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EXISTENCE OF WEAK SOLUTIONS FOR THREE-POINT BOUNDARY-VALUE PROBLEMS OF KIRCHHOFF-TYPE

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Abstract. We show the existence of at least one weak solution for a three-point boundary-value problem of Kirchhoff-type. Our technical approach is based on variational methods. In addition, an example to illustrate our results is given.

1. Introduction

The purpose of this paper is to establish the existence of at least one weak solution for the three-point boundary-value problem of Kirchhoff-type

$$\begin{gathered} -K\Big(\int_a^b |u'(t)|^2 dt\Big)u''(t) = f(t,u(t)) + h(u(t)), \quad t\in(a,b), \\ u(a)=0, \quad u(b)=\alpha u(\eta) \end{gathered} \tag{1.1}$$

where $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m \le K(x) \le M$ for all $x \ge 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|h(\xi_1) - h(\xi_2)| \le L|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}$ and $h(0) = 0$, $\alpha \in \mathbb{R}$ and $\eta \in (a,b)$.

Multi-point boundary-value problems of ordinary differential equations play an important role in applied mathematics, physics and the vibration of cables with nonuniform weights [34], and as a consequence, have attracted a great deal of interest over the years. The study of these problems for linear second-order ordinary differential equations was initiated by Ii'in and Moiseev [22]. Motivated by the study of Ii'in and Moiseev [22], Gupta [14] studied certain three-point boundary-value problems for nonlinear ordinary differential equations.

In the past few years, there has been much attention focused on questions of solutions of three-point boundary-value problems for nonlinear ordinary differential equations. For background and recent results, we refer the reader to [3, 11, 12, 14, 23, 24, 25, 40, 41] and the references therein for details. For example, Xu in [41] by employing the fixed point index method, obtained some multiplicity results for positive solutions of some singular semi-positone three-point boundary-value

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problem. Sun in [40] by using a fixed point theorem of cone expansion-compression type due to Krasnosel'skii, established various results on the existence of single and multiple positive solutions for the nonlinear singular third-order three-point boundary-value problem

$$\begin{gathered} u''(t) - \lambda a(t)F(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0 \end{gathered}$$

with $\lambda > 1$, $\eta \in [\frac{1}{2}, 1)$ where $a(t)$ is a non-negative continuous function defined on $(0,1)$ and $F : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Du et al. in [11] based upon Leray-Schauder degree theory, ensured the existence of at least three solutions for the problem

$$\begin{gathered} u''(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \xi u(\eta) \end{gathered}$$

where $\xi > 0$, $0 < \eta < 1$ such that $\xi\eta < 1$ and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Lin [23] by using variational method and three-critical-point theorem, studied the existence of at least three solutions for a three-point boundary-value problem

$$\begin{gathered} u''(t) + \lambda f(t,u) = 0, \quad t \in [0,1], \\ u(0) = 0, \quad u(1) = \alpha u(\eta). \end{gathered}$$

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [2, 7, 15, 17, 19, 35, 39] and the references therein. For example in [17] based on a three critical point theorem, the existence of an interval of positive real parameters λ for which the boundary-value problem of Kirchhoff-type

$$\begin{gathered} -K\Big(\int_a^b |u'(x)|^2 dx\Big)u'' = \lambda f(x,u), \quad t \in [a,b], \\ u(a) = u(b) = 0 \end{gathered}$$

where $K : [0,+\infty[\to \mathbb{R}$ is a continuous function, $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\lambda > 0$, was discussed. Also, in [19] by using variational methods and critical point theory, multiplicity results of nontrivial solutions for one-dimensional fourth-order Kirchhoff-type equations was studied. In recent years, the existence and multiplicity of stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) has been studied, via variational methods like the symmetric mountain pass theorem in [10] and via a three critical point theorem in [6]. Furthermore, in [4, 5] some evolutionary higher order Kirchhoff problems were studied, largely concentrate on the qualitative properties of the solutions. In [28] Molica Bisci and Rădulescu, applying mountain pass results studied the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. In [26], they also by using an abstract linking theorem for smooth functionals established a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator.

Inspired by the above results, in the present paper, we study the existence of at least one weak solution for (1.1). Precisely, in Theorem 3.1 we establish the existence of at least one weak solution for (1.1) requiring an algebraic condition on f. Example 3.2 illustrates Theorem 3.1. Also in Theorem 3.3 a parametric version of this result is successively discussed in which, for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero, the existence of at least one weak solution is established. We also list some consequences the main results. As a consequence of Theorem 3.3, we obtain Theorem 3.11 for the autonomous case. Finally, we present Example 3.12 in which the hypotheses of Theorem 3.11 are fulfilled.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [29] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems. For a thorough discussion of this subject we refer the reader to [9].

2. Preliminary results

We shall prove the existence of at least one weak solution to the problem (1.1) applying the following version of Ricceri's variational principle [38, Theorem 2.1] that we now recall as follows (For a refinement see also [8]):

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

The above result is related to the celebrated *three critical points theorem* of Pucci and Serrin [36, 37]. We refer the interested reader to the papers [1, 13, 16, 18, 21, 27, 30, 31, 32, 33] in which Theorem 2.1 has been successfully employed to the existence of at least one nontrivial solution for boundary-value problems.

Here and in the sequel, we take

$$X = W_1^{1,2}(a, b) := \{u \in W^{1,2}(a, b) : \ u(a) = 0, \ u(b) = \alpha u(\eta)\}.$$

The space X, equipped with the norm

$$\|u\| := \Big(\int_a^b |u'(t)|^2 dt\Big)^{1/2}.$$

Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, that means:

- (a) $t \mapsto f(t, x)$ is measurable for every $x \in \mathbb{R}$,
- (b) $x \mapsto f(t, x)$ is continuous for a.e. $t \in [a, b]$,
- (c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([a, b])$ such that

$$\sup_{|x|\le\rho} |f(t, x)| \le l_\rho(t)$$

for a.e. $t \in [a, b]$.

Corresponding to the functions f, K and h, we introduce the functions $F : [a, b] \times \mathbb{R} \to \mathbb{R}$, $\tilde{K} : [0, +\infty[\to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, defined as follows

$$F(t, x) := \int_0^x f(t, \xi) d\xi \quad \text{for every } (t, x) \in [a, b] \times \mathbb{R},$$
$$\tilde{K}(x) := \int_0^x K(\xi) d\xi \quad \text{for every } x \geq 0,$$
$$H(x) := \int_0^x h(\xi) d\xi \quad \text{for every } x \in \mathbb{R}.$$

We say that a function $u \in X$ is a weak solution of (1.1) if

$$K\Big(\int_a^b |u'(t)|^2 dt\Big) \int_a^b u'(t)v'(t)dt - \int_a^b h(u(t))v(t)dt - \int_a^b f(t, u(t))v(t)dt = 0$$

holds for all $v \in X$.

We assume throughout and without further mention, that the following condition holds:

(H1) $m > \frac{L(1+|\alpha|)^2(b-a)^2}{4}$.

Theorem 2.2 ([23, Theorem 3.2]). *The set X is a separable and reflexive real Banach space.*

The following lemma is needed in the proof of our main result.

Lemma 2.3 ([20, Lemma 2.3]). *For all $u \in X$, we have*

$$\max_{t \in [a,b]} |u(t)| \leq \frac{(1 + |\alpha|)\sqrt{b - a}}{2} \|u\|. \tag{2.1}$$

3. Main results

Our main result reads as follows.

Theorem 3.1. *Assume that*

$$\sup_{\gamma > 0} \frac{\gamma^2}{\int_a^b \sup_{|x| \leq \gamma} F(t, x) dt} > \frac{2(1 + |\alpha|)^2(b - a)}{4m - L(1 + |\alpha|)^2(b - a)^2}. \tag{3.1}$$

Then, problem (1.1) admits at least one weak solution in X.

Proof. Our goal is to apply Theorem 2.1 to (1.1). We define the functionals Φ, Ψ for every $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\tilde{K}(\|u\|^2) - \int_a^b H(u(t))dt, \tag{3.2}$$

$$\Psi(u) = \int_a^b F(t, u(t))dt, \tag{3.3}$$

and we put $I(u) = \Phi(u) - \Psi(u)$ for every $u \in X$. Let us show that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b f(t, u(t))v(t)dt$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Moreover, since $m \leq K(s) \leq M$ for all $s \in [0,+\infty[$, from (3.2) we have

$$\frac{(4m - L(1+|\alpha|)^2(b-a)^2)}{8}\|u\|^2 \leq \Phi(u) \leq \frac{(4M + L(1+|\alpha|)^2(b-a)^2)}{8}\|u\|^2 \quad (3.4)$$

for all $u \in X$ and bearing (H1) in mind, it follows that $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$, namely Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = K\Big(\int_a^b |u'(t)|^2 dt\Big) \int_a^b u'(t)v'(t)dt - \int_a^b h(u(t))v(t)dt$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we see that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified. Note that the critical points of the functional I are the solutions of the problem (1.1). We now look on the existence of a critical point of the functional I in X. The condition (3.1) ensures that there exists $\bar\gamma > 0$ such that

$$\frac{\bar\gamma^2}{\int_a^b \sup_{|x|\leq\bar\gamma} F(t,x)dt} > \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2}. \quad (3.5)$$

Choose

$$r = \frac{4m - L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)}\bar\gamma^2.$$

Bearing in mind relation (2.1), we see that

$$\begin{aligned}\Phi^{-1}(-\infty, r) = \{u \in X; \Phi(u) < r\} &\subseteq \Big\{u \in X; \|u\| \leq \sqrt{\frac{2r(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2}}\Big\} \\ &\subseteq \{u \in X; |u| \leq \bar\gamma\},\end{aligned}$$

and it follows that

$$\Psi(u) \leq \sup_{u\in\Phi^{-1}(-\infty,r)} \int_a^b F(t,u(t))dt \leq \int_a^b \sup_{|x|\leq\bar\gamma} F(t,x)dt$$

for every $u \in X$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u)<r} \Psi(u) \leq \int_a^b \sup_{|x|\leq\bar\gamma} F(t,x)dt.$$

By simple calculations and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)}{r} \\ &\leq \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2} \frac{\int_a^b \sup_{|x|\leq\bar\gamma} F(t,x)dt}{\bar\gamma^2}.\end{aligned}$$

At this point, we observe that

$$\varphi(r) \leq \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2} \frac{\int_a^b \sup_{|x|\leq\bar\gamma} F(t,x)dt}{\bar\gamma^2}. \quad (3.6)$$

Consequently, from (3.5) and (3.6) one has $\varphi(r) < 1$. Hence, since $1 \in (0, \frac{1}{\varphi(r)})$, applying Theorem 2.1 the functional I admits at least one critical point (local minima) $\tilde{u} \in \Phi^{-1}(-\infty, r)$. The proof is complete. $\square$

Now we present an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2. Consider the problem

$$\begin{gathered}
-K\Big(\int_0^1 |u'(t)|^2 dt\Big) u''(t) = f(u) + h(u), \quad t \in (0,1), \\
u(0) = 0, \quad u(1) = \frac{1}{4} u(\frac{1}{2})
\end{gathered} \tag{3.7}$$

where $K(x) = \frac{3}{2} + \frac{\arctan(x))}{\pi}$ for all $x \in \mathbb{R}$,

$$f(x) = \frac{1}{10^3}(2x + e^x)$$

and $h(x) = \sin(x)$ for every $x \in \mathbb{R}$. By the expression of f we have

$$F(x) = \frac{1}{10^3}(x^2 + e^x - 1)$$

for every $x \in \mathbb{R}$. By simple calculations, we obtain $m = 1$. Since

$$\sup_{\gamma>0} \frac{\gamma^2}{\sup_{|x|\le\gamma} F(x)} > \frac{50}{39},$$

we observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that problem (3.7), admits at least one weak solution in $W_1^{1,2}(0,1)$.

We note that Theorem 3.1 can be exploited showing the existence of at least one solution for the following parametric version of (1.1),

$$\begin{gathered}
-K\Big(\int_a^b |u'(t)|^2 dt\Big) u''(t) = \lambda f(t, u(t)) + h(u(t)), \quad t \in (a,b), \\
u(a) = 0, \quad u(b) = \alpha u(\eta)
\end{gathered} \tag{3.8}$$

where λ is a positive parameter. More precisely, we have the following existence result.

Theorem 3.3. *For every λ small enough, more precisely,*

$$\lambda \in \Big(0, \frac{4m - L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)} \sup_{\gamma>0} \frac{\gamma^2}{\int_a^b \sup_{|x|\le\gamma} F(t,x)dt}\Big),$$

problem (3.8) *admits at least one weak solution $u_\lambda \in X$.*

Proof. Fix λ as in the conclusion. Take Φ and Ψ as given in the proof of Theorem 3.1, and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Let us pick

$$0 < \lambda < \frac{4m - L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)} \sup_{\gamma>0} \frac{\gamma^2}{\int_a^b \sup_{|x|\le\gamma} F(t,x)dt}.$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$\lambda \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2} < \frac{\bar{\gamma}^2}{\int_a^b \sup_{|x|\le\bar{\gamma}} F(t,x)dt}.$$

Choose
$$r = \frac{4m - L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)}\bar{\gamma}^2.$$
With the same notation as in the proof of Theorem 3.1, one has
$$\begin{aligned}\varphi(r) &\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)}{r}\\ &\le \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2}\frac{\int_a^b \sup_{|x|\le\bar{\gamma}} F(t,x)dt}{\bar{\gamma}^2} < \frac{1}{\lambda}.\end{aligned}$$
Hence, since $\lambda \in (0, \frac{1}{\varphi(r)})$, Theorem 2.1 ensures that the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$ and since the critical points of the functional I_λ are the solutions of the problem (3.8) we have the conclusion. □

Remark 3.4. In Theorem 3.3 we looked for the critical points of the functional I_λ naturally associated with the problem (3.8). We note that, in general, I_λ can be unbounded from the following in X. Indeed, for example, when $f(t,\xi) = 1+|\xi|^{\gamma-2}\xi$ for $(t,\xi) \in [a,b]\times\mathbb{R}$ with $\gamma > 2$, for any fixed $u \in X\backslash\{0\}$ and $\iota \in \mathbb{R}$, we obtain
$$\begin{aligned}I_\lambda(\iota u) &= \Phi(\iota u) - \lambda\int_a^b F(t,\iota u(t))dt\\ &\le \iota^2\frac{(4M + L(1+|\alpha|)^2(b-a)^2)}{8}\|u\|^2 - \lambda\iota\|u\|_{L^1} - \lambda\frac{\iota^\gamma}{\gamma}\|u\|_{L^\gamma}^\gamma\\ &\to -\infty\end{aligned}$$
as $\iota \to +\infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Remark 3.5. For a fixed $\bar{\gamma} > 0$ let
$$\frac{\bar{\gamma}^2}{\int_a^b \sup_{|x|\le\bar{\gamma}} F(t,x)dt} > \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2}.$$
Then the result of Theorem 3.3 holds with $\|u_\lambda\|_\infty \le \bar{\gamma}$ where u_λ is the ensured weak solution in X.

Remark 3.6. If in Theorem 3.1 the function $f(t,\xi) \ge 0$ for every $t \in [a,b]$ and $\xi \in \mathbb{R}$, then the condition (3.1) assumes the simpler form
$$\sup_{\gamma>0}\frac{\gamma^2}{\int_a^b F(t,\gamma)dt} > \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2}. \tag{3.9}$$
Moreover, if the assumption
$$\limsup_{\gamma\to+\infty}\frac{\gamma^2}{\int_a^b F(t,\gamma)dt} > \frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2},$$
is satisfied, then condition (3.9) automatically holds.

Remark 3.7. If in Theorem 3.3, $f(t,0) \ne 0$ for all $t \in [a,b]$, then the ensured weak solution is obviously non-trivial. On the other hand, the non-triviality of the weak solution can be achieved also in the case $f(t,0) = 0$ for a.e. $t \in [a,b]$ requiring the extra condition at zero, that is there are a non-empty open set $D \subseteq [a,b]$ and a set $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in B} F(t,\xi)}{|\xi|^2} = +\infty, \tag{3.10}$$

$$\liminf_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in D} F(t,\xi)}{|\xi|^2} > -\infty. \tag{3.11}$$

Indeed, let $0 < \bar{\lambda} < \lambda^*$ where

$$\lambda^* = \frac{4m - L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)} \sup_{\gamma>0} \frac{\gamma^2}{\int_a^b \sup_{|x|\le\gamma} F(t,x)dt}.$$

Then, there exists $\bar{\gamma} > 0$ such that

$$\bar{\lambda}\frac{2(1+|\alpha|)^2(b-a)}{4m - L(1+|\alpha|)^2(b-a)^2} < \frac{\bar{\gamma}^2}{\int_a^b \sup_{|x|\le\bar{\gamma}} F(t,x)dt}.$$

Let Φ and Ψ be as given in (3.2) and (3.3), respectively. Due to Theorem 2.1, for every $\lambda \in (0,\bar{\lambda})$ there exists a critical point of $I_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda = \frac{4m-L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)}\bar{\gamma}^2$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that the function u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.12}$$

Owing to the assumptions (3.10) and (3.11), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{n\to+\infty} \frac{\operatorname{ess\,inf}_{t\in B} F(t,\xi_n)}{|\xi_n|^2} = +\infty,$$
$$\operatorname{ess\,inf}_{t\in D} F(t,\xi) \ge \kappa|\xi|^2$$

for every $\xi \in [0,\sigma]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in X$ such that

(1) $v(t) \in [0,1]$ for every $t \in [a,b]$,
(2) $v(t) = 1$ for every $t \in \mathcal{G}$,
(3) $v(t) = 0$ for every $x \in [a,b] \setminus D$.

Hence, for a fixed $M > 0$ we consider a real positive number η with

$$M < \frac{\eta \operatorname{meas}(\mathcal{G}) + \kappa\int_{D\setminus\mathcal{G}} |v(t)|^2 dt}{\frac{4M+L(1+|\alpha|)^2(b-a)^2}{8}\|v\|^2}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$\operatorname{ess\,inf}_{t\in B} F(t,\xi_n) \ge \eta|\xi_n|^2$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \le \xi_n v(t) < \sigma$ for n large enough), by (3.4), one has

$$\begin{aligned}
\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\int_{\mathcal{G}} F(t,\xi_n)dt + \int_{D\setminus\mathcal{G}} F(t,\xi_n v(t))dt}{\Phi(\xi_n v)} \\
&> \frac{\eta \operatorname{meas}(\mathcal{G}) + \kappa\int_{D\setminus\mathcal{G}} |v(t)|^2 dt}{\frac{4M+L(1+|\alpha|)^2(b-a)^2}{8}\|v\|^2} > M.
\end{aligned}$$

Since M can be arbitrarily large, we obtain

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.12) clearly follows. So, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero such that, for n large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we obtain

$$I_\lambda(u_\lambda) < 0, \tag{3.13}$$

so that u_λ is not trivial.

Remark 3.8. By using (3.13), without difficulty we observe that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.14}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, taking into account the fact that Φ is coercive and for every $\lambda \in (0, \lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \le L$ for every $\lambda \in (0, \lambda^*)$. After that, it is easy to see that there exists a positive constant N such that

$$\Big|\int_a^b f(t, u_\lambda(t))u_\lambda(t)dt\Big| \le N\|u_\lambda\| \le NL \tag{3.15}$$

for every $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in X$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b f(t, u_\lambda(t))u_\lambda(t)dt \tag{3.16}$$

for every $\lambda \in (0, \lambda^*)$. Then, since

$$0 \le (m - \frac{L(1+|\alpha|)^2(b-a)^2}{4})\|u_\lambda\|^2 \le \Phi'(u_\lambda)(u_\lambda),$$

by considering (3.16), it follows that

$$0 \le (m - \frac{L(1+|\alpha|)^2(b-a)^2}{4})\|u_\lambda\|^2 \le \lambda \int_a^b f(t, u_\lambda(t))u_\lambda(t)dt \tag{3.17}$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (3.17) together with (3.15) we obtain

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. At last, we have to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. For our goal we see that for any $u \in X$, one has

$$I_\lambda(u) = \lambda\Big(\frac{\Phi(u)}{\lambda} - \Psi(u)\Big). \tag{3.18}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \Big(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}(-\infty,r)} \Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big)$$

for every $i = 1, 2$. Clearly, (3.14) together with (3.18) and the positivity of λ imply that

$$m_{\lambda i} < 0 \quad \text{for } i = 1, 2. \tag{3.19}$$

Moreover

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{3.20}$$

because $0 < \lambda_1 < \lambda_2$. Then, by (3.18)-(3.20) and again by the fact that $0 < \lambda_1 < \lambda_2$, we obtain that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$.

Remark 3.9. If f is non-negative then the solution ensured in Theorem 3.3 is non-negative. Indeed, let u_* be a non-trivial weak solution of the problem (3.8), then u_* is non-negative. Arguing by a contradiction, assume that the set $\mathcal{A} = \{t \in [a, b]; u_*(t) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(t) = \min\{u_*(t), 0\}$. Using this fact that u_* also is a solution of (3.8), so for every $\bar{v} \in X$ we have

$$K\Big(\int_a^b |u_*'(t)|^2dt\Big)\int_a^b u_*'(t)\bar{v}'(t)dt - \int_a^b h(u_*(t))\bar{v}(t)dt - \lambda\int_a^b f(t, u_*(t))\bar{v}(t)dt = 0$$

and by choosing $\bar{v} = u_*$ and since f is non-negative, we have

$$\begin{aligned}
0 &\le \Big(m - \frac{L(1+|\alpha|)^2(b-a)^2}{4}\Big)\|u_*\|_{\mathcal{A}}^2 \\
&\le K\Big(\int_{\mathcal{A}} |u_*'(t)|^2dt\Big)\int_{\mathcal{A}} |u_*'(t)|^2dt - \int_{\mathcal{A}} h(u_*(t))u_*(t)dt \\
&= \lambda\int_{\mathcal{A}} f(t, u_*(t))u_*(t)dt \le 0
\end{aligned}$$

since $m > \frac{L(1+|\alpha|)^2(b-a)^2}{4}$, we have $\|u_*\|_{\mathcal{A}}^2 \le 0$ which contradicts fact that u_* is a non-trivial solution. Hence, u_* is positive.

Remark 3.10. We observe that Theorem 3.3 is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times (0, +\infty) : u_\lambda \text{ is a non-trivial weak solution of (3.8)}\}$$

in $X \times \mathbb{R}$. Indeed, by Theorem 3.3 we have that

$$\|u_\lambda\| \to 0 \quad \text{as} \quad \lambda \to 0.$$

Hence, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\lambda_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to +\infty$. Moreover, we emphasis that due to the fact that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \ne \lambda_2$, the solutions u_{λ_1} and u_{λ_2} ensured by Theorem 3.3 are different.

When f does not depend on t, we obtain the following consequence of Theorem 3.3.

Theorem 3.11. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function. Put $F(x)=\int_0^x f(\xi)d\xi$ for all $x\in\mathbb{R}$. Assume that*

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=+\infty.$$

Then, for each

$$\lambda\in\Lambda=\Big(0,\frac{4m-L(1+|\alpha|)^2(b-a)^2}{2(1+|\alpha|)^2(b-a)^2}\sup_{\gamma>0}\frac{\gamma^2}{F(\gamma)}\Big),$$

the problem

$$-K\Big(\int_a^b|u'(t)|^2dt\Big)u''(t)=\lambda f(u(t))+h(u(t)),\ t\in[a,b],$$
$$u(a)=0,\ u(b)=\alpha u(\eta)$$

admits at least one positive weak solution $u_\lambda\in X$ such that

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0$$

and the real function

$$\lambda\to\frac{1}{2}\tilde{K}(\|u_\lambda\|^2)-\int_a^b H(u_\lambda(t))dt-\int_a^b F(u_\lambda(t))dt$$

is negative and strictly decreasing in Λ.

We conclude this paper by giving an example that illustrates Theorem 3.11.

Example 3.12. We consider the problem

$$\begin{gathered}-K\Big(\int_0^1|u'(t)|^2dt\Big)u''(t)=f(u)+h(u),\quad t\in(0,1),\\ u(0)=0,\quad u(1)=\frac{1}{3}u(\frac{1}{2})\end{gathered}\tag{3.21}$$

where

$$K(x)=\begin{cases}1+x-[x], & [x]\text{ is even},\\ 1+|x-[x+1]|, & [x]\text{ is odd},\end{cases}$$

where $[x]$ is the integer part of x,

$$f(x)=2x+e^x+\frac{2x}{1+x^2}$$

and $h(x)=1-\cos(x)$ for every $x\in\mathbb{R}$. By the expression of f we have

$$F(x)=x^2+e^x+\ln(1+x^2)-1\,.$$

Direct calculations give $m=1$ and $\sup_{\gamma>0}\frac{\gamma^2}{F(\gamma)}=1$. Then all conditions in Theorem 3.11 are satisfied. Hence, for each $\lambda\in\big(0,\frac{5}{8}\big)$, problem (3.21) admits at least one positive weak solution in $u_\lambda\in W_1^{1,2}(0,1)$ such that $\lim_{\lambda\to 0^+}\|u_\lambda\|=0$ and the real function

$$\lambda\to\frac{1}{2}\tilde{K}(\|u_\lambda\|^2)-\int_0^1(u_\lambda(t)-\sin(u_\lambda(t)))\,dt-\int_0^1(u_\lambda^2(t)+e^{u_\lambda(t)}+\ln(1+u_\lambda^2(t))-1)\,dt$$

is negative and strictly decreasing in $(0,5/8)$.

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