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FINITE TIME EXTINCTION FOR NONLINEAR FRACTIONAL EVOLUTION EQUATIONS AND RELATED PROPERTIES

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Abstract. The finite time extinction phenomenon (the solution reaches an equilibrium after a finite time) is peculiar to certain nonlinear problems whose solutions exhibit an asymptotic behavior entirely different from the typical behavior of solutions associated to linear problems. The main goal of this work is twofold. Firstly, we extend some of the results known in the literature to the case in which the ordinary time derivative is considered jointly with a fractional time differentiation. Secondly, we consider the limit case when only the fractional derivative remains. The latter is the most extraordinary case, since we prove that the finite time extinction phenomenon still appears, even with a non-smooth profile near the extinction time.

Some concrete examples of quasi-linear partial differential operators are proposed. Our results can also be applied in the framework of suitable nonlinear Volterra integro-differential equations.

1. Introduction

The aim of this work is to extend some of the results already known in the literature on the finite time extinction phenomenon [2] to the case in which the ordinary time derivative is considered jointly with a real order differential operator in time. To fix ideas, let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a general open set, let $Q_\infty = \Omega \times (0,+\infty)$, $\Sigma_\infty = \partial\Omega \times (0,+\infty)$, and consider a fractional evolution initial and boundary-value problem formulated as follows:

$$\begin{gathered} a_1 \frac{\partial}{\partial t} u + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} u + Au = f(x,t) \quad \text{in } Q_\infty, \\ Bu = g(x,t) \quad \text{on } \Sigma_\infty, \\ u(x,0) = u_0(x) \quad \text{on } \Omega. \end{gathered} \tag{1.1}$$

Here, $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and the operator $\partial^\alpha/\partial t^\alpha$ is a real order partial derivative called *fractional derivative* in time; it coincides with the classical derivative for $\alpha = 1$ and it is a non local in time (with delay) functional when $\alpha \in (0,1)$. Among the different definitions of real order differential operators given in the literature (see e.g. [20, 25, 27, 29]), we use the so called Riemann-Liouville fractional

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derivative:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = (\frac{\partial}{\partial t}I_t^{1-\alpha}u)(x,t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha}d\tau, \tag{1.2}$$

where $t > 0$ and $(I_t^{1-\alpha}u)(x,t)$ is the Riemann-Liouville fractional integral of order $(1-\alpha)$. A sufficient condition under which (1.2) exists (for a scalar function $u(t)$) is $u \in C^0([0,T]:\mathbb{R})$ and $u' \in L^1(0,T:\mathbb{R})$.

The term Au denotes a nonlinear operator (usually in terms of u and the partial differentials of u), Bu denotes a boundary operator and the data f, g and u_0 are given functions. For simplicity, we are assuming that A and B are *autonomous operators*, i.e., with time independent coefficients; nevertheless our treatment will allow the case of systems of equations (where $\mathbf{u}(x,t) \in \mathbb{R}^m$ with $m > 1$).

The most recurrent approach (see, e.g., [10] and [13] and the references therein) is to study a possible stabilization as $t \to \infty$ of a solution of this problem to a time-independent state, as it it turns out to be of significant interest.

In the actual fact, the stationary solution to many other nonlocal evolution equations with different nonlinearities have been derived and studied in the literature; see, e.g. the study of the non local evolution equation arising in population dispersal in [3].

In this context it is usually assumed that:

$$f(x,t) \to f_\infty(x) \quad \text{and} \quad g(x,t) \to g_\infty(x) \quad \text{as } t \to +\infty, \tag{1.3}$$

in some functional spaces and the main task is to prove that

$$u(x,t) \to u_\infty(x) \quad \text{as } t \to +\infty, \tag{1.4}$$

in some topology of a suitable functional space, with $u_\infty(x)$ solution of the stationary problem

$$\begin{gathered} Au_\infty = f_\infty(x) \quad \text{in } \Omega, \\ Bu_\infty = g_\infty(x) \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Here we are interested in a stronger property. Starting by assuming that $A0 = 0$, $B0 = 0$ and

$$\begin{gathered} f(x,t) = 0 \quad \forall t \geq T_f, \\ g(x,t) = 0 \quad \forall t \geq T_g, \end{gathered} \tag{1.6}$$

for some $T_f < \infty$ and $T_g < \infty$, we arrive to the following natural phenomenon of the *extinction in finite time:*

Definition 1.1. Let u be a solution of the evolution boundary value problem (1.1). We will say that $u(x,t)$ possesses the property of extinction in a finite time if there exists $t^* < \infty$ such that

$$u(x,t) \equiv 0 \quad \text{on } \Omega,\ \forall t \geq t^*. \tag{1.7}$$

Concretely, we will first prove the occurrence of the extinction in finite time for (1.1) with $a_1 > 0$ and $a_\alpha > 0$. Then, we will pass to consider the limit problem obtained when $a_1 = 0$ and $a_\alpha > 0$. The latter is the most extraordinary case, since we prove that the finite time extinction phenomenon still appears, even with a non-smooth profile near the extinction time.

The technique we will employ to derive (1.7) is an energy method [2, 15, 16], whose main idea consists in deriving and studying suitable ordinary differential inequalities for various types of energy.

The plan of our paper is as follows. In the next section we introduce the concrete model we will take under study. In Section 3 we prepare some material needed to prove our main result included in Section 4, where the existence of the finite time extinction phenomenon is demonstrated. Finally, we propose some other problems to which our results on the finite time extinction can be applied. This paper contain the details of a previous presentation by the authors in [17].

2. Model problem

Let us consider the following family of general problems:

$$
\begin{gathered}
a_1 \frac{\partial}{\partial t} u + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda\beta(u) = f \quad \text{in } Q_\infty \\
u = 0 \quad \text{on } \Sigma_\infty \\
u(x,0) = u_0(x) \quad \text{in } \Omega.
\end{gathered}
\tag{2.1}
$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and $\lambda > 0$, $p > 1$. In actual fact, if $a_1 = 0$ the initial condition must be understood as follows (see e.g. [20, Sec.3.2.2]):

$$
\lim_{t\to 0} \Gamma(\alpha) t^{1-\alpha} u(x,t) = \lim_{t\to 0} (I_t^{1-\alpha} u)(x,t) = u_0(x).
$$

Here, $\beta(u)$ is the equivalent of the "feedback" term in the control theory.

There is a wide literature concerning problems like (2.1), also due to their relevance in applications. Actually, Volterra integro-differential equations of convolution type with completely monotone kernel arise naturally in several fields, as in the theory of thermo-viscoelasticity, in the heat conduction in materials with memory [24, 11, 23] or in the study of the nonlinear reaction-diffusion equation with absorption: see, for instance, the monograph of Prüss [28] and the references therein.

Under suitable conditions, we shall prove that the solution to (2.1) satisfies an integral energy inequality leading to its extinction in a finite time.

3. Preliminaries

In this section, we present some lemmas establishing certain inequalities valid for the Riemann-Liouville fractional time derivative, which will be used hereinafter.

Lemma 3.1. *Let $\alpha \in (0,1)$ and $u \in C^0([0,T] : \mathbb{R})$, $u' \in L^1(0,T : \mathbb{R})$ and u monotone. Then*

$$
2u(t)\frac{d^\alpha u}{dt^\alpha}(t) \geq \frac{d^\alpha u^2}{dt^\alpha}(t), \quad \text{a.e. } t \in (0,T]. \tag{3.1}
$$

Proof. Let us write the following equalities:

$$
\begin{aligned}
\frac{d^\alpha u^2}{dt^\alpha}(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(u^2)'(\tau)}{(t-\tau)^\alpha}\, d\tau + \frac{u^2(0)t^{-\alpha}}{\Gamma(1-\alpha)} \\
&= u^2(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} \int_\tau^t (u^2)'(\xi) d\xi d\tau;
\end{aligned}
$$

$$
\begin{aligned}
u(t)\frac{d^\alpha u}{dt^\alpha}(t) &= \Big[\frac{u(t)}{\Gamma(1-\alpha)} \int_0^t \frac{(u)'(\tau)}{(t-\tau)^\alpha}\, d\tau + \frac{u(t)u(0)t^{-\alpha}}{\Gamma(1-\alpha)}\Big] \\
&= u^2(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{u(t)}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} \int_\tau^t u'(\xi) d\xi d\tau.
\end{aligned}
$$

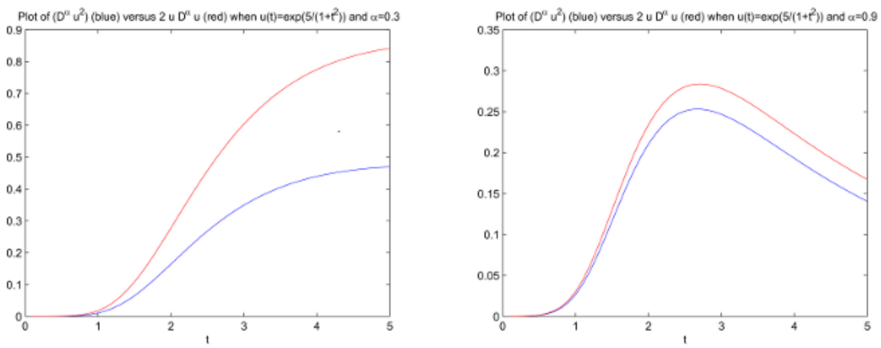

FIGURE 1. Function $\frac{d^\alpha u^2}{dt^\alpha}(t)$ vs. $2u(t)\frac{d^\alpha u}{dt^\alpha}(t)$ for $t \in (0,5]$, $u(t) = e^{5/(1+t^2)}$ and different values of α.

Now, from $u(\xi)u'(\xi) \le u(t)u'(\xi)$, a.e. $\xi \in (0,t)$, we obtain

$$\begin{aligned} \frac{d^\alpha u^2}{dt^\alpha}(t) &= u^2(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t 2u(\xi)u'(\xi)d\xi d\tau \\ &\le 2u^2(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{2u(t)}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t u'(\xi)d\xi d\tau \\ &= 2u(t)\frac{d^\alpha u}{dt^\alpha}(t) \quad \text{a.e. } t \in (0,T). \end{aligned}$$

□

Remark 3.2. We notice that the inequality (3.1) can be trivially checked if $\alpha = 1$.

Remark 3.3. We point out that (3.1), together with the initial condition $u(0) = 0$, allows to conclude the monotonicity (or accretiveness) of the fractional differential operator in a very direct way. This is in agreement with the result stated by Stankovic and Atanackovic [31]. As well, it has to be highlighted that more sophisticated proofs of said accretiveness had already been provided in the literature. At this aim, see, for instance, the studies carried out in [8, 19] (also in [9, 11, 12]) on the linear Volterra operator

$$(Lu)(t) = \frac{d}{dt}\Big[k_0 u(t) + \int_0^t k_1(t-s)u(s)ds\Big], \quad t > 0$$

in certain function spaces, where $k_0 \ge 0$ and $k_1 \in L^1_{\rm loc}(\mathbb{R}_+)$ is nonnegative and nonincreasing. It has been shown that the operator L is m-accretive in $L^p(0,T;X)$ and in $L^p(\mathbb{R}_+;X)$, for any $1 \le p < \infty$ and where X denotes any Banach space.

Conjecture 3.4. *Inequality* (3.1) *still holds under weaker hypothesis on u. Some answers can be found in* [1].

In fact, we considered a huge number of examples in the class of non-monotone functions with just the regularity properties as in Lemma 3.1, and we observed numerically that the validity of (3.1) is preserved. Figures 1–3 illustrate the results of numerical simulations for some concrete functions (monotone and not). The analytical approach is under study.

Now, we shall provide a more general version of Lemma 3.1:

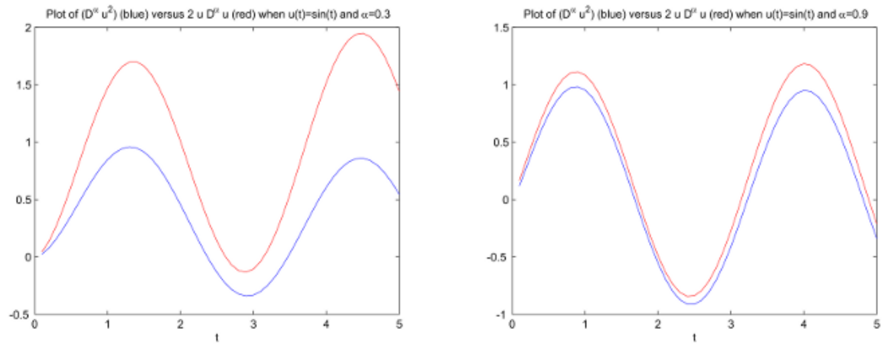

FIGURE 2. Function $\frac{d^\alpha u^2}{dt^\alpha}(t)$ vs. $2u(t)\frac{d^\alpha u}{dt^\alpha}(t)$ for $t \in (0,5]$, $u(t) = \sin(t)$ and different values of α.

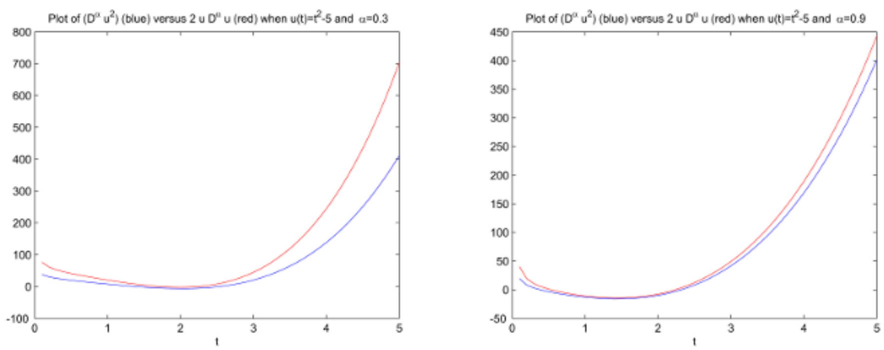

FIGURE 3. Function $\frac{d^\alpha u^2}{dt^\alpha}(t)$ vs. $2u(t)\frac{d^\alpha u}{dt^\alpha}(t)$ for $t \in (0,5]$, $u(t) = (t-3)^2$ and different values of α.

Lemma 3.5. *Given the Hilbert space H, let $\alpha \in (0,1)$ and $u \in L^\infty(0,T:H)$ such that $\frac{d^\alpha}{dt^\alpha}u \in L^1(0,T:H)$. Assume that $\|u(\cdot)\|_H$ is non-increasing (i.e. $\|u(t_2)\|_H \leq \|u(t_1)\|_H$ for a.e. $t_1, t_2 \in (0,T)$ such that $t_1 \leq t_2$). Then, there exists $k(\alpha) > 0$ such that for almost every $t \in (0,T)$ we have that*

$$\Big(u(t), \frac{d^\alpha}{dt^\alpha}u(t)\Big) \geq k(\alpha)\big(\frac{d^\alpha}{dt^\alpha}\|u(t)\|_H^2\big). \tag{3.2}$$

Proof. We shall only give the details for $H = \mathbb{R}$, since the general case can be deduced from this one through easy generalizations of the arguments here employed. Moreover, we shall refer to Lemma 3.1, which proves (3.2) for $k(\alpha) = 1/2$ and u satisfying some additional regularity hypothesis, to come to our conclusion.

Indeed, let us suppose we are given a non-increasing u satisfying $u \in L^\infty(0,T:\mathbb{R})$ such that $\frac{d^\alpha}{dt^\alpha}u \in L^1(0,T:\mathbb{R})$. We always can construct a sequence of functions $u_n \in C^\infty([0,T]:\mathbb{R})$ as follows:

$$u_n(t) = \int_{\mathbb{R}} \rho_n(t-s)\overline{u}(s)ds,$$

where $\overline{u}$ denotes the extension by zero of u:

$$\overline{u}(t) = \begin{cases} u(t) & \text{if } t \in [0,T] \\ 0 & \text{if } t \in \mathbb{R} \setminus [0,T] \end{cases} \tag{3.3}$$

and $(\rho_n)_{n\geq 1}$ is a regularizing sequence called "mollifiers" (see, e.g., [7, Ch.IV]):

$$\rho_n \in C_c^\infty(\mathbb{R}), \quad \text{Supp } \rho_n \subset \mathbf{B}(0, \frac{1}{n}),$$

$$\int \rho_n = 1, \quad \rho_n \geq 0 \text{ in } \mathbb{R}.$$

Then, it is well known that, in particular, $u_n \in C^0([0,T]:\mathbb{R})$, $u_n' \in L^1(0,T:\mathbb{R})$ and $u_n \to u$. Moreover, u non-increasing implies that the distributional derivative satisfies $u'(t) \leq 0$ in $\mathcal{D}'(0,T)$. Then, as $\rho_n \geq 0$, taking convolutions in $\mathcal{D}'(0,T)$, we obtain

$$u_n' = \rho_n * \overline{u}' \leq 0 \quad \text{in } \mathcal{D}'(0,T),$$

and, since we know that $u_n' \in L^1(0,T)$, we can conclude that $u_n'(t) \leq 0$ for a.e. $t \in (0,T)$. Moreover, it holds

$$\frac{d^\alpha}{dt^\alpha} u_n(t) = \int_{\mathbb{R}} \rho_n(s) \frac{d^\alpha}{dt^\alpha} \overline{u}(t-s) ds,$$

and, because of that, we can conclude that $\frac{d^\alpha}{dt^\alpha} u_n \in L^1(0,T)$ and

$$\frac{d^\alpha}{dt^\alpha} u_n \to \frac{d^\alpha}{dt^\alpha} u \text{ in } L^1(0,T) \quad \text{as } n \to +\infty.$$

Then, according to Lemma 3.1, for any n we have

$$u_n(t) \frac{d^\alpha}{dt^\alpha} u_n(t) \geq k(\alpha) (\frac{d^\alpha}{dt^\alpha} u_n(t)^2); \tag{3.4}$$

as a consequence, since

$$u_n \frac{d^\alpha}{dt^\alpha} u_n \to u \frac{d^\alpha}{dt^\alpha} u$$

strongly in $L^1(0,T)$, we obtain, from (3.4) that $\frac{d^\alpha}{dt^\alpha} u(\cdot)^2 \in L^1(0,T)$ and (by the dominated Lebesgue Theorem) that $\frac{d^\alpha}{dt^\alpha} u_n(\cdot)^2 \to \frac{d^\alpha}{dt^\alpha} u(\cdot)^2$ in $L^1(0,T)$. Passing to the limit in (3.4) (since $k(\alpha) = 1/2$ is independent of n) we obtain inequality (3.2). $\square$

Note that (3.2) implies $\frac{d^\alpha}{dt^\alpha} \|u(t)\|_H^2 \in L^1(0,T)$, which is not straightforward to see.

Remark 3.6. It worths to mention that several inequalities very close to (3.2) already existed in the literature. Here, we will include just two examples. The first one is the Shinbrot's inequality [30]:

$$\int_0^T \| \frac{d^{\alpha/2}}{dt^{\alpha/2}} u(t) \|_{L^2(V)}^2 dt \leq \sec \frac{\pi\alpha}{2} \int_0^T \big(\frac{d^\alpha}{dt^\alpha} u(t), u(t) \big) dt \tag{3.5}$$

holding under certain hypothesis on u and the domain V. The second one has been proved by means of different methods by many authors (see, e.g., [10]):

$$\int_0^t u(t) \frac{d^\alpha}{dt^\alpha} u(t) dt \geq \int_0^t |u(t)|^2 dt. \tag{3.6}$$

Note that the independence from α in the right term of (3.6) is the main reason why it could not serve to show the extinction in finite time for our problem.

4. Finite time extinction phenomenon

Now, let us present the main result of this paper:

Theorem 4.1. *Let $\beta(\cdot)$ be any nondecreasing continuous function such that $\beta(0) = 0$. Then, for any $f \in L^1_{\rm loc}(0, +\infty : L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists a weak solution of the problem (2.1). Assume also that $\beta(s) = |s|^{\sigma-1}s$ for some $\sigma > 0$ such that either $p < 2$ and $\lambda \geq 0$ and $\sigma > 0$ arbitrary, or $\sigma < 1$ and $p > 1$ arbitrary. Additionally, let $u_0 \in H^2(\Omega)$, $u_0 \in L^{2\sigma}(\Omega)$ and $f \in H^1_{\rm loc}(0, +\infty : L^2(\Omega))$ satisfying that $\exists t_f \geq 0$ such that $f(x,t) \equiv 0$ a.e. $x \in \Omega$ and a.e. $t > t_f$. Then, there exists $t_0 \geq t_f \geq 0$ such that $u(x,t) \equiv 0$ for a.e. $x \in \Omega$ and for any $t \geq t_0$.*

Proof. The existence of a weak solution $u \in C([0, +\infty) : L^2(\Omega))$ can be deduced from the abstract results on Volterra intregro-differential equations with accretive operators (see [4] and the long list of references herein included, which seems to be started in 1963 by Friedman [18]), if you take $k_0 = a_1$, $k_1(t) = a_\alpha/t^\alpha$, $G(u(t)) = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \beta(u)$ and $F(t, u(t)) = f(\cdot, t)$. The operator G is m-accretive (or, equivalently, maximal monotone) in $H = L^2(\Omega)$, as it is already well known in the literature (see, e.g., [15, Ch.IV]).

Now, let us start by considering the case $a_1 > 0$. If we define the energy function

$$y(t) := \int_\Omega u(x,t)^2 dx = \|u(\cdot,t)\|^2_{L^2(\Omega)}, \tag{4.1}$$

then, multiplying by u and integrating on Ω the equation appearing in (2.1), (as in [2, Sec.2, Ch.2]), we obtain, due to the Sobolev, Hölder and Young inequalities:

$$\frac{a_1}{2}\frac{dy}{dt} + a_\alpha \int_\Omega \frac{\partial^\alpha u}{\partial t^\alpha}(x,t)u(x,t)dx + Cy(t)^\nu \leq 0, \tag{4.2}$$

for some $C > 0$ and $\nu \in (0,1)$ (this is implied by the hypothesis on σ and p) and for a.e. $t \in (t_f, +\infty)$.

Also, we know [12, p.98] that the operator:

$$u \mapsto a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} \tag{4.3}$$

generates contraction semigroups in $E = L^r(0, +\infty : L^q(\Omega))$, with $1 < r$, $q < \infty$ which are positive with respect to the usual cone E^+ of positive functions.

In particular (4.3) generates a contraction semigroup in $L^2(\Omega)$. So, since $a_1 > 0$ we obtain that, for any $t \geq t_f$, the application $t \mapsto y(t)$ is non increasing, $y \in C([t_f, +\infty])$ and $\frac{d^\alpha y}{dt^\alpha} \in L^1(t_f, T)$.

Therefore, we are in conditions as to apply Lemma 3.5, and we obtain:

$$\begin{gathered} \frac{a_1}{2}\frac{d}{dt}y + \frac{a_\alpha}{2}\frac{d^\alpha}{dt^\alpha}y(t) + C\,y(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = Y_0. \end{gathered} \tag{4.4}$$

Moreover, since the semigroup generated by the operator (4.3) is positive [although it is non local], we have that

$$0 \leq y(t) \leq Y(t) \quad \text{for any } t \in [t_f, +\infty), \tag{4.5}$$

where $Y(t)$ is a *supersolution*, i.e, $Y(t)$ satisfies the inequality:

$$\begin{gathered}\frac{a_1}{2}\frac{d}{dt}Y+\frac{a_\alpha}{2}\frac{d^\alpha}{dt^\alpha}Y(t)+CY(t)^\nu\geq 0 \quad \text{on } (t_f,+\infty)\\ Y(t_f)\geq Y_0.\end{gathered}\tag{4.6}$$

Now, our conclusion comes from the fact that we can construct $Y(t)$ satisfying (4.6) and such that $Y(t)\equiv 0$ for all $t\geq t_Y$, for some $t_Y>t_f$. Indeed, let $Y(t)$ be a function satisfying

$$\begin{gathered}\frac{a_1}{2}Y'(t)+\frac{C}{2}Y(t)^\nu=0\\ Y(t_f)=Y_0,\end{gathered}\tag{4.7}$$

for instance,

$$Y(t)=k\,(t_Y-t)_+^{\frac{1}{1-\nu}}$$

for some $t_Y>t_f$ and some $k>0$.

Then, we may conclude that

$$\frac{a_1}{2}\frac{dY}{dt}+\frac{a_\alpha}{2}\frac{d^\alpha Y}{dt^\alpha}(t)+\frac{C}{2}Y(t)^\nu+\frac{C}{2}Y(t)^\nu\geq\frac{a_\alpha}{2}\frac{d^\alpha Y}{dt^\alpha}(t)+\frac{C}{2}Y(t)^\nu\geq 0$$

since

$$\frac{a_\alpha}{2}\frac{d^\alpha Y}{dt^\alpha}(t)\geq k^*(t_Y-t)_+^{\frac{1}{1-\nu}-\alpha},\tag{4.8}$$

holds with $k^*<0$, and it is

$$\frac{\nu}{1-\nu}\leq\frac{(1-\alpha)+\alpha\nu}{(1-\nu)},$$

where $\nu\in(0,1)$.

Let us prove (4.8) when $t\leq t_Y$, since it is trivial when $t_Y<t$. To do that, we have to write again the Riemann-Liouville fractional derivative in this equivalent form

$$\begin{aligned}&\frac{d^\alpha}{dt^\alpha}(t_Y-t)_+^{\frac{1}{1-\nu}}\\ &=-\frac{1/(1-\nu)}{\Gamma(1-\alpha)}\int_0^t\frac{(t_Y-\tau)_+^{\frac{\nu}{1-\nu}}}{(t-\tau)^\alpha}\,d\tau+\frac{t_Y\,t^{-\alpha}}{\Gamma(1-\alpha)}\\ &\geq-\frac{1/(1-\nu)}{\Gamma(1-\alpha)}\int_0^t\frac{(t_Y-\tau)_+^{\frac{\nu}{1-\nu}}}{(t-\tau)^\alpha}\,d\tau\\ &\geq\frac{2^\alpha/(1-\nu)}{\Gamma(1-\alpha)}\int_0^t\frac{(t_Y-\tau)_+^{\frac{\nu}{1-\nu}}}{(t_Y-\tau)^\alpha}\,d\tau\\ &=-\frac{2^\alpha/(1-\nu)}{\Gamma(1-\alpha)(\frac{1}{1-\nu}-\alpha)}(t_Y-t)_+^{\frac{1}{1-\nu}-\alpha}+\frac{2^\alpha/(1-\nu)}{\Gamma(1-\alpha)(\frac{1}{1-\nu}-\alpha)}t_Y^{\frac{1}{1-\nu}-\alpha}\\ &\geq-\frac{2^\alpha/(1-\nu)}{\Gamma(1-\alpha)(\frac{1}{1-\nu}-\alpha)}(t_Y-t)_+^{\frac{1}{1-\nu}-\alpha}=k^*(t_Y-t)_+^{\frac{1}{1-\nu}-\alpha},\end{aligned}$$

where we used the inequality

$$(t-\tau^\alpha)=[(t_Y-\tau)-(t_Y-t)]^\alpha\geq-2^\alpha(t_Y-\tau)^\alpha,$$

and that $\frac{1}{1-\nu}>1$. So, we have shown that when $a_1>0$,

$$\|u(\cdot,t)\|^2_{L^2(\Omega)}\leq k\,(t_Y-t)_+^{\frac{1}{1-\nu}}\quad\forall t\geq t_f\,,\ \nu\in(0,1).\tag{4.9}$$

Now let us pass to consider the limit case $a_1 = 0$. At this aim, let u_ε be the solution of (2.1) with $a_1 = \varepsilon > 0$. Then, as we can prove that

$$u_\varepsilon \to u^* \quad \text{in } L^2(0, +\infty : L^2(\Omega)) \quad \text{as } \varepsilon \to 0,$$

we obtain that the mapping

$$t \mapsto y^*(t) := \|u^*(\cdot, t)\|^2_{L^2(\Omega)}$$

is also decreasing. Then, we can apply Lemma 3.1 and write for y^* the same inequality as in (4.4):

$$\begin{gathered} \frac{a_\alpha}{2} \frac{d^\alpha}{dt^\alpha} y^*(t) + C\, y^*(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = W_0. \end{gathered} \tag{4.10}$$

As before, the conclusion comes now from the fact that we can construct a supersolution $W(t)$ satisfying

$$\begin{gathered} \frac{a_\alpha}{2} \frac{d^\alpha}{dt^\alpha} W(t) + CW(t)^\nu \geq 0 \quad \text{on } (t_f, +\infty) \\ W(t_f) = W_0, \end{gathered} \tag{4.11}$$

and such that $W(t) \equiv 0$ for all $t \geq t_W$, for some $t_W > t_F$.

Indeed, let $W(t) = h\,(t_W - t)_+^{\frac{\alpha}{1-\nu}}$ for some $t_W > t_f$ and some $h > 0$. Then, as before,

$$\frac{a_\alpha}{2} \frac{d^\alpha W}{dt^\alpha}(t) \geq h^*(t_W - t)_+^{\frac{\alpha\nu}{1-\nu}},$$

with $h^* < 0$, and

$$\|u^*(\cdot, t)\|^2_{L^2(\Omega)} \leq h\,(t_W - t)_+^{\frac{\alpha}{1-\nu}} \quad \forall t \geq t_f. \tag{4.12}$$

□

Remark 4.2. The decreasing behavior of the norm appearing in (4.9) for the case $a_1 > 0$ is actually the same as when the fractional derivative is not included in the problem (2.1). However, what is more extraordinary is the decreasing behavior of the norm (4.12) when $a_1 = 0$ as, if $\alpha < 1 - \nu$, we are dealing with a function $W(t)$ such that $\frac{d^\alpha W}{dt^\alpha}(t) \in L^\infty(0, +\infty)$ whereas $W'(t) \notin L^\infty(0, +\infty)$ although $W'(t) \in L^1(0, +\infty)$.

It should be highlighted that, even if in the literature [20, 27, 29] lots of calculations refer to the exact expression of the fractional derivative of polynomial functions, as far as we know, none of them leads to inequality (4.8) in the form as we understand.

5. Other applications

5.1. **Nonlinear heat equation with absorption for porous media.** We consider the model initial-boundary value problem for a nonlinear degenerate parabolic equation with a single space variable [2]. Denote $Q_T = \Omega \times (0, T)$, $\Omega = (-L, L)$, $T \in R_+$. Let $u(x, t)$ be a solution of the problem

$$\begin{gathered} a_1 \frac{\partial}{\partial t}\left(u|u|^{\gamma-1}\right) + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(u|u|^{\gamma-1}\right) - \left(|u_x|^{p-2}u_x|\right)_x + \lambda u|u|^{\sigma-1} = f \quad \text{in } Q_T \\ u(\pm L, t) = 0 \quad t \in (0, T); \\ u(x, 0) = u_0(x) \quad \text{in } \Omega. \end{gathered} \tag{5.1}$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$, $\lambda > 0$, $0 < \gamma < \infty$, $1 \leq p < \infty$ and $\sigma > 0$.

Notice that the equation

$$a_1 \frac{\partial}{\partial t} v + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} v - (\gamma^{1-p}|v|^{m-1}|v_x|^{p-2}v_x)_x + \lambda v|v|^{q-1} = f(x,t) \tag{5.2}$$

with the parameters

$$m = 1 + \frac{(1-\gamma)(p-1)}{\gamma}, \quad q = \frac{\sigma}{\gamma}$$

can be transformed into (5.1) after the change of the unknown function $v = u|u|^{\gamma-1}$. Equation (5.2) with $\alpha = 1$ is usually referred to as the *nonlinear heat equation with absorption*.

The existence of a weak solution $v \in C([0,+\infty) : L^{1+\gamma}(\Omega))$ to (5.2) can be deduced for any $f \in H^1_{\rm loc}(0,+\infty : L^{1+\gamma}(\Omega))$ and $v_0 \in L^{1+\gamma}(\Omega)$, as in Theorem 4.1 from the abstract results on Volterra intregro-differential equations with accretive operators (see [4]), taking $k_0 = a_1$, $k_1(t) = a_\alpha/t^\alpha$, $G(v(t)) = -(\gamma^{1-p}|v|^{m-1}|v_x|^{p-2}v_x)_x + \lambda v|v|^{q-1}$ and $F(t, v(t)) = f(\cdot, t)$. This operator G, as the usual p-Laplacian is also m-accretive (or, equivalently, maximal monotone) in $H = L^1(\Omega)$ (see, e.g., [15, Ch.IV]).

Let us also assume that the solution $u(x,t)$ of problem (5.1) is a weak solution from a suitable functional space, $V(Q_T)$, such that for almost all $t \in (0,T)$ the following energy equality, obtained multiplying by u and integrating on Ω the equation included in (5.1), holds:

$$\begin{aligned} &a_1 \frac{\gamma}{\gamma+1} \frac{d}{dt} \int_\Omega |u|^{\gamma+1} dx + a_\alpha \int_\Omega u \frac{\partial^\alpha}{\partial t^\alpha} \left(u|u|^{\gamma-1}\right) dx \\ &+ \int_\Omega \left(|u_x|^p + \lambda |u|^{1+\sigma} - fu\right) dx = 0. \end{aligned} \tag{5.3}$$

Now, let us introduce the energy functions

$$\begin{gathered} y(t) = \int_\Omega |u(x,t)^{1+\gamma}| dx = \|u(\cdot,t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)}, \\ D(t) = \int_\Omega |u_x(x,t)^p| dx = \|u(\cdot,t)\|^p_{L^p(\Omega)}, \\ A(t) = \int_\Omega |u(x,t)^{1+\sigma}| dx = \|u(\cdot,t)\|^{1+\sigma}_{L^{1+\sigma}(\Omega)} \end{gathered}$$

which, for any given function $u \in V(Q_T)$, are defined for almost all $t \in (0,T)$ and are in $L^1(0,T)$.

With this notation, the energy equality (5.3) takes the form

$$a_1 \frac{\gamma}{\gamma+1} \frac{d}{dt} y + a_\alpha \int_\Omega u \frac{\partial^\alpha}{\partial t^\alpha} \left(u|u|^{\gamma-1}\right) dx + D(t) + \lambda A(t) = \int_\Omega fu\, dx \tag{5.4}$$

and in [2, pp 72-73] it is shown how to pass, when $\alpha = 1$ and $f(x,t) \equiv 0$, from this to the following ordinary differential inequality:

$$a_1 y' + C y^\nu \leq 0, \tag{5.5}$$

where $C > 0$ and $0 < \nu < 1$.

Also, we recall that from [12, p.98] the operator

$$u \mapsto a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha}$$

generates a contraction semigroup in $L^{1+\gamma}(\Omega)$. So, since $a_1 > 0$ we obtain that, for any $t \geq t_f$ (t_f such that $f(x,t) \equiv 0$ for all $t \geq t_f$), the mapping $t \mapsto y(t)$ is non increasing, $y \in C([t_f, +\infty])$ and $\frac{d^\alpha y}{dt^\alpha} \in L^1(t_f, T)$.

Therefore, to obtain a fractional ordinary differential inequality for the general case of $\alpha \in (0,1)$, we just need a slightly different version of previous lemmas 3.1 and 3.5.

Lemma 5.1. *Let $\alpha \in (0,1)$ and $u \in C^0([0,T] : \mathbb{R})$, $u' \in L^1(0,T : \mathbb{R})$ with u monotone. Then*

$$\Big(\frac{\gamma+1}{\gamma}\Big)u(t)\frac{d^\alpha u|u|^{\gamma-1}}{dt^\alpha}(t) \geq \frac{d^\alpha |u|^{\gamma+1}}{dt^\alpha}(t), \quad a.e.\ t \in (0,T]. \tag{5.6}$$

Proof. Note that the following equalities hold:

$$\begin{aligned}
\frac{d^\alpha |u|^{\gamma+1}}{dt^\alpha}(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{(|u|^{\gamma+1})'(\tau)}{(t-\tau)^\alpha}\,d\tau + \frac{|u|^{\gamma+1}(0)t^{-\alpha}}{\Gamma(1-\alpha)} \\
&= |u|^{\gamma+1}(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t (|u|^{\gamma+1})'(\xi)d\xi d\tau;
\end{aligned}$$

$$\begin{aligned}
u(t)\frac{d^\alpha u|u|^{\gamma-1}}{dt^\alpha}(t) &= \Big[\frac{u(t)}{\Gamma(1-\alpha)}\int_0^t \frac{(u|u|^{\gamma-1})'(\tau)}{(t-\tau)^\alpha}\,d\tau + \frac{u(t)u(0)|u|^{\gamma-1}(0)t^{-\alpha}}{\Gamma(1-\alpha)}\Big] \\
&= |u|^{\gamma+1}(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{u(t)}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t (u|u|^{\gamma-1})'(\xi)d\xi d\tau.
\end{aligned}$$

Now, from $|u(\xi)|^{\gamma-1}u(\xi)u'(\xi) \leq |u(\xi)|^{\gamma-1}u(t)u'(\xi)$, a.e. $\xi \in (0,t)$, we obtain

$$\begin{aligned}
&\frac{d^\alpha |u|^{\gamma+1}}{dt^\alpha}(t) \\
&= |u|^{\gamma+1}(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{\gamma+1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t |u(\xi)|^{\gamma-1}u((\xi))u'(\xi)d\xi d\tau \\
&\leq \frac{\gamma+1}{\gamma}\Big[|u|^{\gamma+1}(t)\frac{d^\alpha 1}{dt^\alpha} - \frac{\gamma u(t)}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\int_\tau^t |u(\xi)|^{\gamma-1}u'(\xi)d\xi d\tau\Big] \\
&= \frac{\gamma+1}{\gamma}u(t)\frac{d^\alpha u|u|^{\gamma-1}}{dt^\alpha}(t) \quad \text{a.e. } t \in (0,T).
\end{aligned}$$

∎

Then, gathering (5.4) and (5.5), by applying Lemma 5.1 we can write

$$a_1 y' + a_\alpha \frac{d^\alpha}{dt^\alpha}y + Cy^\nu \leq 0, \tag{5.7}$$

whenever $f(x,t) \equiv 0$. From this, it is implied that the weak solution vanishes in a finite time.

5.2. **Higher-order parabolic equations.** The non linear operator Au may contain derivatives of order higher than two. Let us consider, for example, the initial and boundary value problem

$$\begin{gathered}
a_1\frac{\partial}{\partial t}u + a_\alpha\frac{\partial^\alpha}{\partial t^\alpha}u + \Delta\left(|\Delta u|^{p-2}\Delta u\right) + \beta(u) = f \quad \text{in } Q_\infty \\
u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma_\infty \\
u(x,0) = u_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{5.8}$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$, $p \in (1,2)$ and ν is the unit normal outer vector to $\partial\Omega$.

In fact, the existence of a weak solution $u \in C([0,+\infty) : L^2(\Omega))$ can be proved as in Theorem 4.1, when $\beta(\cdot)$ is any nondecreasing continuous function such that $\beta(0) = 0$, for any $f \in L^1_{\rm loc}(0,+\infty : L^2(\Omega))$ and $u_0 \in L^2(\Omega)$.

Also, because of the embedding $W_0^{2,2}(\Omega) \subset L^2(\Omega)$, it can be written

$$-(\Delta\left(|\Delta u|^{p-2}\,\Delta u\right), u)_\Omega = -\|\Delta u\|^p_{L^p(\Omega)} \leq -C\|u\|^p_{L^2(\Omega)},$$

for some $C < 0$. Then assuming $\beta(s) = |s|^{\sigma-1}s$ for some $\sigma > 0$ and $f \in H^1_{\rm loc}(0,+\infty : L^2(\Omega))$ satisfying that $\exists t_f \geq 0$ such that $f(x,t) \equiv 0$ a.e. $x \in \Omega$ and a.e. $t > t_f$, it follows that the nonlinear differential equation

$$\frac{a_1}{2}\frac{dy}{dt} + \frac{a_\alpha}{2}\frac{d^\alpha y}{dt^\alpha}(t) + C^*\,y(t)^\rho \leq 0$$

holds, for some $C^* > 0$ and $\rho \in (0,1)$ (this is implied by the assumptions on σ and p) and for a.e. $t \in (t_f,+\infty)$. Therefore, the finite time extinction of weak solutions is provided.

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