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EXISTENCE AND REGULARITY OF SOLUTIONS TO THE LERAY-α MODEL WITH NAVIER SLIP BOUNDARY CONDITIONS

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Abstract. We establish the existence and regularity of a unique weak solution to turbulent flows in a bounded domain $\Omega \subset \mathbb{R}^3$ governed by the Leray-α model with Navier slip boundary condition for the velocity. Furthermore, we show that when the filter coefficient α tends to zero, these weak solutions converge to a suitable weak solution to the incompressible Navier Stokes equations subject to the Navier boundary conditions. Finally, we discuss the relation between the Leray-α model and the Navier-Stokes equations with homogeneous Dirichlet boundary condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞ boundary, $T \in (0,\infty)$, and $\alpha > 0$. Our goal is to study properties of the Leray-α model ($\mathcal{L}(\alpha)$)

$$\operatorname{div} \mathbf{v} = 0, \tag{1.1}$$

$$\mathbf{v}_t + \operatorname{div}(\mathbf{v} \otimes \overline{\mathbf{v}}) - 2\nu \operatorname{div} \mathbf{D}(\mathbf{v}) = -\nabla p + \mathbf{f}, \tag{1.2}$$

$$-\alpha^2 \operatorname{div} \mathbf{D}(\overline{\mathbf{v}}) + \overline{\mathbf{v}} + \nabla \pi = \mathbf{v}, \quad \operatorname{div} \overline{\mathbf{v}} = 0 \tag{1.3}$$

in $(0,T) \times \Omega$. The unknown functions are the fluid velocity field $\mathbf{v}$, the smoothed velocity $\overline{\mathbf{v}}$ and the pressure p. The external body force $\mathbf{f}$ and the viscosity $\nu > 0$ are given. In the above system, $\mathbf{D}$ denotes the symmetric part of the velocity gradient, that is $2\mathbf{D}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^T$.

We complement the system (1.1)-(1.3) to the initial condition

$$\mathbf{v}(0,x) = \mathbf{v}_0(x) \quad \text{in } \Omega, \tag{1.4}$$

and the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \lambda \mathbf{v}_{\boldsymbol{\tau}} + (1-\lambda)(\mathbf{D}(\mathbf{v})\mathbf{n})_{\boldsymbol{\tau}} = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{1.5}$$

$$\overline{\mathbf{v}} \cdot \mathbf{n} = 0, \quad \lambda \overline{\mathbf{v}}_{\boldsymbol{\tau}} + (1-\lambda)(\mathbf{D}(\overline{\mathbf{v}})\mathbf{n})_{\boldsymbol{\tau}} = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{1.6}$$

Here, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the outer normal located at $\mathbf{x} \in \partial\Omega$ to the boundary, $\mathbf{w}_{\boldsymbol{\tau}} := \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\mathbf{n}$ is the projection of a vector $\mathbf{w} = \mathbf{w}(\mathbf{x})$ onto the tangent plane of the boundary at $\mathbf{x}$, and the parameter $\lambda \in [0,1]$ homotopically connects perfect slip boundary condition when $\lambda = 0$ with no-slip boundary conditions when $\lambda = 1$. If

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$0 < \lambda < 1$, then (1.5) is called the Navier slip boundary conditions. In this paper we assume that λ is any number from $[0, 1)$.

We start our investigation by showing that the problem (1.1)-(1.6) has a unique weak solution. Since existence and regularity theory of the problem (1.3) with boundary condition (1.6) is well known (see Lemma 2.1 and Corollary 2.2) $\overline{\mathbf{v}}$ can always be uniquely reconstructed from $\mathbf{v}$. In this sense we understand $\overline{\mathbf{v}}$ in the whole article and we concentrate only on the properties of $(\mathbf{v}, p)$.

We use standard notation for Lebesgue, Sobolev and Besov spaces on a domain O and their norms, e.g. $L^2(O)$, $W^{1,2}(O)$, $B^1_{2,2}(O)$ $(= W^{1,2}(O)$ if O is smooth). If $O = \Omega$ we drop (Ω), e.g. $L^{5/2}$. We denote the inner product in $L^2(O)$ by $(\cdot,\cdot)_O$, while $\langle \cdot,\cdot\rangle$ stands for a duality pairing. We do not distinguish between scalar and vector spaces; the correct meaning is always clear from the context. Next we define the relevant function spaces for the velocity field. Let $k \in \mathbb{N}$, $p, q \geq 1$, then

$$
\begin{gathered}
W^{k,p}_{\mathbf{n}} := \{\mathbf{v} \in W^{k,p} : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
W^{k,p}_{\mathbf{n},\mathrm{div}} := \{\mathbf{v} \in W^{k,p}_{\mathbf{n}} : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\},\\
W^{-k,p'}_{\mathbf{n}} := (W^{k,p}_{\mathbf{n}})^*, \quad W^{-k,p'}_{\mathbf{n},\mathrm{div}} := (W^{k,p}_{\mathbf{n},\mathrm{div}})^*,\\
L^q_{\mathbf{n},\mathrm{div}} := \overline{W^{1,q}_{\mathbf{n},\mathrm{div}}}^{\|\,\|_q}.
\end{gathered}
$$

Our first result is the following theorem.

Theorem 1.1. *Let* $\mathbf{f} \in L^2(0,T; W^{-1,2}_{\mathbf{n}})$, $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$. *Then there exists a unique solution* $(\mathbf{v}, p)$ *to the system* (1.1)–(1.3) *such that*

$$
\begin{gathered}
\mathbf{v} \in \mathcal{C}(0,T; L^2_{\mathbf{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{n},\mathrm{div}}), \qquad (1.7)\\
\mathbf{v}_{,t} \in L^2(0,T; W^{-1,2}_{\mathbf{n}}), \qquad (1.8)\\
p \in L^2(0,T; L^2) \qquad (1.9)\\
\int_\Omega p\,d\mathbf{x} = 0 \quad \text{for a.e. } t \in (0,T) \qquad (1.10)
\end{gathered}
$$

and

$$
\begin{aligned}
&\int_0^T \langle \mathbf{v}_{,t}, \mathbf{w}\rangle - (\mathbf{v}\otimes\overline{\mathbf{v}}, \nabla\mathbf{w}) + \frac{2\nu\lambda}{1-\lambda}(\mathbf{v},\mathbf{w})_{\partial\Omega} + 2\nu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{w}))\,dt\\
&= \int_0^T (p, \operatorname{div}\mathbf{w}) + \langle \mathbf{f}, \mathbf{w}\rangle\,dt \quad \text{for all } \mathbf{w} \in L^2(0,T; W^{1,2}_{\mathbf{n}}),
\end{aligned} \qquad (1.11)
$$

where the unique strong solution $(\overline{\mathbf{v}}, \pi)$ *to* (1.3) *with* (1.6) *satisfies*

$$
\begin{gathered}
\overline{\mathbf{v}} \in \mathcal{C}(0,T; W^{2,2}_{\mathbf{n},\mathrm{div}}) \cap L^2(0,T; W^{3,2}_{\mathbf{n},\mathrm{div}}),\\
\pi \in \mathcal{C}(0,T; W^{1,2}) \cap L^2(0,T; W^{2,2}).
\end{gathered}
$$

The initial conditions are attained in the following sense

$$
\lim_{t\to 0+} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 = 0. \qquad (1.12)
$$

Moreover, the solution $(\mathbf{v}, p)$ *satisfies the local energy equality*

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|\mathbf{v}|^2\phi)(t,\mathbf{x})\,d\mathbf{x} + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\phi\,d\mathbf{x}\,dt \\ &= \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\phi(0,\mathbf{x})\,d\mathbf{x} + \int_0^t\int_\Omega \frac{|\mathbf{v}|^2}{2}(\phi_t + \nu\Delta\phi)\,d\mathbf{x}\,dt \\ &\quad + \int_0^t\int_\Omega (\frac{|\mathbf{v}|^2}{2}\overline{\mathbf{v}} + p\mathbf{v})\cdot\nabla\phi\,d\mathbf{x}dt + \int_0^t \langle \mathbf{f}, \mathbf{v}\phi\rangle\,dt, \end{aligned} \tag{1.13}$$

for all $t \in (0,T)$ *and for all non-negative functions* $\phi \in C^\infty(\Omega\times\mathbb{R})$ *and* $\operatorname{spt}\phi \subset\subset \mathbb{R}\times\Omega$.

In the next theorem we focus our attention on the regularity of the unique weak solution of (1.1)-(1.6). First, we define the spaces of initial conditions. We follow [29]. For $q \geq 2$ we set

$$\mathcal{D}_q := \{\varphi \in B_{q,q}^{2(1-\frac{1}{q})} \cap L^q_{\mathbf{n},\mathrm{div}} : (1.5) \text{ holds if } q > 3\}.$$

Here the spaces $B^\alpha_{p,p}$ are the standard Besov spaces, see [29, Section 2.2]. Note that $\mathcal{D}_2 = W^{1,2}_{\mathbf{n},\mathrm{div}}$.

Now we can formulate the maximal regularity result.

Theorem 1.2. *Assume* $q \geq 2$, $q \neq 3$, $\mathbf{f} \in L^q(0,T;L^q_{\mathbf{n},\mathrm{div}})$ *and* $\mathbf{v}_0 \in \mathcal{D}_q$. *Then the unique weak solution to the problem* $\mathcal{L}(\alpha)$ *with initial boundary condition* (1.4) *and boundary condition* (1.5), (1.6) *is regular, i.e.* $\mathbf{v} \in L^q(0,T;W^{2,q}_{\mathbf{n},\mathrm{div}})$, $\mathbf{v}_{,t} \in L^q(0,T;L^q_{\mathbf{n},\mathrm{div}})$ *and* $p \in L^q(0,T;W^{1,q})$.

Further we are interested in the behavior of the unique weak solution to (1.1)-(1.6) as $\alpha \to 0+$, see Theorem 4.2; as $\lambda \to 1-$, see Theorem 5.1; and as $\lambda \to 1-$ and $\alpha \to 0+$ simultaneously in Theorem 6.1.

Leray [22] was the first who regularized the Navier Stokes equations by smoothing the convective velocity with regularization made by convolution. The α models are based on a smoothing obtained by applying with the application of the inverse of the Helmholtz operator $I - \alpha^2\Delta$. There exists a large family of α models, see for example [2, 4, 14, 10, 12, 15, 16, 20, 21].

One of the first α models is the Lagrangian averaged Navier Stokes equations (LANS-α) [11] that was introduced as a sub-grid scale turbulence model. In [15] the authors suggest the LANS-α as a closure model for the Reynolds averaged equations. The Leray-α model [12], as the other family of α models, enjoys the same results of existence and uniqueness of solutions and was also used as a closure model for the Reynolds averaged equations. The Leray-α was tested numerically in [12, 18]. In this numerical simulation the authors showed that large scales of motion bigger than α in flow are captured. It was shown also that for scales of motion smaller than α, the energy spectra decays faster in comparison to that of the Navier Stokes equations. In [12], the convergence of a weak solution of the Leray-α to a weak solution of the Navier-Stokes equations as $\alpha \to 0$ was established. It is shown in [2] that the Leray-α equations give rise to a suitable weak solution to the Navier-Stokes equations. All previously mentioned results were derived under periodic boundary conditions.

The existence and uniqueness of global weak solutions to the LANS-α on bounded domain with no-slip boundary condition is given in [13]. The fact that we are able to establish such results of existence, uniqueness and convergence with Navier slip boundary conditions to the $\mathcal{L}(\alpha)$ model is a novel feature of the present study.

Finally, one may ask questions about other closure models of turbulence on bounded domains with usual boundary conditions, such as the Navier slip conditions. This is a crucial problem, because the filter in this case does not commute with the differential operators [3, 6, 14, 17, 21].

This article is organized as follows. In Sect. 2 we recall some preliminary results concerning solutions of elliptic equations with Navier boundary conditions. Then, in Sect. 3, inspired by the result in [7], we give the proofs of Theorems 1.1 and 1.2. In Sect. 4 we concentrate on an analysis of the behavior of the solutions $(\mathbf{v}^\alpha, p^\alpha)$ as $\alpha \to 0+$, where we show that α regularization gives rise to a suitable weak solution to the Navier-Stokes equations. In Sect. 5 we take care of the dependence of the solution of the parameter λ in order to pass to the limit as $\lambda \to 1-$ and in the last section we pass to the limit as $\alpha \to 0+$ and $\lambda \to 1-$ simultaneously.

2. Auxiliary results

2.1. **Stokes problem.** In this subsection we collect some known results concerning properties of solutions to the Stokes problem with Navier boundary condition (1.5).

Let us first consider the stationary Stokes problem for some fixed function $\mathbf{v}$.

$$-\alpha^2 \operatorname{div} \mathbf{D}(\overline{\mathbf{v}}) + \overline{\mathbf{v}} + \nabla \pi = \mathbf{v}, \quad \operatorname{div} \overline{\mathbf{v}} = 0 \quad \text{on } \Omega, \tag{2.1}$$

$$\overline{\mathbf{v}} \cdot \mathbf{n} = 0, \quad \lambda \overline{\mathbf{v}}_{\boldsymbol{\tau}} + (1-\lambda)(\mathbf{D}(\overline{\mathbf{v}})\mathbf{n})_{\boldsymbol{\tau}} = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\int_\Omega \pi d\mathbf{x} = 0. \tag{2.3}$$

We have the following lemma about existence and regularity of solutions.

Lemma 2.1. *Assume that $\alpha_0 > 0$, $\alpha \in (0, \alpha_0)$, $q > 1$, $\mathbf{v} \in L^q$. Then the unique solution $(\overline{\mathbf{v}}, \pi)$ of system (2.1)-(2.3) is in $W^{2,q}_{\mathbf{n},\mathrm{div}} \times W^{1,q}$ and satisfies the estimates*

$$\|\overline{\mathbf{v}}\|_{2,q} + \|\pi\|_{1,q} \le C(\alpha)\|\mathbf{v}\|_q, \quad \|\overline{\mathbf{v}}\|_q \le C(\alpha_0)\|\mathbf{v}\|_q.$$

The constant $C(\alpha) > 0$ depends on α, while $C(\alpha_0) > 0$ may depend on α only through α_0.

If moreover $k \in \mathbb{N}$, $k > 1$ and $\mathbf{v} \in W^{k,q}$, then $(\overline{\mathbf{v}}, \pi) \in W^{k+2,q}_{\mathbf{n},\mathrm{div}} \times W^{k+1,q}$ and the following estimate holds

$$\|\overline{\mathbf{v}}\|_{k+2,q} + \|\pi\|_{k+1,q} \le C(\alpha)(\|\mathbf{v}\|_{k,q} + \|\overline{\mathbf{v}}\|_q + \|\pi\|_q).$$

Proof. The first part of the lemma is proved in [24, Theorem 1.3, (1)]. The second part follows from the result [1, Theorem 10.5], since the Stokes operator satisfies the ellipticity condition [1, Section I.1] and the Navier boundary condition is a complementary one, see [1, Section I.2]. □

Corollary 2.2. *Let $k \in \mathbb{N} \cup \{0\}$, $r \in [1, +\infty)$, $q > 1$. Assume $\mathbf{v} \in L^r(0,T; W^{k,q})$. Then the unique solution $(\overline{\mathbf{v}}, \pi)$ to problem (1.3) with boundary conditions (1.6) and (2.3) satisfies $\overline{\mathbf{v}} \in L^r(0,T; W^{k+2,q}_{\mathbf{n},\mathrm{div}})$, $\pi \in L^r(0,T; W^{k+1,q})$.*

Now we turn our attention to the evolutionary variant of the problem (2.1).

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}_{,t} - 2\nu \operatorname{div} \mathbf{D}(\mathbf{v}) = -\nabla p + \mathbf{f}. \tag{2.4}$$

Lemma 2.3. *Let $2 \le q < +\infty$, $q \neq 3$. If $\mathbf{v}_0 \in \mathcal{D}_q$ and $\mathbf{f} \in L^q(0,T;L^q)$ then problem* (2.4) *with* (1.10)*, boundary condition* (1.5) *and initial condition* (1.4) *admits a unique solution $(\mathbf{v}, p)$ such that*

$$\mathbf{v} \in L^q(0,T;W^{2,q}_{\mathbf{n},\mathrm{div}}) \cap W^{1,q}(0,T;L^q), \quad p \in L^q(0,T;W^{1,q}).$$

The above theorem is proved in [25, Theorem 1.2]. We finish this section with the following interpolation lemma.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $r > 1$ and f belong to $L^\infty(0,T;L^r) \cap L^r(0,T;W^{2,r})$. Then $\nabla f \in L^s(Q)$ for $s = r + r^2/(n+r)$.*

Proof. First we realize that the inequality

$$\|\nabla f\|_s \le C\|f\|_r^{1-\theta}\|f\|_{2,r}^{\theta}\,,$$

with $\theta = (n+r)/(n+2r)$ holds as a consequence of [30, 4.2.1/3], [30, 2.4.2/11 and 4.3.2/Theorem 2], [30, Theorem 4.6.2a]. Taking the s power of this inequality the statement of the lemma then follows since $\theta s = r$. $\square$

3. Proof of main theorems

Proof of Theorem 1.1. We prove the theorem using the Schauder fixed point theorem. To this end we fix $r > 1$, $q > 1$ (the exact values of r and q will be determined later) and study properties of the mapping

$$M_2 : L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}}) \cap L^r(0,T;L^q) \to L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}}) \cap L^\infty(0,T;L^2),$$
$$M_2(\overline{\mathbf{v}}) = \mathbf{u},$$

where $\mathbf{u} \in L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}}) \cap L^\infty(0,T;L^2)$ is the unique solution to the problem

$$\operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}_{,t} + \operatorname{div}(\mathbf{u}\otimes\overline{\mathbf{v}}) - 2\nu\operatorname{div}\mathbf{D}(\mathbf{u}) = -\nabla p + \mathbf{f},$$

with the initial condition

$$\mathbf{u}(0,x) = \mathbf{v}_0(x) \quad \text{in } \Omega,$$

and boundary condition

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \lambda\mathbf{u}_{\boldsymbol{\tau}} + (1-\lambda)(\mathbf{D}(\mathbf{u})\mathbf{n})_{\boldsymbol{\tau}} = 0 \quad \text{on } (0,T)\times\partial\Omega.$$

Our first goal is to determine the constants r, q such that the mapping M_2 is well defined and continuous. Since for any $\gamma \ge 2$,

$$L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2}) \hookrightarrow L^\gamma(0,T;L^{\frac{6\gamma}{3\gamma-4}}) \tag{3.1}$$

it is enough to assume for some $\gamma > 2$ that

$$r \ge \frac{2\gamma}{\gamma-2}, \quad q \ge \frac{3\gamma}{2}. \tag{3.2}$$

Under these assumptions, $|\mathbf{u}||\overline{\mathbf{u}}| \in L^2(0,T;L^2)$. The correctness of the definition of M_2 and its continuity follow by standard technique. Moreover, it is also seen that there exists $C > 0$ independent of $\overline{\mathbf{v}}$ such that

$$\|\mathbf{u}\|_{L^\gamma(0,T;L^{\frac{6\gamma}{3\gamma-4}})} + \|\mathbf{u}\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}\|_{L^2(0,T;W^{1,2})} \le C. \tag{3.3}$$

Condition (3.2) also assures that

$$\mathbf{u}_t \in L^2(0,T;\big(W^{1,2}_{n,\mathrm{div}}\big)^*)$$

and the Aubin-Lions compactness lemma provides that

$$M_2 : L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}}) \cap L^r(0,T;L^q) \hookrightarrow L^\gamma(0,T;L^s) \tag{3.4}$$

is compact for any $\gamma > 2$ and $s \in (1, 6\gamma/(3\gamma-4))$. Compare (3.1).

For $s \in (1,3/2)$ we introduce a mapping

$$M_1 : L^\gamma(0,T;L^s) \hookrightarrow L^\gamma(0,T;W^{2,s}), \quad M_1(\mathbf{v}) = \overline{\mathbf{v}},$$

where $\overline{\mathbf{v}}$ is the unique solution to the problem (2.1)-(2.3). Its existence and regularity is assured by Corollary 2.2. Here γ, s, r and q are sought such that

$$L^\gamma(0,T;W^{2,s}) \hookrightarrow L^r(0,T;L^q) \cap L^2(0,T;W^{1,2}). \tag{3.5}$$

We need $\gamma \ge r$, $\gamma \ge 2$ and $3s/(3-2s) \ge q$, $3s/(3-s) \ge 2$.

Finally we want to apply the Schauder fixed point theorem to $M = M_2 \circ M_1$. To this end we set $\gamma = r = q = 5$. In order to have M well defined we need (3.5) which is verified if $s > 6/5$. The compactness of M follows from (3.4) provided $s < 30/11$. It is seen that we can fix $s \in (6/5, 3/2)$. Altogether we obtain that

$$M : L^5(0,T;L^s) \hookrightarrow L^5(0,T;L^s)$$

is a continuous, compact mapping that maps a certain ball into itself, see (3.3). The Schauder fixed point theorem gives a fixed point of M which solves (1.1)-(1.6) in the weak sense and satisfies (1.7), (1.8) and (1.12). It remains to reconstruct pressure. This can be done as in [8, Section 3.2] since in $W^{1,2}_n$ the Helmholtz decomposition holds, compare [8, Section 2.3]. The procedure gives (1.9)-(1.11). Properties of $\overline{\mathbf{v}}$ and π follow from Lemma 2.1 and Corollary 2.2.

Up to now we have proved the existence of the solution. Now we concentrate on its uniqueness. Let $(\mathbf{v}_1, p_1)$ and $(\mathbf{v}_2, p_2)$ be any two solutions to $\mathcal{L}(\alpha)$ on the interval $[0,T]$, with initial values $\mathbf{v}_1(0)$ and $\mathbf{v}_2(0)$. Let $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$ and $\overline{\mathbf{w}} = \overline{\mathbf{v}_1} - \overline{\mathbf{v}_2}$. We subtract the equation for $\mathbf{v}_2$ from the equation for $\mathbf{v}_1$ and test it with $\mathbf{w}$. Using Korn's inequality, the embedding theorem and Lemma 2.1 successively we obtain

$$\begin{aligned} \frac{d}{dt}\|\mathbf{w}\|_2^2 + 4\nu\|\mathbf{D}(\mathbf{w})\|_2^2 &\le \frac{C}{\nu}\|\mathbf{v}_1\overline{\mathbf{w}}\|_2^2 + \nu(\|\mathbf{w}\|_2^2 + \|\mathbf{D}(\mathbf{w})\|_2^2) \\ &\le \frac{C}{\nu}\|\overline{\mathbf{w}}\|_{2,2}^2\|\mathbf{v}_1\|_{1,2}^2 + \nu(\|\mathbf{w}\|_2^2 + \|\mathbf{D}(\mathbf{w})\|_2^2) \\ &\le \|\mathbf{w}\|_2^2(\frac{C}{\nu}\|\mathbf{v}_1\|_{1,2}^2 + \nu) + \nu\|\mathbf{D}(\mathbf{w})\|_2^2. \end{aligned} \tag{3.6}$$

Using Gronwall's inequality we prove the continuous dependence of the solutions on the initial data in the $L^\infty(0,T,L^2_{\mathbf{n},\mathrm{div}})$ norm. In particular, if $\mathbf{w}_0 = 0$ then $\mathbf{w} = 0$ and the solution $\mathbf{v}$ is unique. Since the pressure part of the solution is uniquely determined by the velocity part and the condition (1.10), the proof of the uniqueness is complete.

It remains to prove that the unique solution $(\mathbf{v}, p)$ satisfies the local energy equality (1.13). To this end let us take $\phi\mathbf{v}$ as the test function in (1.11). We note that the regularity of $\overline{\mathbf{v}}$ ensure that all the terms are well defined. In particular the integral

$$\int_0^T \int_\Omega \mathbf{v} \otimes \overline{\mathbf{v}} \cdot \nabla(\mathbf{v}\phi)\, d\mathbf{x}dt$$

is finite by using the fact that $\mathbf{v}\otimes\overline{\mathbf{v}}\in L^2(0,T;L^2)$ and $\phi\mathbf{v}\in L^2(0,T;W^{1,2})$. Integration by parts combined with the identity

$$\int_\Omega \mathbf{v}\otimes\overline{\mathbf{v}}\cdot\nabla(\mathbf{v}\phi)\,d\mathbf{x}=\frac{1}{2}\int_\Omega \overline{\mathbf{v}}|\mathbf{v}|^2\cdot\nabla\phi\,d\mathbf{x} \tag{3.7}$$

yields that for all $t\in(0,T)$ and for all non-negative functions $\phi\in C^\infty$ and $\operatorname{spt}\phi\subset\subset\Omega\times(0,T)$, $(\mathbf{v},p)$ satisfies

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\mathbf{v}(t)|^2\phi(t,\mathbf{x})\,d\mathbf{x}+\nu\int_0^t\int_\Omega|\nabla\mathbf{v}|^2\phi\,d\mathbf{x}dt\\
&=\frac{1}{2}\int_\Omega|\mathbf{v}_0|^2\phi(0,\mathbf{x})\,d\mathbf{x}+\int_0^t\int_\Omega\frac{|\mathbf{v}|^2}{2}\phi_t\,d\mathbf{x}dt\\
&\quad+\int_0^t\int_\Omega(\frac{|\mathbf{v}|^2}{2}\overline{\mathbf{v}}+p\mathbf{v}-\nu[\nabla\mathbf{v}]\mathbf{v})\cdot\nabla\phi\,d\mathbf{x}dt+\int_0^t\langle\mathbf{f},\mathbf{v}\phi\rangle\,dt.
\end{aligned}\tag{3.8}$$

Integrating by parts once more in the above equality, we obtain (1.13) and the proof of Theorem 1.1 is complete. □

Remark 3.1. Since $T>0$ was arbitrary the solution constructed in Theorem 1.1 may be uniquely extended for all time.

Proof of Theorem 1.2. First we realize that by Theorem 1.1 we know the existence of a solution $\mathbf{v}$ to the problem $\mathcal{L}(\alpha)$ such that $\mathbf{v}\in\mathcal{C}(0,T;L^2_{\mathbf{n},\mathrm{div}})\cap L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}})$. By Corollary 2.2 we obtain that $\overline{\mathbf{v}}\in L^\infty(0,T;W^{2,2}_{\mathbf{n},\mathrm{div}})\cap L^2(0,T;W^{3,2}_{\mathbf{n},\mathrm{div}})$. The embedding theorem gives $\overline{\mathbf{v}}\in L^\infty(Q)$. We know that $\nabla\mathbf{v}\in L^2(Q)$. From the regularity of $\overline{\mathbf{v}}$ it follows that $\operatorname{div}(\mathbf{v}\otimes\overline{\mathbf{v}})=[\nabla\mathbf{v}]\overline{\mathbf{v}}\in L^2(Q)$. Applying Lemma 2.3 we obtain $\mathbf{v}\in W^{1,2}(0,T;L^2_{\mathbf{n},\mathrm{div}})\cap L^2(0,T;W^{2,2}_{\mathbf{n},\mathrm{div}})$ and by Lemma 2.4 $\nabla\mathbf{v}\in L^{s(2)}(Q)$ with function $s(r):=r+r^2/(3+r)$.

Let us assume $\nabla\mathbf{v}\in L^r(Q)$ with $r\in[2,q]$. Then $\operatorname{div}(\mathbf{v}\otimes\overline{\mathbf{v}})\in L^r(Q)$ and by Lemma 2.3 $\mathbf{v}\in W^{1,r}(0,T;L^r_{\mathbf{n},\mathrm{div}})\cap L^r(0,T;W^{2,r}_{\mathbf{n},\mathrm{div}})$. Lemma 2.4 gives $\nabla\mathbf{v}\in L^{s(r)}(Q)$. Since for all $r\geq 2$ it holds that $s(r)>r$. The statement of the theorem follows by iterating this procedure. □

4. Passage to the limit as $\alpha\to 0+$

If we set $\alpha=0$ and π constant in $\mathcal{L}(\alpha)$ we obtain the Navier Stokes system $\mathcal{NS}$

$$\operatorname{div}\mathbf{v}=0, \tag{4.1}$$

$$\mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-2\nu\operatorname{div}\mathbf{D}(\mathbf{v})=-\nabla p+\mathbf{f}, \tag{4.2}$$

$$\mathbf{v}(0,x)=\mathbf{v}_0(x). \tag{4.3}$$

Our aim here is to show that the solutions of $\mathcal{L}(\alpha)$ from Theorem 1.1 with $\alpha>0$ converge to a suitable weak solution to $\mathcal{NS}$. The notion of a suitable weak solution of $\mathcal{NS}$ was introduced by Scheffer [23]. It is related to the notion of the weak solution. However, in addition, a local energy inequality is required (see (4.10) below). First we examine the connection between $\mathbf{v}$ and $\overline{\mathbf{v}}$.

Lemma 4.1. *Assume that* $\mathbf{v} \in W^{1,2}_{\mathbf{n},\mathrm{div}}$ *and* $\overline{\mathbf{v}}$ *is a solution to* (1.3) *with boundary conditions* (1.6). *Then*

$$\begin{aligned} &\alpha^2\|\mathbf{D}(\mathbf{v}-\overline{\mathbf{v}})\|_2^2 + \frac{\alpha^2\lambda}{1-\lambda}\|\mathbf{v}-\overline{\mathbf{v}}\|_{2,\partial\Omega}^2 + 2\|\overline{\mathbf{v}}-\mathbf{v}\|_2^2 \\ &\le \alpha^2(\|\mathbf{D}(\mathbf{v})\|_2^2 + \frac{\lambda}{1-\lambda}(\mathbf{v},\mathbf{v})_{\partial\Omega}). \end{aligned} \tag{4.4}$$

Proof. Testing the weak formulation of (1.3) with $\mathbf{v}-\overline{\mathbf{v}}$ yields

$$\begin{aligned} &\alpha^2\|\mathbf{D}(\mathbf{v})-\mathbf{D}(\overline{\mathbf{v}})\|_2^2 + \alpha^2\frac{\lambda}{1-\lambda}(\mathbf{v}-\overline{\mathbf{v}},\mathbf{v}-\overline{\mathbf{v}})_{\partial\Omega} + \|\mathbf{v}-\overline{\mathbf{v}}\|_2^2 \\ &= \alpha^2(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{v}-\overline{\mathbf{v}}))_\Omega + \alpha^2\frac{\lambda}{1-\lambda}(\mathbf{v},(\mathbf{v}-\overline{\mathbf{v}}))_{\partial\Omega} \\ &\le \frac{1}{2}\Big(\alpha^2\|\mathbf{D}(\mathbf{v})\|_2^2 + \alpha^2\|\mathbf{D}(\mathbf{v})-\mathbf{D}(\overline{\mathbf{v}})\|_2^2 \\ &+ \alpha^2\frac{\lambda}{1-\lambda}(\mathbf{v},\mathbf{v})_{\partial\Omega} + \alpha^2\frac{\lambda}{1-\lambda}(\mathbf{v}-\overline{\mathbf{v}},\mathbf{v}-\overline{\mathbf{v}})_{\partial\Omega}\Big) \end{aligned}$$

and the result follows. □

Theorem 4.2. *Let* $\alpha_j \to 0+$ *as* $j \to +\infty$, $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$, $\mathbf{f} \in L^2(0,T;W^{-1,2}_{\mathbf{n}})$. *Let* $\mathbf{v}^{\alpha_j}$ *be the unique solution to* $\mathcal{L}(\alpha)$ *with* (1.4)-(1.6) *and* $\alpha = \alpha_j$. *Then there is a subsequence of* $\{\alpha_j\}$, *which we denote again by* $\{\alpha_j\}$, $\mathbf{v} \in C_{weak}(0,T;L^2_{\mathbf{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}})$, $p \in L^{5/3}(\Omega\times(0,T))$ *with* $\mathbf{v}_t \in (L^{5/2}(0,T;W^{1,5/2}_{\mathbf{n}}))^*$ *and* $\mathbf{v}(0) = \mathbf{v}_0$ *such that as* $j \to +\infty$,

$$\mathbf{v}^{\alpha_j} \rightharpoonup \mathbf{v} \quad \textit{weakly in } L^2(0,T;W^{1,2}), \tag{4.5}$$

$$\mathbf{v}^{\alpha_j}_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \textit{weakly in } (L^{5/2}(0,T;W^{1,5/2}_{\mathbf{n}}))^*, \tag{4.6}$$

$$\mathbf{v}^{\alpha_j} \to \mathbf{v} \quad \textit{strongly in } L^q(0,T;L^q),\ \textit{for all } 1 \le q < 10/3 \tag{4.7}$$

$$p^{\alpha_j} \rightharpoonup p \quad \textit{weakly in } L^{5/3}(0,T;L^{5/3}). \tag{4.8}$$

Consequently, $(\mathbf{v},p)$ *is a weak dissipative solution of* $\mathcal{NS}$ *with Navier boundary condition* (1.5) *and the initial condition* (1.4), *i.e.*

$$\begin{aligned} &\int_0^T \langle \mathbf{v}_{,t},\mathbf{w}\rangle - (\mathbf{v}\otimes\mathbf{v},\nabla\mathbf{w}) + \frac{2\nu\lambda}{1-\lambda}(\mathbf{v},\mathbf{w})_{\partial\Omega} + 2\nu(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{w}))\,dt \\ &= \int_0^T (p,\operatorname{div}\mathbf{w}) + \langle \mathbf{f},\mathbf{w}\rangle\,dt \quad \textit{for all } \mathbf{w} \in L^{\frac{5}{2}}(0,T;W^{1,\frac{5}{2}}_{\mathbf{n}}). \end{aligned} \tag{4.9}$$

Moreover, the solution $(\mathbf{v},p)$ *satisfies the following local energy inequality*

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|\mathbf{v}|^2\phi)(t,\mathbf{x})\,d\mathbf{x} + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\phi\,d\mathbf{x}dt \\ &\le \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\phi(0,\mathbf{x})\,d\mathbf{x} + \int_0^t\int_\Omega \frac{|\mathbf{v}|^2}{2}(\phi_t+\nu\Delta\phi) \\ &\quad + \int_0^t\int_\Omega \Big(\frac{|\mathbf{v}|^2}{2}\mathbf{v} + p\mathbf{v}\Big)\cdot\nabla\phi\,d\mathbf{x}dt + \int_0^t \langle \mathbf{f},\mathbf{v}\phi\rangle\,dt \end{aligned} \tag{4.10}$$

for a.e. $t \in (0,T)$ *and for all non-negative functions* $\phi \in C^\infty$ *and supp* $\phi \subset\subset \Omega\times(0,T)$.

Proof. We need to find estimates that are independent of α. In this proof, the constant $C > 0$ is independent of α.

First we obtain, testing (1.11) by $\mathbf{v}^\alpha$, the existence of $C > 0$ such that for all α we have

$$\frac{2\nu\lambda}{1-\lambda}\|\mathbf{v}^\alpha\|_{L^2(0,T;L^2(\partial\Omega))} + \|\mathbf{v}^\alpha\|_{L^\infty(0,T,L^2)} + \|\mathbf{v}^\alpha\|_{L^2(0,T,W^{1,2})} \le C. \tag{4.11}$$

By standard interpolation we obtain

$$\|\mathbf{v}^\alpha\|_{L^{10/3}(0,T;L^{10/3})} \le C. \tag{4.12}$$

Lemma 2.1 gives

$$\|\overline{\mathbf{v}}^\alpha\|_{L^{10/3}(0,T;L^{10/3})} \le C. \tag{4.13}$$

Since we are considering Navier boundary conditions and in $W^{1,5/2}_{\mathbf{n}}$ there holds Helmholtz decomposition (compare [8, Section 2.3]) we can conclude from (4.11), (4.12) and (4.13) a uniform bound

$$\|\mathbf{v}^\alpha_{,t}\|_{(L^{5/2}(0,T;W^{1,5/2}_{\mathbf{n}}))^*} \le C. \tag{4.14}$$

From [7, Remark 3.1] we know that for all $h \in L^\infty$ and a.e. $t \in (0,T)$

$$\begin{aligned}(p^\alpha(t), h) = &-(\mathbf{v}^\alpha(t)\otimes\overline{\mathbf{v}}^\alpha(t), \nabla^2 H) + \frac{2\nu\lambda}{1-\lambda}(\mathbf{v}^\alpha(t), \nabla H)_{\partial\Omega}\\ &+ 2\nu(\mathbf{D}(\mathbf{v}^\alpha(t)), \nabla^2 H) - \langle \mathbf{f}(t), \nabla H\rangle,\end{aligned}$$

holds, where H is a solution of $-\Delta H = h$ in Ω, $\partial H/\partial\mathbf{n} = 0$ on $\partial\Omega$, $\int_\Omega H = 0$. It is seen that integrability of the pressure follows from the integrability of $\mathbf{v}\otimes\overline{\mathbf{v}}$, $\mathbf{D}(\mathbf{v})$, $\mathbf{f}$ and $\mathbf{v}$. It is standard to show from (4.11), (4.12) and (4.13) that

$$\|p^\alpha\|_{L^{5/3}(0,T;L^{5/3})} \le C. \tag{4.15}$$

It follows from (4.11), (4.14) and (4.15) that we can find a subsequence of $\{\alpha^j\}$ and $(\mathbf{v}, p)$ such that (4.5), (4.6), (4.8) hold and $\mathbf{v} \in L^\infty(0,T;L^2)$. Another subsequence can be extracted such that (4.7) holds due to (4.11) and (4.14) by the Aubin-Lions lemma.

To show that $(\mathbf{v}, p)$ solves (4.9) and (4.10) it is necessary to pass to the limit $\alpha^j \to 0$ as $j \to +\infty$ in (1.11) and (1.13). This is standard if we realize that by Lemma 4.1 and (4.11) we know that there exists $C > 0$ such that

$$\|\overline{\mathbf{v}}^\alpha - \mathbf{v}^\alpha\|^2_{L^2(0,T;L^2)} \le C\alpha^2, \tag{4.16}$$

and that this fact implies (together with (4.7) and (4.13)) that, up to a subsequence, $\overline{\mathbf{v}}^{\alpha_j} \to \mathbf{v}$ in $L^q(0,T;L^q)$ for all $q \in [2, \frac{10}{3})$ as $j \to +\infty$.

It remains to show weak continuity of $\mathbf{v}$, which however follows from the fact that $\mathbf{v} \in C(0,T;(W^{1,5/2}_{\mathbf{n}})^*)$ by (5.2) and $\mathbf{v} \in L^\infty(0,T;L^2)$. □

5. Passage to the limit as $\lambda \to 1-$

Now we want to take care of dependence of the solution on the parameter λ from (1.5) and (1.6). We will denote this dependence by superscript λ.

When $\lambda \to 1-$ in (1.5) we obtain the homogeneous Dirichlet boundary condition (i.e. the condition $\mathbf{v} = 0$ on $(0,T)\times\partial\Omega$). In this case the problem $\mathcal{L}(\alpha)$ with homogeneous Dirichlet boundary condition can be obtained as a limit from $\mathcal{L}(\alpha)$ with Navier slip boundary conditions for any $\alpha > 0$ by letting λ in (1.5) and (1.6) tend to $1-$.

Theorem 5.1. *Let $\lambda_j \to 1-$ as $j \to +\infty$, $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$, $\mathbf{f} \in L^2(0,T;W^{-1,2}_{\mathbf{n}})$. Let $\mathbf{v}^{\lambda_j}$ be the unique solution to $\mathcal{L}(\alpha)$ with (1.4)-(1.6) and $\lambda = \lambda_j$.*

Then there is a subsequence of $\{\lambda_j\}$, which we denote again by $\{\lambda_j\}$, $\mathbf{v} \in C(0,T;L^2_{\mathbf{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}})$ with $\mathbf{v}_t \in (L^2(0,T;W^{1,2}_{0,\mathrm{div}})^*$ and $\mathbf{v}(0) = \mathbf{v}_0$ such that as $j \to +\infty$,

$$\mathbf{v}^{\lambda_j} \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T;W^{1,2}), \tag{5.1}$$

$$\mathbf{v}^{\lambda_j}_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } (L^2(0,T;W^{1,2}_{0,\mathrm{div}}))^*, \tag{5.2}$$

$$\mathbf{v}^{\lambda_j} \to \mathbf{v} \quad \text{strongly in } L^q(0,T;L^q), \text{ for all } 1 \le q < 10/3, \tag{5.3}$$

$\mathbf{v}$ is the unique weak solution to $\mathcal{L}(\alpha)$ with homogeneous Dirichlet boundary condition and initial condition (1.4), i.e.

$$\int_0^T \langle \mathbf{v}_{,t}, \mathbf{w}\rangle - (\mathbf{v} \otimes \overline{\mathbf{v}}, \nabla \mathbf{w}) + 2\nu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{w}))\, dt = \int_0^T \langle \mathbf{f}, \mathbf{w}\rangle\, dt \tag{5.4}$$

for all $\mathbf{w} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}})$.

Moreover let $\mathbf{f} \in L^q(0,T;L^q_{\mathbf{n},\mathrm{div}})$ for some $q \ge 2$, $\mathbf{v}_0 \in W^{2-2/q,q}$ with $\mathbf{v}_0 = 0$ on $\partial\Omega$ and $\operatorname{div} \mathbf{v}_0 = 0$ is Ω. Then

$$\mathbf{v} \in L^q(0,T;W^{2,q}_{0,\mathrm{div}}) \cap W^{1,q}(0,T;L^q_{\mathbf{n},\mathrm{div}}) \tag{5.5}$$

and the pressure can be reconstructed in such a way that $p \in L^q(0,T;W^{1,q})$ and (1.10) holds.

Proof. Testing (1.11) with $\mathbf{v}^\lambda$ we know that

$$\sup_{t\in(0,T)} \|\mathbf{v}^\lambda(t)\|_2^2 + \nu \int_0^T \|\mathbf{v}^\lambda(t)\|_{1,2}^2 dt + \nu \frac{\lambda}{1-\lambda} \int_0^T (\mathbf{v}^\lambda, \mathbf{v}^\lambda)_{\partial\Omega} \le C(\mathbf{v}_0, \mathbf{f}) < \infty. \tag{5.6}$$

Testing (1.3) by $\mathbf{v}^\lambda$ we obtain using (5.6) the estimate

$$\|\overline{\mathbf{v}}^\lambda\|_{L^\infty(0,T;W^{1,2})} + \|\overline{\mathbf{v}}^\lambda\|_{L^\infty(0,T;L^6)} \le C(\mathbf{v}_0, \mathbf{f}). \tag{5.7}$$

From (5.6) and (5.7) we obtain that

$$\|\mathbf{v}^\lambda \overline{\mathbf{v}}^\lambda\|_{L^{5/2}(Q)} \le C(\mathbf{v}_0, \mathbf{f}),$$

and consequently

$$\|\mathbf{v}^\lambda_{,t}\|_{(L^2(0,T;W^{1,2}_{0,\mathrm{div}}))^*} \le C(\mathbf{v}_0, \mathbf{f}). \tag{5.8}$$

Using (5.6) and (5.8) it is standard to find a subsequence $\{\lambda_j\}$ and $\mathbf{v}$ such that (5.1)-(5.3) and (5.4) hold. The equation (5.4) is obtained letting $\lambda^j \to 1-$ in (1.11). The boundary terms disappear since the test functions vanish on the boundary and the term with pressure is not present because the test functions are divergence free.

Now we show that the trace of $\mathbf{v}$ is zero. It follows from (5.6) since

$$\int_0^T \|\mathbf{v}^\lambda\|^2_{2,\partial\Omega} \le C \frac{1-\lambda}{\lambda} \to 0 \quad \text{as } \lambda \to 0+.$$

Last, we need that $\mathbf{v}(0) = \mathbf{v}_0$. This follows from the initial condition for $\mathbf{v}^{\lambda_j}(0) = \mathbf{v}(0)$ since $\mathbf{v}, \mathbf{v}^{\lambda_j} \in C_{weak}(0,T;L^2_{\mathbf{n},\mathrm{div}})$. (The last statement follows from the fact that $\mathbf{v}, \mathbf{v}^{\lambda_j} \in C(0,T;(W^{1,5/2}_{\mathbf{n},\mathrm{div}})^*) \cap L^\infty(0,T;L^2) \hookrightarrow C_{weak}(0,T;L^2_{\mathbf{n},\mathrm{div}}))$.

In the situation where $\mathbf{f} \in (L^2(0,T;W^{1,2}_{0,\mathrm{div}}))^*$ only it is not known how to construct pressure as a function $p \in L^2((0,T)\times\Omega)$, compare [28, Section IV.2.6]. A different situation occurs if $\mathbf{f} \in L^q(Q)$, $q \geq 2$. Then the regularity (5.5) of the solution $\mathbf{v}$ can be shown as in Theorem 1.2 since Lemmas 2.1 and 2.3 hold also under homogeneous Dirichlet boundary conditions, compare [5], [19]. Having (5.5) the pressure can be reconstructed on a.e. time level by de Rham's theorem and its regularity can be read from the equation. □

6. Passage to the limit as $\lambda \to 1-$ and $\alpha \to 0+$

When $\lambda \to 1-$ and $\alpha \to 0+$ a theorem similar to Theorem 5.1 can be proved.

Theorem 6.1. *Let $\lambda_j \to 1-$, $\alpha_j \to 0+$, $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$, $\mathbf{f} \in L^2(0,T;W^{-1,2}_{\mathbf{n}})$. Let $\mathbf{v}^{\lambda_j,\alpha_j}$ be the unique solution to $\mathcal{L}(\alpha)$ with (1.4)-(1.6), $\lambda_j = \lambda$ and $\alpha = \alpha_j$. Then there is a subsequence of $\{\lambda_j,\alpha_j\}$, which we denote again by $\{\lambda_j,\alpha_j\}$, $\mathbf{v} \in C_{weak}(0,T;L^2_{\mathbf{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}})$, with $\mathbf{v}_t \in (L^2(0,T;W^{1,3}_{0,\mathrm{div}}))^*$ and $\mathbf{v}(0) = \mathbf{v}_0$ such that as $j \to +\infty$*

$$\mathbf{v}^{\lambda_j,\alpha_j} \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T;W^{1,2}), \tag{6.1}$$

$$\mathbf{v}^{\lambda_j,\alpha_j}_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } (L^2(0,T;W^{1,3}_{0,\mathrm{div}}))^*, \tag{6.2}$$

$$\mathbf{v}^{\lambda_j,\alpha_j} \to \mathbf{v} \quad \text{strongly in } L^q(0,T;L^q),\ \text{for all } 1 \leq q < 10/3 \tag{6.3}$$

Consequently, the velocity part $\mathbf{v}$ is a weak dissipative solution to the Navier Stokes equations with homogeneous Dirichlet boundary condition and the initial condition $\mathbf{v}_0$, i.e.

$$\int_0^T \langle \mathbf{v}_{,t},\mathbf{w}\rangle - (\mathbf{v}\otimes\mathbf{v},\nabla\mathbf{w}) + 2\nu(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{w}))\,dt = \int_0^T \langle \mathbf{f},\mathbf{w}\rangle\,dt \tag{6.4}$$

for all $\mathbf{w} \in L^2(0,T;W^{1,3}_{0,\mathrm{div}})$.

Proof. The proof of this theorem follows the lines of the proof of Theorem 4.2 and Theorem 5.1. First we obtain uniform estimates (4.11) and (4.12). Now we need to reconstruct a uniform estimate for $\overline{\mathbf{v}}^{\lambda_j,\alpha_j}$. Since in Lemma 2.1 the dependence of constants on λ is not addressed we cannot use it. Instead we test (1.3) with $\overline{\mathbf{v}}^{\lambda_j,\alpha_j}$ and get a uniform estimate

$$\|\overline{\mathbf{v}}^{\lambda_j,\alpha_j}\|_{L^\infty(0,T;L^2)} < C. \tag{6.5}$$

It follows that

$$\big\||\mathbf{v}^{\lambda_j,\alpha_j}||\overline{\mathbf{v}}^{\lambda_j,\alpha_j}|\big\|_{L^2(0,T;L^{\frac{3}{2}})} < C \quad \text{and} \quad \|\mathbf{v}^{\lambda_j,\alpha_j}_{,t}\|_{(L^2(0,T;W^{1,3}_{0,\mathrm{div}}))^*} < C.$$

Consequently we can extract a subsequence (λ_j,α_j) such that (6.1), (6.2) and by the Aubin-Lions lemma also (6.3) hold. Combining Lemma 4.1 with the estimate (4.11) we obtain that $\overline{\mathbf{v}}^{\lambda_j,\alpha_j} \to \mathbf{v}$ in $L^2(Q)$ and by (6.5) also in $L^s(0,T;L^2)$ for all $s > 2$ as $j \to +\infty$. The limit function $\mathbf{v}$ must be traceless due to (4.11). With this information it is standard to pass to the limit as $j \to +\infty$ in (1.11) to get (6.4). □

Remark 6.2. Generally, with homogeneous Dirichlet boundary condition, the existence and regularity of the pressure term p of the Navier-Stokes equations is not obvious, compare [9, 26, 27].

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