}\sum_{l=1}^n (q_l - q_{l+n})\omega_l(v)\Big]Z_{\mathbf{q}}(v) = Q_{\mathbf{q}}(v) - S_{\mathbf{q}}(v).$$

Taking into account the definitions of $R^{\pm}_{k,\mathbf{q}}(v)$ and $S_{\mathbf{q}}(v)$, these equations are soluble for any $\varepsilon \in [0, \varepsilon_0]$ with sufficiently small $\varepsilon_0 > 0$, and, as a consequence, the same is true for equations (8.3). Although the coefficients $Y^{\pm}_{k,\mathbf{q}}(v)$ and $Z_{\mathbf{q}}(v)$ are complex-valued, one can make the forms $\mathcal{Y}_j(v)y^j$ and $\mathcal{Z}_j(v)y^j$ take values, respectively, in $\mathbb{R}^{2n}$ and $\mathbb{R}^m$ by setting $Y^{\pm}_{k,\mathbf{q}}(v) \equiv 0$ for any $\mathbf{q}$, such that $\sum_{l=1}^n |q_l - q_{l+n} \mp \delta_{kl}| = 0$, and $Z_{\mathbf{q}}(v) \equiv 0$ for any $\mathbf{q}$, such that $\sum_{l=1}^n |q_l - q_{l+n}| = 0$. The smoothness properties of $\mathcal{Y}_j(v)y^j, \mathcal{Z}_j(v)y^j$ are obvious. □

Remark 8.2. For $\mathbf{q} \in \mathbb{Z}^{2n}_+$ with $|\mathbf{q}| = 1$, the equality $\sum_{l=1}^n |q_l - q_{l+n} \mp \delta_{kl}| = 0$ is satisfied if and only if $\mathbf{q} = \mathbf{e}^{\pm}_k$, where $\mathbf{e}^{+}_k \in \mathbb{Z}^{2n}_+$ ($\mathbf{e}^{-}_k \in \mathbb{Z}^{2n}_+$) is a vector whose k-th ($(k+n)$-th) coordinate equals 1 while the other are 0. Hence, in such case,

$$R^{\pm}_{k,\mathbf{q}}(v) \neq 0 \quad \text{if and only if} \quad \mathbf{q} = \mathbf{e}^{\pm}_k.$$

Let

$$\begin{aligned} \dot{y} &= J(v)y + \sum_{i\geq 0} \mu^i H_i(y,v), \\ \dot{v} &= \mu\Big[g(v) + \sum_{i\geq 0} \mu^i C_i(y,v)\Big], \end{aligned}$$

be the system obtained from (8.1) by means of the formal change of variables (8.2). In view of (1.3) and the definition of $g(v)$, we have

$$F_0(y,v) = O(y^2), \quad G_0(y,v) = O(y).$$

Therefore, we require that

$$X_0(y,v) = O(y^2), \quad H_0(y,v) = O(y^2), \quad C_0(y,v) = O(y), \quad U_0(y,v) = O(y).$$

Substituting (8.2) in (8.1) and equating coefficients near like powers of μ, we obtain the following chain of homological equations for the unknown coefficients

$$\begin{aligned} \mathfrak{L}_{J(v)y} X_0(y,v) &= F_0(y + X_0(y,v), v) - \frac{\partial X_0(y,v)}{\partial y} H_0(y,v) - H_0(y,v), \\ \partial_{J(v)y} U_0(y,v) &= G_0(y + X_0(y,v), v) - \frac{\partial U_0(y,v)}{\partial y} H_0(y,v) - C_0(y,v), \end{aligned}$$

and, for $i > 0$,

$$\begin{aligned} &\mathfrak{L}_{J(v)y} X_i(y,v) \\ &= \frac{\partial X_{i-k}(y,v)}{\partial y} H_k(y,v) - \frac{\partial X_{i-1}(y,v)}{\partial v} g(v) \\ &\quad - \sum_{k=0}^{i-1} \frac{\partial X_{i-k-1}(y,v)}{\partial v} C_k(y,v) + \big[\frac{\partial F_0(x,v)}{\partial x} X_i(y,v)\big]_{x=y+X_0(y,v)} \\ &\quad + \frac{1}{i!}\frac{\partial^i}{\partial \mu^i}\Big|_{\mu=0} \Big[J\Big(v + \mu\sum_{j=0}^{i-1} \mu^j U_j(y,v)\Big)\Big(y + \sum_{k=0}^{i-1} \mu^k X_k(y,v)\Big)\Big] \\ &\quad + \frac{1}{i!}\frac{\partial^i}{\partial \mu^i}\Big|_{\mu=0} \sum_{l=0}^{i} \mu^l F_l\Big(y + \sum_{k=0}^{i-1} \mu^k X_k(y,v), v + \mu\sum_{j=0}^{i-1} \mu^j U_j(y,v)\Big) - H_i(y,v), \end{aligned}$$

$$\begin{aligned} &\partial_{J(v)y} U_i(y,v) \\ &= -\sum_{k=0}^{i} \frac{\partial U_{i-k}(y,v)}{\partial y} H_k(y,v) - \frac{\partial U_{i-1}(y,v)}{\partial v} g(v) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=0}^{i-1}\frac{\partial U_{i-k-1}(y,v)}{\partial v}C_k(y,v)+\big[\frac{\partial G_0(v,x)}{\partial x}X_i(y,v)\big]_{x=y+X_0(y,v)}\\
&+\frac{1}{i!}\frac{\partial^i}{\partial\mu^i}\big|_{\mu=0}g\Big(v+\mu\sum_{j=0}^{i-1}\mu^jU_j(y,v)\Big)\\
&+\frac{1}{i!}\frac{\partial^i}{\partial\mu^i}\big|_{\mu=0}\sum_{l=0}^{i}\mu^lG_l\Big(y+\sum_{k=0}^{i-1}\mu^kX_k(y,v),v+\mu\sum_{j=0}^{i-1}\mu^jU_j(y,v)\Big)-C_i(y,v).
\end{aligned}
$$

On account of (1.3) and the definition of $g(v)$, we have the following expansions.

$$
\begin{gathered}
F_0(y,v)=\sum_{j\ge 2}F_{0j}(v)y^j,\quad F_i(y,v)=\sum_{j\ge 0}F_{ij}(v)y^j,\quad i\ge 1,\\
G_0(y,v)=\sum_{j\ge 1}G_{0j}(v)y^j,\quad G_i(y,v)=\sum_{j\ge 0}G_{ij}(v)y^j,\quad i\ge 1.
\end{gathered}
$$

Thus, the unknown functions can be sought in the form

$$
\begin{gathered}
X_0(y,v)=\sum_{j\ge 2}X_{0j}(v)y^j,\quad X_i(y,v)=\sum_{j\ge 0}X_{ij}(v)y^j,\quad i\ge 1,\\
H_0(y,v)=\sum_{j\ge 2}H_{0j}(v)y^j,\quad H_i(y,v)=\sum_{j\ge 0}H_{ij}(v)y^j,\quad i\ge 1,\\
U_0(y,v)=\sum_{j\ge 1}U_{0j}(v)y^j,\quad U_i(y,v)=\sum_{j\ge 0}U_{ij}(v)y^j,\quad i\ge 1,\\
C_0(y,v)=\sum_{j\ge 1}C_{0j}(v)y^j,\quad C_i(y,v)=\sum_{j\ge 0}C_{ij}(v)y^j,\quad i\ge 1,
\end{gathered}
$$

It is easily seen that the coefficients of these expansions satisfy the chain of equations

$$
\mathfrak{L}_{J(v)y}[X_{ij}(v)y^j]=P_{ij}(v)y^j-H_{ij}(v)y^j,\quad \partial_{J(v)y}[U_{ij}(v)y^j]=Q_{ij}(v)y^j-C_{ij}(v)y^j,
$$

where $P_{ij}(v)$ and $Q_{ij}(v)$ can be determined subsequently. In fact, $P_{02}(v):=F_{02}(v)$, $Q_{01}(v):=G_{01}(v)$, and now, if $v\in\mathcal{D}_N$, one can use Proposition 8.1 to get $H_{02}=0$, X_{02}, $C_{01}=0$, U_{01}, and then subsequently find P_{0j}, H_{0j}, X_{0j} for $j=3,\dots,N$, and Q_{0j}, C_{0j}, U_{0j} for $j=2,\dots,N$. If $0\le k<i$, $l\le N$ and the coefficients $X_{kl}(v)$, $H_{kl}(v)$, $U_{kl}(v)$, $C_{kl}(v)$ are already known, then one can determine $P_{i0}(v)$ and subsequently find $H_{ij}(v)$, $X_{ij}(v)$, $P_{i,j+1}(v)$ for $j=0,\dots,N$. (Note that (8.5) yields $H_{i0}(v)=0$.) After that, $Q_{i0}(v)$ can be determined, and subsequently $C_{ij}(v)$, $U_{ij}(v)$, $Q_{i,j+1}(v)$ may be found for $j=0,\dots,N$. At last, for any $j>N$, we assign $H_{ij}(v)\equiv P_{ij}(v)$, $X_{ij}(v)\equiv 0$, $C_{ij}(v)\equiv Q_{ij}(v)$ and $U_{ij}(v)\equiv 0$.

This result can be summed up as the following proposition.

Proposition 8.3. *Suppose that $P\ge 2$, $N\ge 3$ and $\mathcal{D}_N$ satisfy the conditions of Proposition 8.1. Then there exist $\delta_0>0$ and $\mu_0>0$, such that for any $\varepsilon\in[0,\varepsilon_0]$ the smooth diffeomorphic change of variables*

$$
\begin{gathered}
x=y+\sum_{j=2}^{N}X_{0j}(v)y^j+\sum_{i=1}^{P}\mu^i\sum_{j=0}^{N}X_{ij}(v)y^j,\\
u=v+\sum_{j=1}^{N}U_{0j}(v)y^j+\sum_{i=1}^{P}\mu^i\sum_{j=0}^{N}U_{ij}(v)y^j
\end{gathered}
$$

defined on the set $\{(y,v,\mu)\colon \|y\|<\delta_0, v\in\mathcal{D}_N, \mu\in[0,\mu_0]\}$ *transforms system* (2.3) *into*

$$\begin{aligned} \dot y &= J(v)y+\sum_{j=3}^{N}H_{0j}(v)y^j+\sum_{i=1}^{P}\mu^i\sum_{j=1}^{N}H_{ij}(v)y^j+O(\|y\|^{N+1}+\mu^{P+1}),\\ \dot v &= \mu\Big[g(v)+\sum_{j=2}^{N}C_{0j}(v)y^j+\sum_{i=1}^{P}\mu^i\sum_{j=0}^{N}C_{ij}(v)y^j+O(\|y\|^{N+1}+\mu^{P+1})\Big]. \end{aligned} \tag{8.6}$$

Here the homogeneous forms in the right-hand sides admit the expansions

$$H_{ij}(v)y^j=\sum_{k=1}^{n}\Big[{\sum_{|\mathbf{q}|=j}}^{+}h^{+}_{i,k,\mathbf{q}}(v)e^{+}_{k,\mathbf{q}}(y)+{\sum_{|\mathbf{q}|=j}}^{-}h^{-}_{i,k,\mathbf{q}}(v)e^{-}_{k,\mathbf{q}}(y)\Big],$$

$$C_{ij}(v)y^j=\sum_{k=1}^{n}{\sum_{|\mathbf{q}|=j}}'\mathfrak{s}_{\mathbf{q}}(y)c_{i,\mathbf{q}}(v),$$

where $h^{\pm}_{i,k,\mathbf{q}}(\cdot)\in \mathrm{C}^\infty(\mathcal{D}_N;\mathbb{C})$, $c_{i,\mathbf{q}}(\cdot)\in \mathrm{C}^\infty(\mathcal{D}_N;\mathbb{C}^m)$ *and the summations* $\sum^{+}_{|\mathbf{q}|=j}$, $\sum^{-}_{|\mathbf{q}|=j}$, $\sum'_{|\mathbf{q}|=j}$ *are performed on all vectors* $\mathbf{q}\in\mathbb{Z}^{2n}_{+}$ *with* $|\mathbf{q}|=j$ *whose components satisfy, respectively, the equalities* $q_l=q_{l+n}+1$, $q_{l+n}=q_l+1$ *and* $q_l=q_{l+n}$ *for all* $l\in\{1,\dots,n\}$.

Remark 8.4. In view of Remark 2.1, under hypothesis (H2) the assertion of Proposition 8.3 is true for $N=3$ and $\mathcal{D}_3=\mathcal{V}$. If hypothesis (H3) is valid, then this proposition is correct in a sufficiently small neighborhood of any stationary point $v_*\in\mathcal{W}$.

On account of Remark 8.2,

$$\mathfrak{s}_{\mathbf{q}}(S(z,\bar z))=(z,\bar z)^{\mathbf{q}},\quad S^{-1}e^{+}_{k,\mathbf{q}}(Sz)=(z,\bar z)^{\mathbf{q}}\mathbf{e}^{+}_k,\quad S^{-1}e^{+}_{k,\mathbf{q}}(Sz)=(z,\bar z)^{\mathbf{q}}\mathbf{e}^{-}_k,$$

$$S^{-1}H_{i1}(v)(S(z,\bar z))^1=\sum_{k=1}^{n}\left[z_k h^{+}_{i,k,\mathbf{e}^{+}_k}(v)\mathbf{e}^{+}_k+\bar z_k h^{-}_{i,k,\mathbf{e}^{-}_k}(v)\mathbf{e}^{-}_k\right],$$

which means that the matrices of each of the linear form $S^{-1}H_{i1}(v)(S(z,\bar z))^1$ are diagonal and the change of variables

$$y=S(z,\bar z)=\sum_{k=1}^{n}(z_k s_k+\bar z_k\bar s_k),\quad \|z\|<\|S\|\delta_0=:\delta_1,$$

reduces system (8.6) to the form (2.4), where

$$\eta_{j,k}(v):=h^{+}_{i,k,\mathbf{e}^{+}_k}(v),\quad h_{j,k,\mathbf{p}}(v):=h^{+}_{j,k,(\mathbf{p},\mathbf{p})+\mathbf{e}^{+}_k}(v),\quad g_{j,\mathbf{p}}(v):=c_{j,(\mathbf{p},\mathbf{p})}(v),$$

and $(\mathbf{p},\mathbf{p}):=(p_1,\dots,p_n,p_1,\dots,p_n)$.

9. Summary

In this article we have examined a system of oscillators with weak and slow coupling which demonstrates a dynamic bifurcation of multi-frequency oscillations. Having adopted results of the static bifurcation theory, we have shown that when the system's parameters slowly evolve and the static parameters are sufficiently small, then certain general conditions guarantee occurrence of the following transient process for typical forward trajectories within a small neighborhood of the slow surface. While the slow component $u(t)$ is far from the stationary points of the Morse function V and lies inside the stability zone of the fast subsystem, the fast component $(w(t), \dot{w}(t))$ exhibits damping oscillations with the amplitude approaching zero. However, as soon as $u(t)$ leaves the stability zone and later enters the zone of instability of the fast subsystem, the amplitude starts to grow and eventually the forward trajectory is attracted by an invariant torus, which means establishment of some multi-frequency oscillatory regime.

It was shown that almost all forward trajectories, in terms of the Lebesgue measure, starting from the neighborhood of the slow surface demonstrate such a behavior. More than that, they are attracted by trajectories on the stable n-dimensional invariant tori, whereas all other forward trajectories of the system lie on the stable manifolds of hyperbolic tori of dimensions less than n. This enables us to easily categorize the trajectories by the type of their ultimate behavior.

At last, we have also considered a practical example which depicts occurrence of the multi-frequency bifurcation in a circuit of two coupled oscillators.

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