Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 241, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF MILD SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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Abstract. In this article, we study the existence of piecewise-continuous mild solutions for the initial value problems for a class of semilinear evolution equations. These equations have non-instantaneous impulses in Banach spaces and the corresponding solution semigroup is noncompact. We assume that the nonlinear term satisfies certain local growth condition and a noncompactness measure condition. Also we assume the non-instantaneous impulsive functions satisfy some Lipschitz conditions. An example is given to illustrate our results.

1. Introduction

In this article, we study the existence of piecewise-continuous mild solutions (PC-mild solutions) for the initial value problem (IVP) of semi-linear evolution equations with non-instantaneous impulses in Banach space E,

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t)), \quad t \in \cup_{k=0}^{m}(s_k, t_{k+1}], \\ u(t) = \gamma_k(t, u(t)), \quad t \in \cup_{k=1}^{m}(t_k, s_k], \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where $A : \mathcal{D}(A) \subset E \to E$ is a closed linear operator, $-A$ is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) $T(t)(t \geq 0)$ in E, $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} := a$, $a > 0$ is a constant, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \dots, m$, $f : [0, a] \times E \to E$ is a given nonlinear function satisfying some assumptions, $\gamma_k : (t_k, s_k] \times E \to E$ is non-instantaneous impulsive function for all $k = 1, 2, \dots, m$, $u_0 \in E$.

The theory of instantaneous impulsive differential equations describes processes which experience a sudden change in their states at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemistry, biology, population and dynamics, engineering and economics, see the monographs by Benchohra et al [6], Lakshmikantham et al [17] and the papers of Guo [14], Li and Liu [19] for more comments and citations. Particularly, the theory of instantaneous impulsive evolution equations has become more important in resent years because of its wide applicability in control, mechanics, electrical engineering, biological and medical fields. The theory of instantaneous

2010 *Mathematics Subject Classification.* 34K45, 47D06.
Key words and phrases. Evolution equation; initial value problem; non-instantaneous impulse; mild solution; equicontinuous semigroup; existence of solutions.

Submitted March 7, 2016. Published September 1, 2016.

impulsive evolution equations in Banach spaces has been emerging as an important area of investigation in the last few decades. For more details on this theory and its applications, we refer to the the references [1, 3, 7, 9, 12, 20, 21], where numerous properties of their solutions are studied and detailed bibliographies are given.

The most important feature of instantaneous impulsive ordinary and partial differential equations is this class of equations is linked to their utility in simulating processes and phenomena subject to short time perturbations during their evolution, and the perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena when construct mathematical models. In short, these ordinary and partial differential equations with instantaneous impulses consider basically problems for which the impulses are abrupt and instantaneous. However, one can see that the models with instantaneous impulses could not explain the certain dynamics of evolution processes in pharmacotherapy. Just as pointed out by Hernández and O'Regan in [16], when we consider the simplified situation concerning the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Therefore, one can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval. We call such phenomenon non-instantaneous impulses during construct mathematical models. It is reported that many models arising from realistic models can be described as partial differential equations with non-instantaneous impulses.

In the past two years, nonlinear differential equations with non-instantaneous impulses have been studied by several authors and some interesting results have been obtained, see [2, 10, 13, 16, 23, 24, 25, 26]. In 2013, Hernández and O'Regan [16] firstly studied the initial value problem for a new class of abstract evolution equations with non-instantaneous impulses in Banach spaces. In the same year, Pierri et al [23] obtained the existence of mild solutions for a class of semi-linear abstract differential equations with non-instantaneous impulses by using the theory of analytic semigroup. Gautam and Dabas [13] studied the existence, uniqueness and continuous dependence results of mild solution for fractional functional integro-differential equations with non-instantaneous impulses by using the theory of analytic α-resolvent family and fixed point theorems. Colao et al [10] obtained the existence of solutions for a second-order differential equation with non-instantaneous impulses and delay on an unbounded interval by establish a compactness criterion in a certain class of functions. Yu and Wang [26] investigated the existence of solutions to periodic boundary value problems for nonlinear evolution equations with non-instantaneous impulses on Banach spaces by using the theory of semigroup and fixed point methods. Wang and Li [24] obtained the existence of solutions for periodic boundary value problem of nonlinear ordinary differential equations with non-instantaneous impulses. In addition, fractional ordinary and partial differential equations with non-instantaneous impulses have also been studied in [2, 25].

The motivation of this article is as follows: to the best of the authors knowledge, all the existing articles (see, for example [10, 13, 16, 23, 26]) used various fixed point theorems to study abstract evolution equations with non-instantaneous impulses when the corresponding semigroup $T(t)(t \geq 0)$ is compact, this is very convenient to the equations with compact resolvent. But for the case that the corresponding semigroup $T(t)(t \geq 0)$ is noncompact, we have not seen the relevant papers to study abstract evolution equations with non-instantaneous impulses. Therefore, inspired by the previous works, we study the existence of PC-mild solution for (1.1) under the assumption that the corresponding solution semigroup is noncompact. By using the properties of Kuratowski measure of noncompactness, k-set-contraction mapping fixed point theorem (see Lemma 2.5) and a new estimation technique for the measure of noncompactness (see Lemma 2.6), we obtained the existence of PC-mild solution for (1.1). Our work can be considered as a supplement for the case that the corresponding solution semigroup is compact.

In the following section we first introduce some notations and preliminary lemmas which will be used in this paper, at the same time the definition of PC-mild solution for (1.1) has been given. In section 3 we state and prove the existence of PC-mild solution for (1.1) under the appropriate assumptions. In the last paragraph we give an example to illustrate the feasibility of our abstract results.

2. Preliminaries

We begin by giving some notation. Let E be a Banach space with the norm $\|\cdot\|$. We use θ to present the zero element in E. For any constant $a > 0$, denote $J = [0, a]$. Let $C(J, E)$ be the Banach space of all continuous functions from J into E endowed with the supremum-norm $\|u\|_C = \sup_{t\in J} \|u(t)\|$ for every $u \in C(J, E)$. From the associate literature, we consider the following space of piecewise continuous functions,

$$PC(J,E) = \big\{u : J \to E : u \text{ is continuous for } t \neq t_k, \\ \text{left continuous at } t = t_k \text{ and } u(t_k^+) \text{ exists for } k = 1, 2, \dots, m\big\}.$$

It is easy to see that $PC(J, E)$ is a Banach space endowed with the PC-norm

$$\|u\|_{PC} = \max\Big\{\sup_{t\in J}\|u(t^+)\|,\ \sup_{t\in J}\|u(t^-)\|\Big\},\quad u \in PC(J,E),$$

where $u(t^+)$ and $u(t^-)$ represent respectively the right and left limits of $u(t)$ at $t \in J$.

For each finite constant $r > 0$, let

$$\Omega_r = \{u \in PC(J,E) : \|u(t)\| \le r,\ t \in J\},$$

then Ω_r is a bounded closed and convex set in $PC(J, E)$.

Let $\mathcal{L}(E)$ be the Banach space of all linear and bounded operators on E. Since the semigroup $T(t)(t \ge 0)$ generated by $-A$ is a C_0-semigroup in E, denote

$$M := \sup_{t\in J}\|T(t)\|_{\mathcal{L}(E)}, \tag{2.1}$$

then $M \ge 1$ is a finite number.

Definition 2.1. A C_0-semigroup $T(t)(t \ge 0)$ in E is said to be equicontinuous if $T(t)$ is continuous by operator norm for every $t > 0$.

Now we introduce some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.2 ([5, 11])**.** *The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set S of Banach space E is*

$$\alpha(S) := \inf\{\delta > 0 : S = \cup_{i=1}^m S_i \text{ with } \operatorname{diam}(S_i) \le \delta \text{ for } i = 1, 2, \dots, m\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.3 ([5, 11])**.** *Let E be a Banach space and S, $U \subset E$ be bounded. The following properties are satisfied:*

(i) $\alpha(S) = 0$ *if and only if* $\overline{S}$ *is compact, where* $\overline{S}$ *means the closure hull of* S*;*
(ii) $\alpha(S) = \alpha(\overline{S}) = \alpha(\operatorname{conv} S)$*, where* $\operatorname{conv} S$ *means the convex hull of* S*;*
(iii) $\alpha(\lambda S) = |\lambda|\alpha(S)$ *for any* $\lambda \in \mathbb{R}$*;*
(iv) $S \subset U$ *implies* $\alpha(S) \le \alpha(U)$*;*
(v) $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$*;*
(vi) $\alpha(S + U) \le \alpha(S) + \alpha(U)$*, where* $S + U = \{x \mid x = y + z, y \in S, z \in U\}$*;*
(vii) *If the map* $Q : \mathcal{D}(Q) \subset E \to X$ *is Lipschitz continuous with constant* k*, then* $\alpha(Q(V)) \le k\alpha(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$*, where* X *is another Banach space.*

In this article, we denote by $\alpha(\cdot)$, $\alpha_C(\cdot)$ and $\alpha_{PC}(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of E, $C(J, E)$ and $PC(J, E)$, respectively. For any $D \subset C(J, E)$ and $t \in J$, set $D(t) = \{u(t) \mid u \in D\}$ then $D(t) \subset E$. If $D \subset C(J, E)$ is bounded, then $D(t)$ is bounded in E and $\alpha(D(t)) \le \alpha_C(D)$. For more details about the properties of the Kuratowski measure of noncompactness, we refer to the monographs [5, 11].

Definition 2.4 ([11])**.** Let E be a Banach space, and let S be a nonempty subset of E. A continuous mapping $Q : S \to E$ is called to be k-set-contractive if there exists a constant $k \in [0, 1)$ such that, for every bounded set $\Omega \subset S$,

$$\alpha(Q(\Omega)) \le k\alpha(\Omega).$$

Lemma 2.5 ([11])**.** *Let E be a Banach space. Assume that $\Omega \subset E$ is a bounded closed and convex set on E, the operator $Q : \Omega \to \Omega$ is k-set-contractive. Then Q has at least one fixed point in Ω.*

Lemma 2.6 ([8, 18])**.** *Let E be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \le 2\alpha(D_0)$.*

Lemma 2.7 ([15])**.** *Let E be a Banach space, and let $D = \{u_n\} \subset PC([b_1, b_2], E)$ be a bounded and countable set for constants $-\infty < b_1 < b_2 < +\infty$. Then $\alpha(D(t))$ is Lebesgue integral on $[b_1, b_2]$, and*

$$\alpha\Big(\Big\{\int_{b_1}^{b_2} u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \le 2\int_{b_1}^{b_2} \alpha(D(t))dt.$$

Lemma 2.8 ([5])**.** *Let E be a Banach space, and let $D \subset C([b_1, b_2], E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $[b_1, b_2]$, and*

$$\alpha_C(D) = \max_{t\in[b_1,b_2]} \alpha(D(t)).$$

At last, we give the definition of mild solution for (1.1) according to the developments of Hernández and O'Regan [16].

Definition 2.9. A function $u \in PC(J, E)$ is called a mild solution of (1.1) if u satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(s, u(s))ds, \quad t \in [0, t_1];$$

$$u(t) = \gamma_k(t, u(t)), \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m;$$
$$u(t) = T(t - s_k)\gamma_k(s_k, u(s_k)) + \int_{s_k}^{t} T(t - s)f(s, u(s))ds,$$
$$\text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, m.$$

3. Main results

To obtain the existence of PC-mild solution for (1.1), we introduce the following hypotheses:

- (H1) The nonlinear function $f : J \times E \to E$ is continuous, for some $r > 0$, there exist a constant $\rho > 0$, Lebesgue integrable function $\varphi : J \to [0, +\infty)$ and a nondecreasing continuous function $\Psi : [0, +\infty) \to (0, +\infty)$ such that for all $t \in J$ and $u \in E$ satisfying $\|u\| \le r$,
$$\|f(t, u)\| \le \varphi(t)\Psi(\|u\|) \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Psi(r)}{r} = \rho < +\infty.$$
- (H2) The impulsive function $\gamma_k : [t_k, s_k] \times E \to E$ is continuous, and there exists a constant $K_{\gamma_k} > 0$, $k = 1, 2, \dots, m$, such that for all u, $v \in E$
$$\|\gamma_k(t, u) - \gamma_k(t, v)\| \le K_{\gamma_k}\|u - v\|, \quad \forall\, t \in (t_k, s_k].$$
- (H3) There exist positive constant L_k $(k = 0, 1, \dots, m)$ such that for any countable set $D \subset E$,
$$\alpha(f(t, D)) \le L_k\alpha(D), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, m.$$

For brevity of notation, we denote

$$K := \max_{k=1,2,\dots,m} K_{\gamma_k}, \quad \Lambda := \max_{k=0,1,\dots,m} \|\varphi\|_{L[s_k,t_{k+1}]}, \tag{3.1}$$
$$L := \max_{k=0,1,\dots,m} L_k(t_{k+1} - s_k). \tag{3.2}$$

Theorem 3.1. *Assume that the semigroup $T(t)(t \ge 0)$ generated by $-A$ is equicontinuous, the function $\gamma_k(\cdot, \theta)$ is bounded for $k = 1, 2, \dots, m$. If the conditions* (H1)–(H3) *are satisfied, then* (1.1) *has at least one PC-mild solution $u \in PC(J, E)$ provided that*

$$M \max\{\rho\Lambda + K, K + 4L\} < 1. \tag{3.3}$$

Proof. Define the operator $\mathcal{F}$ on $PC(J, E)$ by

$$(\mathcal{F}u)(t) = (\mathcal{F}_1u)(t) + (\mathcal{F}_2u)(t), \tag{3.4}$$

where

$$(\mathcal{F}_1u)(t) = \begin{cases} T(t)u_0 & t \in [0, t_1]; \\ \gamma_k(t, u(t)), & t \in (t_k, s_k],\ k = 1, 2, \dots, m; \\ T(t - s_k)\gamma_k(s_k, u(s_k)), & t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, m, \end{cases} \tag{3.5}$$

$$(\mathcal{F}_2u)(t) = \begin{cases} \int_{s_k}^{t} T(t - s)f(s, u(s))ds, & t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, m; \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

By direct calculations, it is easy to see that the operator $\mathcal{F}$ is well defined on $PC(J, E)$. From Definition 2.9, one can easily see that the PC-mild solution of (1.1) is equivalent to the fixed point of operator $\mathcal{F}$ defined by (3.4). Next, we will prove that the operator $\mathcal{F}$ has at least one fixed point.

Firstly, we show that $\mathcal{F}u \in PC(J,E)$ for all $u \in PC(J,E)$. For $0 \le \tau < t \le t_1$, by the strongly continuity of the semigroup $T(t)(t \ge 0)$, (2.1) and (3.4), we know that

$$\begin{aligned} &\|(\mathcal{F}u)(t) - (\mathcal{F}u)(\tau)\| \\ &\le \|T(t)u_0 - T(\tau)u_0\| + \Big\| \int_\tau^t T(t-s)f(s,u(s))ds \Big\| \\ &\quad + \Big\| \int_0^\tau [T(t-s) - T(\tau - s)]f(s,u(s))ds \Big\| \\ &\le M\|T(t-\tau)u_0 - u_0\| + M\int_\tau^t \|f(s,u(s))\|ds \\ &\quad + \int_0^\tau \|T(t-\tau)T(\tau-s)f(s,u(s)) - T(\tau-s)f(s,u(s))\|ds \\ &\to 0 \quad \text{as } t \to \tau. \end{aligned} \tag{3.7}$$

From this inequality it follows that $\mathcal{F}u \in C([0,t_1],E)$.

From (3.4) and the continuity of the non-instantaneous impulsive functions $\gamma_k(t,u(t))$, $k = 1,2,\dots,m$, it is easy to know that $\mathcal{F}u \in C((t_k,s_k],E)$ for every $k = 1,2,\dots,m$. Completely similar with the proof for the continuity of $(\mathcal{F}u)(t)$ with respect to t on $[0,t_1]$, we can prove that $\mathcal{F}u \in C((s_k,t_{k+1}],E)$ for $k = 1,2,\dots,m$. Therefore, we have proved that $\mathcal{F}u \in PC(J,E)$ for $u \in PC(J,E)$, namely, $\mathcal{F}$ maps $PC(J,E)$ to $PC(J,E)$.

Next, we prove that there exists a constant $R > 0$, such that $\mathcal{F}(\Omega_R) \subset \Omega_R$. If this is not true, then for each $r > 0$, there would exist $u_r \in \Omega_r$ and $t_r \in J$ such that $\|(\mathcal{F}u_r)(t_r)\| > r$. If $t_r \in [0,t_1]$, then by (2.1), (3.4) and the assumption (H1), we know that

$$\begin{aligned} \|(\mathcal{F}u_r)(t_r)\| &\le M\|u_0\| + M\int_0^{t_r} \|f(s,u_r(s))\|ds \\ &\le M\|u_0\| + M\int_0^{t_r} \Psi(\|u_r\|)\varphi(s)ds \\ &\le M\|u_0\| + M\Psi(r)\|\varphi\|_{L[0,t_1]}. \end{aligned} \tag{3.8}$$

If $t_r \in (t_k,s_k]$, $k = 1,2,\dots,m$, then by (2.1), (3.4) and assumption (H2), we obtain

$$\|(\mathcal{F}u_r)(t_r)\| = \|\gamma_k(t_r,u_r(t_r))\| \le K_{\gamma_k}\|u_r(t_r)\| + \|\gamma_k(t_r,\theta)\| \le K_{\gamma_k}r + N, \tag{3.9}$$

where

$$N = \max_{k=1,2,\dots,m} \sup_{t\in J} \|\gamma_k(t,\theta)\|.$$

If $t_r \in (s_k,t_{k+1}]$, $k = 1,2,\dots,m$, then by (2.1), (3.4) and the assumptions (H1) and (H2), we obtain

$$\begin{aligned} \|(\mathcal{F}u_r)(t_r)\| &\le M\|\gamma_k(s_k,u_r(s_k))\| + M\int_{s_k}^{t_r} \|f(s,u_r(s))\|ds \\ &\le M(K_{\gamma_k}\|u_r(s_k)\| + \|\gamma_k(s_k,\theta)\|) + M\Psi(r)\int_{s_k}^{t_r} \varphi(s)ds \\ &\le M(K_{\gamma_k}r + N) + M\Psi(r)\|\varphi\|_{L[s_k,t_{k+1}]}. \end{aligned} \tag{3.10}$$

Combining (2.1), (3.1), (3.4), (3.8)-(3.10) with the fact $r < \|(\mathcal{F}u_r)(t_r)\|$, we obtain

$$r < \|(\mathcal{F}u_r)(t_r)\| \le M\Big(\|u_0\| + \Psi(r)\Lambda + Kr + N\Big). \tag{3.11}$$

Dividing both side of (3.11) by r and taking the lower limit as $r \to +\infty$, we have

$$1 \le M(\rho\Lambda + K),$$

which contradicts (3.3).

Next, we prove that the operator $\mathcal{F}_1 : \Omega_R \to \Omega_R$ is Lipschitz continuous. For $t \in (t_k, s_k]$, $k = 1, 2, \dots, m$ and $u, v \in \Omega_R$, by (3.5) and the assumption (H2), we obtain

$$\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1 v)(t)\| \le K_{\gamma_k} \|u(t) - v(t)\| \le K_{\gamma_k} \|u - v\|_{PC}. \tag{3.12}$$

For $t \in (s_k, t_{k+1}]$, $k = 1, 2, \dots, m$ and $u, v \in \Omega_R$, by (3.5) and the assumption (H2), we know that

$$\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1 v)(t)\| \le M K_{\gamma_k} \|u(s_k) - v(s_k)\| \le M K_{\gamma_k} \|u - v\|_{PC}. \tag{3.13}$$

From (3.12), (3.13), (2.1) and (3.1), we obtain

$$\|\mathcal{F}_1 u - \mathcal{F}_1 v\|_{PC} \le MK \|u - v\|_{PC}. \tag{3.14}$$

In the following, we prove that $\mathcal{F}_2$ is continuous in Ω_R. To this end, let $u_n \in \Omega_R$ be a sequence such that $\lim_{n\to+\infty} u_n = u$ in Ω_R. By the continuity of nonlinear term f with respect to the second variable, for each $s \in J$ we have

$$\lim_{n\to+\infty} f(s, u_n(s)) = f(s, u(s)). \tag{3.15}$$

By assumption (H1), we obtain that for every $s \in J$,

$$\|f(s, u_n(s)) - f(s, u(s))\| \le 2\varphi(s)\Psi(R). \tag{3.16}$$

Using the fact that the function $s \to 2\varphi(s)\Psi(R)$ is Lebesgue integrable for $s \in [s_k, t]$ and $t \in (s_k, t_{k+1}]$, $k = 0, 1, \dots, m$, by (2.1), (3.6), (3.15), (3.16) and the Lebesgue dominated convergence theorem, we know that

$$\begin{aligned} \|(\mathcal{F}_2 u_n)(t) - (\mathcal{F}_2 u)(t)\| &\le M \int_{s_k}^{t} \|f(s, u_n(s)) - f(s, u(s))\| ds \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{3.17}$$

Then we infer that

$$\|\mathcal{F}_2 u_n - \mathcal{F}_2 u\|_{PC} \to 0 \quad \text{as } n \to +\infty,$$

which means that $\mathcal{F}_2$ defined by (3.6) is continuous in Ω_R.

Now, we demonstrate that the operator $\mathcal{F}_2 : \Omega_R \to \Omega_R$ is equicontinuous. For any $u \in \Omega_R$ and $s_k < t' < t'' \le t_{k+1}$ for $k = 0, 1, \dots, m$, we obtain that

$$\begin{aligned} &\|(\mathcal{F}_2 u)(t'') - (\mathcal{F}_2 u)(t')\| \\ &\le \Big\| \int_{t'}^{t''} T(t'' - s) f(s, u(s)) ds \Big\| + \Big\| \int_{s_k}^{t'} \big[T(t'' - s) - T(t' - s)\big] f(s, u(s)) ds \Big\| \\ &:= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \Big\| \int_{t'}^{t''} T(t'' - s) f(s, u(s)) ds \Big\|,$$

$$I_2 = \Big\| \int_{s_k}^{t'} \big[T(t'' - s) - T(t' - s)\big] f(s, u(s)) ds \Big\|.$$

Therefore, we only need to check I_1 and I_2 tend to 0 independently of $u \in \Omega_R$ when $t'' - t' \to 0$. For I_1, by (2.1) and the assumption (H1), we can easily see that

$$I_1 \le M\Psi(R) \int_{t'}^{t''} \varphi(s)ds \to 0 \quad \text{as } t'' - t' \to 0.$$

For $\epsilon > 0$ small enough, by (2.1), the assumption (H1), equicontinuity of the C_0-semigroup $T(t)(t \ge 0)$ and the Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned} I_2 &\le \Big\| \int_{s_k}^{t'-\epsilon} \big[T(t''-s) - T(t'-s)\big] f(s,u(s))ds \Big\| \\ &\quad + \Big\| \int_{t'-\epsilon}^{t'} \big[T(t''-s) - T(t'-s)\big] f(s,u(s))ds \Big\| \\ &\le \Psi(R) \int_{\epsilon}^{t'-s_k} \|T(t''-t'+s) - T(s)\| \varphi(t'-s)ds + 2M\Psi(R) \int_{t'-\epsilon}^{t'} \varphi(s)ds \\ &\to 0 \quad \text{as } t''-t' \to 0 \text{ and } \epsilon \to 0. \end{aligned}$$

As a result, $\|(\mathcal{F}_2 u)(t'') - (\mathcal{F}_2 u)(t')\|$ tends to 0 independently of $u \in \Omega_R$ as $t'' - t' \to 0$, which means that $\mathcal{F}_2 : \Omega_R \to \Omega_R$ is equicontinuous.

For any bounded $D \subset \Omega_R$, by Lemma 2.6, we know that there exists a countable set $D_0 = \{u_n\} \subset D$, such that

$$\alpha(\mathcal{F}_2(D))_{PC} \le 2\alpha(\mathcal{F}_2(D_0))_{PC}. \tag{3.18}$$

Since $\mathcal{F}_2(D_0) \subset \mathcal{F}_2(\Omega_R)$ is bounded and equicontinuous, we know from Lemma 2.8 that

$$\alpha(\mathcal{F}_2(D_0))_{PC} = \max_{t \in [s_k, t_{k+1}],\, k=0,1,\dots,m} \alpha(\mathcal{F}_2(D_0)(t)). \tag{3.19}$$

For every $t \in [s_k, t_{k+1}]$, $k = 0, 1, \dots, m$, by Lemma 2.7, the assumption (H3) and (3.6), we have

$$\begin{aligned} \alpha(\mathcal{F}_2(D_0)(t)) &= \alpha\Big(\Big\{ \int_{s_k}^{t} T(t-s) f(s, u_n(s))ds \Big\}\Big) \\ &\le 2M \int_{s_k}^{t} \alpha(\{f(s, u_n(s))\})ds \\ &\le 2M \int_{s_k}^{t} L_k \alpha(D_0(s))ds \\ &\le 2M L_k (t_{k+1} - s_k) \alpha(D)_{PC}. \end{aligned} \tag{3.20}$$

Therefore, from (3.3), (3.18), (3.19) and (3.20) we know that

$$\alpha(\mathcal{F}_2(D))_{PC} \le 4ML\alpha(D)_{PC}. \tag{3.21}$$

From (3.14) and Lemma 2.3 (vii), we know that for any bounded $D \subset \Omega_R$,

$$\alpha(\mathcal{F}_1(D))_{PC} \le MK\alpha(D)_{PC}. \tag{3.22}$$

By (3.21), (3.22) and Lemma 2.3 (vi), we have

$$\alpha(\mathcal{F}(D))_{PC} \le \alpha(\mathcal{F}_1(D))_{PC} + \alpha(\mathcal{F}_2(D))_{PC} \le M(K + 4L)\alpha(D)_{PC}. \tag{3.23}$$

(3.23) combining with (3.3) and Definition 2.4 we know that the operator $\mathcal{F} : \Omega_R \to \Omega_R$ is a k-set-contractive. It follows from Lemma 2.5 that $\mathcal{F}$ has at least one fixed point $u \in \Omega_R$, which is just a PC-mild solution of (1.1). This completes the proof $\square$

Remark 3.2. The analytic semigroup and differentiable semigroup are equicontinuous semigroup [22]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. Therefore, Theorem 3.1 has a broad applicability.

Remark 3.3. Theorem 3.1 complements results in [16] and [23]. Theorem 3.1 can be applied to a class of partial differential equations of evolution type for which the corresponding solution semigroup are not compact.

4. An example

To illustrate the applicability of our main results, we consider the parabolic partial differential equation with non-instantaneous impulses,

$$\begin{gathered} \frac{\partial}{\partial t}u(x,t)+\mathcal{A}u(x,t)=\frac{e^{-t}}{2+|u(x,t)|}, \quad x\in\Omega,\ t\in[0,\frac{1}{3})\cup(\frac{2}{3},1], \\ \mathcal{B}u(x,t)=0, \quad x\in\partial\Omega, \quad t\in[0,1], \\ u(x,t)=\frac{e^{-(t-\frac{1}{3})}}{4}\frac{|u(x,t)|}{1+|u(x,t)|}, \quad x\in\Omega,\ t\in[\frac{1}{3},\frac{2}{3}), \\ u(x,0)=\varphi(x), \quad x\in\Omega, \end{gathered} \tag{4.1}$$

where $J=[0,1]$, integer $N\geq 1$, $\Omega\subset\mathbb{R}^N$ is a bounded domain, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional $C^{2+\mu}$-manifold for some $0<\mu<1$,

$$\mathcal{A}u := -\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)+a_0(x)u$$

is a uniformly elliptic differential operator on $\overline{\Omega}$ with the coefficients $a_{ij}\in C^{1+\mu}(\overline{\Omega})$ $(i,j=1,2,\dots,N)$ and $a_0\in C^{\mu}(\overline{\Omega})$ for some $\mu\in(0,1)$, $a_0(x)\geq 0$ on $\overline{\Omega}$. That is, $[a_{ij}(x)]_{N\times N}$ is a positive definite symmetric matrix for every $x\in\overline{\Omega}$ and there exists a constant $\mu_0>0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(x)\eta_i\eta_j\geq\mu_0\mid\eta\mid^2, \quad \forall\,\eta=(\eta_1,\eta_2,\dots,\eta_N)\in\mathbb{R}^N,\ x\in\overline{\Omega};$$

$$\mathcal{B}u := \delta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(x)\cos(\nu,x_i)\frac{\partial u}{\partial x_j}+(1-\delta)u$$

is a boundary operator on $\partial\Omega$, where ν is an outer unit normal on $\partial\Omega$, $\delta=0$ or 1; $\varphi\in L^p(\Omega)$ with $p\geq 2$.

Let $E=L^p(\Omega)$ with $p\geq 2$. Then E is a Banach space equipped with the L^p-norm $\|\cdot\|_p$. Consider the operator $A: D(A)\subset E\to E$ defined by

$$D(A)=\Big\{u\in W^{2,p}(\Omega):\mathcal{B}u=0\Big\}, \quad Au=\mathcal{A}u. \tag{4.2}$$

It is well known from [4] and [22] that $-A$ generates an analytic C_0-semigroup $T(t)$ $(t\geq 0)$ on E, and

$$\|T(t)\|\leq 1, \quad \text{for any } t\geq 0, \tag{4.3}$$

which means that the C_0-semigroup $T(t)$ $(t\geq 0)$ is contraction on E.

Let $a = t_2 = 1$, $t_0 = s_0 = 0$, $t_1 = \frac{1}{3}$, $s_1 = \frac{2}{3}$, $u(t) = u(\cdot,t)$, $f(t,u(t)) = \frac{e^{-t}}{2+|u(\cdot,t)|}$, $\gamma_1(t,u(t)) = \frac{e^{-(t-\frac{1}{3})}}{4}\frac{|u(\cdot,t)|}{1+|u(\cdot,t)|}$, $u_0 = \varphi(\cdot)$, then the parabolic partial differential equation (4.1) can be rewritten into the abstract form of (1.1) in $L^p(\Omega)$ for $m = 1$.

From the definition of nonlinear term f and non-instantaneous impulsive function γ_1, we can easily to verify that the assumptions (H1)-(H3) and the condition (3.3) hold with

$$M = 1, \quad \varphi(t) = \frac{|\Omega|}{2}e^{-t}, \quad \Psi \equiv 1, \quad \Lambda = \frac{|\Omega|}{6}, \quad K = K_{\gamma_1} = \frac{1}{4}, \quad L = \frac{1}{12}.$$

Therefore, by Theorem 3.1, the parabolic partial differential equation (4.1) has at least one PC-mild solution.

Acknowledgments. This research was supported by NNSFs of China (11501455, 11261053), by the Science Research Project for Colleges and Universities of Gansu Province (2015A-003, 2015A-213) and by Key Project of Gansu Provincial National Science Foundation (1606RJZA015).

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