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EXISTENCE, UNIQUENESS AND EXPONENTIAL DECAY OF SOLUTIONS TO KIRCHHOFF EQUATION IN $\mathbb{R}^n$

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Abstract. We discuss the global well-posedness and uniform exponential stability for the Kirchhoff equation in $\mathbb{R}^n$

$$u_{tt} - M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u + \lambda u_t = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

The global solvability is proved when the initial data are taken small enough and the exponential decay of the energy is obtained in the strong topology $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, which is a different feature of the present article when compared with the prior literature. We also dedicate a section to discuss a model with the frictional damping term λu_t, is replaced by a viscoelastic damping term $\int_0^t g(t-s)\Delta u(s)ds$.

1. Introduction

1.1. Description of the problem and main difficulties. This article addresses the global well-posedness and uniform exponential stability to the Kirchhoff equation

$$\begin{gathered} u'' - M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u + \lambda u' = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u'(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $' = \frac{\partial}{\partial t}$; $\nabla\cdot = (\frac{\partial\cdot}{\partial x_1}, \dots, \frac{\partial\cdot}{\partial x_n})$ and $\Delta\cdot = \sum_{i=1}^n \frac{\partial^2\cdot}{\partial x_i^2}$ are the gradient and Laplace operator on the spatial variable, respectively; $M : \mathbb{R}^+ \to \mathbb{R}^+$, with $M(s) \ge m_0 > 0$, for all $s \ge 0$; $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$ are given functions and λ is a real positive parameter.

In the simple case when $M(s) = 1$, for all $s \in \mathbb{R}^+$, the wave the equation

$$\begin{gathered} u'' - \Delta u + \lambda u' = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u'(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \end{gathered} \tag{1.2}$$

The well-posedness to problem (1.2) is well-known for all initial data $(u_0, u_1) \in H^{m+1}(\mathbb{R}^n) \times H^m(\mathbb{R}^n)$, $m = 0, 1, 2 \dots$ and the exponential decay never holds for

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the topology

$$\sum_{i=0}^{m+1} \|D_t^i D_x^{m+1-i} u(t)\|_{L^2(\mathbb{R}^n)},$$

see, for instance [12, 13, 17, 30, 35, 36] and references therein. Instead, one has a wide assortment of polynomial decay rate estimates, in some cases sharp estimates as in the recent paper [9]. However, under some smallness on the initial data, Feireisl [10] proved that for the semi-linear wave equation, $(u(t), u'(t))$ decays exponentially to zero in the weak topology $X := H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, but, the semigroup $S_t : (u, u')(0) \mapsto (u, u')(t)$ is not dissipative in X, because $\mathbb{R}^n$ is not compact. One the main ingredients to recover the exponential stability to problem (1.2) without restrictions on the initial data is the existence of the Poincaré's inequality which is well-known to be true for bounded domains or unbounded ones with finite measure. Nevertheless, this also holds for the specific case of $\mathbb{R}^n$, provided, roughly speaking, that the Fourier transform of the initial data is zero in bounded sets of $\mathbb{R}^n$, see, the nice paper due to Bjorland and Schonbek [3], which will be clarified in section 2. Inspired in the work [3], if we look for a nonlinear model such that it is invariant under the flow of S_t in light of the previous comments (namely, such that the Poincaré's inequality remains true under the flow), the first equation that comes into our mind is precisely the Kirchhoff model given in (1.1). Nevertheless, due to the nonlinear character of this type of equation, it is expected its solvability in the strong topology $Y := H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ provided the initial data are taken small enough. From the above considerations the main task of the present article is twofold: (i) to prove the existence and uniqueness of regular global solutions to problem (1.1); (ii) to show that these solutions decay exponentially to zero in the natural strong topology Y, which is much more difficult that to prove that regular solutions decay exponentially in the weak topology X. In order to achieve (i) we need to define suitable Hilbert spaces V, H and an operator $A = -\Delta$ define by the triple $\{V, H, a(u, v)\}$, where $a(u, v)$ is a bilinear, continuous and coercive form defined in V. All this spectral analysis necessary to development of the paper is presented in section 2. Section 3 is devoted to the prove of existence and uniqueness of regular solutions to (1.1). Indeed, the strategy is the following: First, we consider the linear auxiliary problem

$$\begin{gathered} u'' - \mu(t)\Delta u + \lambda u' = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u'(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.3}$$

where

$$\mu \in W^{2,1}_{\rm loc}(0, \infty); \quad \mu(t) \geq m_0 > 0, \quad \text{for all } t \geq 0.$$

Thus, we prove that problem (1.3) possesses a unique global regular solution which implies, by employing the Banach contraction theorem combined with a priori estimates for the linearized problem (1.3), and for a complete metric space suitably chosen, that problem (1.1) possesses a unique local regular solution for a certain interval $[0, T_0]$, $T_0 > 0$. By Zorn's lemma we derive the existence of a regular maximal solution on $[0, T_{\max})$, which can be be extended to the whole interval $[0, +\infty)$ by considering the initial data sufficiently small. All this will be clarified in section 3. In Section 4 we prove the item (ii) above mentioned, namely, the exponential stability to problem (1.1) in its strong topology, which is also one the main novelties of the present article. For this purpose we employ Nakao's lemma twice: first to obtain the exponential stability for regular solutions in the weak topology X and then, from the previous decay, to obtain the analogous one now for the strong topology Y. It is know that the Nakao's lemma is appropriate to deal with decay properties of the Kirchhoff model, see [31, 32, 33, 34, 37] and references therein. See, in particular, Nishihara [31] and Ono [34, 37] where the stability in strong topology was proved. We highlight [37] where this technique was used and it was obtained polynomial decay to Kirchhoff equation in $\mathbb{R}^n$.

Finally, we dedicate the section 5 to discuss the problem when the frictional damping term is replaced by a viscoelastic damping term, precisely, we study

$$\begin{gathered} u'' - M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u'(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.4}$$

where g is a know function. In this case, we only describe what are the technical differences between the two cases (frictional or viscoelastic). The decay is obtained by the same strategy of Messaoudi [22].

1.2. **Literature overview.** There is a lot of literature in what concerns the well-posedness and decay rate estimates for the Kirchhoff equation in a general setting. However, our focus of interest are those models posed in the whole $\mathbb{R}^n$. It seems that one the pioneers in establishing the local well-posedness in the absence of a damping term u' was Perla Menzala [21]. For the damped Kirchhoff model we would like to quote the following ones: Yamada [39], whom seems to be the one the pioneers in investigating the (polynomial) asymptotic stability for global solutions of equation (1.1); Ikehata and Okazawa [11] give a different treatment to the same equation by employing the Yosida approximation method together with compactness argument, which allow them to treat simultaneously the global solvability for Dirichlet and Neumann cases. Finally we would like to quote the important contribution of Manfrin [20] also in the context of damped Kirchhoff models. The author studies the Cauchy problem for the damped Kirchhoff equation in the phase space $\mathbb{H}^r \times \mathbb{H}^{r-1}$, with $r \geq 3/2$. The author proves global solvability and like polynomial decay of solutions when the initial data belong to an open, dense subset B of the phase space such that $B + B = \mathbb{H}^r \times \mathbb{H}^{r-1}$. From the above comments, a distinctive feature of the present paper, as we have already mentioned before, is to establish in what conditions the exponential stability holds for the Kirchhoff model posed in $\mathbb{R}^n$. Definitely the source of the inspiration of the present article comes from the work of Bjorland and Schonbek [3] which allows us to create appropriate spaces to develop the associate spectral theory that is necessary to solve our problem. We dedicate the section 2 to describe these ideas. In addition, we can not forget to say that some of ideas contained here were previously nicely presented in Milla Miranda and Jutuca [27] obviously adapted to the present context. Basically, they combine the Faedo-Galerkin method to solve a linearized problem with a point fix theorem. The Faedo-Galerkin method is a traditional method which have been used to get existence of solution when the problem involves the Kirchhoff equation. Proofs using Faedo-Galerkin method can be found in Lourêdo, Oliveira and Clark [24], Lourêdo and Milla Miranda [25, 26], Lourêdo, Milla Miranda and Medeiros [28] and references therein.

On the other hand, wave equation with viscoelastic damping has been studied by many authors. When the domain is an open bounded of $\mathbb{R}^n$ see, for instance, [1, 2, 6, 7, 8, 19, 22, 23, 38] and reference therein. We highlight the recent works of Cavalcanti *et al.* [5] and Lasiecka, Messaoudi and Mustafa [18] where very general decay rates was obtained. When the domain is whole $\mathbb{R}^n$ there are not many papers in this direction. The difficulty is into deal with the problem (1.4) without the Poincaré's inequality to hold in an appropriate space. Therefore, some authors have used the finite-speed propagation to compensate for the lack of Poincaré's inequality, see the works of Kafini [14, 15] and Kafini and Messaoudi [16]; or they have considered the solutions in spaces weighted by the introduction of an appropriate function to the equation, see the papers of Zennir [40, 41], this strategy also compensates for the lack of Poincaré's inequality.

2. Preliminaries and overview on spectral theory

Now, inspired on work of Bjorland and Schonbek [3], we will introduce the spaces which will be necessary to prove our results.

When it is considered an initial and boundary value problem as

$$\begin{gathered} u'' - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega, \end{gathered}$$

where Ω is an open and bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$, the natural way is take the Sobolev spaces $L^2(\Omega)$, $H_0^1(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$. Here, the main idea is create appropriate Sobolev spaces H, V and W which work like $L^2(\Omega)$, $H_0^1(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$, respectively. These space must have some essential properties like Poincaré's Inequality and Green's Formula. After establishing the spaces, we also will show an overview on spectral theory associated to our problem.

Let $R > 0$ be a fixed real number. Define

$$H = \{u \in L^2(\mathbb{R}^n); \widehat{u}(\xi) = 0 \text{ a.e. in } \|\xi\| \le R\},$$

where $\widehat{u}$ denotes the Fourier Transform of u. We observe that $H \neq \emptyset$ follows from [3, Lemma 5.1], for example, in the case $n = 1$ we can consider

$$u(x) = v(x) - (H_R * v)(x), \quad x \in \mathbb{R},$$

where $v(x) = \exp(\pi|x|^2)$, $H_R(x) = \frac{\sin(2\pi R x)}{\pi x}$ and $*$ denotes the convolution product. We affirm that $u \in H$, in fact, we see that $u \in L^2(\mathbb{R}^n)$ and, moreover,

$$\widehat{u}(\xi) = \widehat{v}(\xi) - \widehat{H}_R(\xi)\widehat{v}(\xi) = \widehat{v}(\xi) - \chi_R(\xi)\widehat{v}(\xi), \quad \text{a.e. in } \mathbb{R},$$

where $\chi_R(\xi)$ is the cut-off function such that $\chi_R(\xi) = 1$ when $|\xi| \le R$ and $\chi_R(\xi) = 0$ when $|\xi| > R$.

We endowed H with the inner product and norm given by

$$(u,v) = \int_{\mathbb{R}^n} u(x)v(x)dx \quad \text{and} \quad \|u\|_H = \Big(\int_{\mathbb{R}^n} |u(x)|^2 dx\Big)^{1/2}.$$

It is not difficult to prove that H is a separate Hilbert space.

It is possible to prove that (see [3, Theorem 4.1]) for each $u \in H^1(\mathbb{R}^n)$ and any $\Lambda > 0$, the following inequality holds

$$\|\nabla u\|_H^2 \ge \Lambda^2 \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 d\xi - \int_{\{\xi;\, |\xi| \le \Lambda\}} (\Lambda^2 - |\xi|^2)|\widehat{u}(\xi)|^2 d\xi.$$

Defining

$$V = \{u \in H^1(\mathbb{R}^n);\ \widehat{u}(\xi) = 0 \text{ a.e. in } \|\xi\| \leq R\}$$

and taking, in particular, $\Lambda = R$ we have, after use the Plancherel Identity, the following version of Poincaré's Inequality:

$$\|u\|_H \leq \frac{1}{R}\|\nabla u\|_V, \quad \text{for all } u \in V.$$

This allows us to consider the following inner product and norm in V:

$$((u,v)) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x)dx \quad \text{and} \quad \|u\|_V = \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx\Big)^{1/2}.$$

We observe that $\|\cdot\|_V$ is equivalent, in V, to usual norm gives by $H^1(\mathbb{R}^n)$. We can prove that the couple $(V, ((\cdot,\cdot)))$ is a separable Hilbert space and V is a dense subspace of H.

Now, we describe the spectral theory associate with (1.1). We define the bilinear, continuous and coercive form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$:

$$(u,v) \mapsto a(u,v) = ((u,v)).$$

We denote by $D(A)$ the set of $u \in V$ such that the linear form $g_u : V \to \mathbb{R}$:

$$g_u(v) = ((u,v)) \tag{2.1}$$

is continuous in V with the topology gives by H. As V is dense in H, we can extend this form to whole H, i.e., there exists $\tilde{g_u} : H \to \mathbb{R}$: such that

$$\tilde{g_u}(v) = ((u,v)), \quad \text{for all } v \in V. \tag{2.2}$$

By Riesz representation theorem, there exists a unique $f_u \in H$ such that

$$\tilde{g_u}(v) = (f_u, v), \quad \text{for all } v \in H. \tag{2.3}$$

From (2.1)–(2.3) we have

$$((u,v)) = (f_u, v), \quad \text{for all } v \in V.$$

This allow us to define the operator $A : D(A) \to H$:

$$Au = f_u.$$

We observe that $D(A)$ has the following characterization

$$\begin{aligned} D(A) = \big\{&u \in V;\ \text{there exists } f \in H \text{ that satisfies} \\ &((u,v)) = (f_u, v),\ \text{for all } v \in V\big\}. \end{aligned} \tag{2.4}$$

From this it follows that $D(A)$ is a subspace of H and the operator A, which is characterized by

$$(Au, v) = ((u,v)), \quad \text{for all } u \in D(A) \text{ and } v \in V.$$

In this case, we say that A is defined by the term$\{V, H, ((\cdot,\cdot))\}$.

The operator A has the following properties:

(a) $A : D(A) \to H$ is bijective;
(b) $D(A)$ is dense in H and A is a closed operator of H;
(c) A is an unbounded operator of H;
(d) $D(A)$ is dense in V;
(e) $A : D(A) \to H$ is self-adjoint operator and satisfies

$$(Au, v) = (u, Av), \quad \text{for all } u, v \in D(A).$$

We introduce in $D(A)$ the inner product:

$$((u,v))_{D(A)} = (u,v) + (Au, Av), \quad \text{for all } v \in D(A),$$

then, as A is closed, we have that $D(A)$ is a Hilbert space. It is possible to prove that there exists $c > 0$ such that

$$\|u\|_V \leq c\|u\|_{D(A)}, \quad \text{for all } u \in D(A),$$

i.e., $D(A) \hookrightarrow V$ continuously. Identifying H with its dual space we have the sequence of continuous and dense embedding

$$D(A) \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow (D(A))'.$$

Now, we define

$$W = \{u \in H^2(\mathbb{R}^n);\ \widehat{u}(\xi) = 0 \text{ a.e. in } \|\xi\| \leq R\}.$$

Since, for all $u \in H^2(\mathbb{R}^n)$, $\widehat{\Delta u}(\xi) = \|\xi\|^2\widehat{u}(\xi)$, we infer: if $u \in W$, then $-\Delta u \in H$. Therefore, for all $u \in W$ we have

$$(-\Delta u, v) = -\int_{\mathbb{R}^n} \widehat{\Delta u}(\xi)\widehat{v}(\xi)d\xi = -\int_{\mathbb{R}^n} \|\xi\|^2\widehat{u}(\xi)\widehat{v}(\xi)d\xi, \quad \text{for all } v \in V \tag{2.5}$$

and

$$((u,v)) = -\int_{\mathbb{R}^n} \nabla u(x)\cdot\nabla v(x)dx = -\int_{\mathbb{R}^n} \|\xi\|^2\widehat{u}(\xi)\widehat{v}(\xi)d\xi, \tag{2.6}$$

for all $v \in V$. Combining (2.5) and (2.6) we obtain

$$((u,v)) = (-\Delta u, v), \quad \text{for all } v \in V, \tag{2.7}$$

observing (2.4), this gives us that $u \in D(A)$, i.e.,

$$W \subset D(A). \tag{2.8}$$

We observe that (2.7) is a Green's Formula.

On the other hand, from the definition of A, we have

$$(Au, v) = ((u,v)), \quad \text{for all } u \in W \text{ and } v \in V. \tag{2.9}$$

From (2.7) and (2.9) and as V is dense in H we obtain

$$(Au, v) = (-\Delta u, v), \quad \text{for all } v \in H,$$

this gives

$$Au = -\Delta u, \quad \text{for all } u \in W.$$

We also can prove that W is a subspace dense and closed of $D(A)$, this combined with (2.8) give that $W = D(A)$. Therefore, W is other characterization of $D(A)$ and A is the know operator $-\Delta$.

3. Existence and uniqueness of a solution

In this section we prove the existence and uniqueness of solution to (1.1). We start by presenting two results concerned with existence of solution to an auxiliary linear problem which will be necessary to prove the result of existence of local solution in time. Therefore, associated to (1.1) we consider the linear problem

$$u'' - \mu(t)\Delta u + \lambda u' = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{3.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{3.2}$$

$$u'(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \tag{3.3}$$

We consider

$$\mu \in W^{2,1}_{\rm loc}(0,\infty); \quad \mu(t) \geq m_0 > 0, \quad \text{for all } t \geq 0. \tag{3.4}$$

Proposition 3.1. *Suppose that assumption* (3.4) *holds. Then for each* $u_0 \in W \cap H^3(\mathbb{R}^n)$ *and* $u_1 \in W$ *there exists a unique solution* u *to* (3.1)-(3.3) *satisfying*

$$\begin{gathered} u \in L^\infty_{\rm loc}(0,\infty; W \cap H^3(\mathbb{R}^n)),\ u' \in L^\infty_{\rm loc}(0,\infty; W), \\ u'' \in L^\infty_{\rm loc}(0,\infty; V),\ u''' \in L^\infty_{\rm loc}(0,\infty; H)\,. \end{gathered} \tag{3.5}$$

Proof. We employ the Faedo-Galerkin method. Let $(w_j)_{j\in\mathbb{N}}$ be an orthonormal bases in $W \cap H^3(\mathbb{R}^n)$. For each $m \in \mathbb{N}$, we denote U_m the m-dimensional subspaces spanned by the first m vectors of $(w_j)_{j\in\mathbb{N}}$. Let $T > 0$ be any fixed positive number. From Ordinary Differential Equations Theory for each $m \in \mathbb{N}$ we can find $0 < T_m \leq T$, $u_m : \mathbb{R}^n \times [0, T_m] \to \mathbb{R}$ of the form

$$u_m(x,t) = \sum_{j=1}^m \rho_{jm}(t) w_j(x),$$

satisfying the following approximate problem:

$$(u''_m(t), w_j) + \mu(t)((u_m(t), w_j)) + \lambda(u'_m(t), w_j) = 0; \tag{3.6}$$

$$u_m(0) = u_{0m} = \sum_{i=1}^m u_0^i w_i \to u_0 \quad \text{in } W \cap H^3(\mathbb{R}^n); \tag{3.7}$$

$$u'_m(0) = u_{1m} = \sum_{i=1}^m u_1^i w_i \to u_1 \quad \text{in } W. \tag{3.8}$$

Here $1 \leq j \leq m$ and u_0^i, u_1^i, $i = 1, \dots, m$, are known scalars. From (3.6) we have the approximate equation

$$(u''_m(t), w) + \mu(t)((u_m(t), w)) + \lambda(u'_m(t), w) = 0, \quad \text{for all } v \in V. \tag{3.9}$$

Estimate I: Setting $w = u'_m(t)$ in the approximate equations (3.9) we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u'_m(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2) + \lambda\|u'_m(t)\|_H^2 = \mu'(t)\|u_m(t)\|_V^2.$$

Integrating from 0 to $t \leq T_m$, we obtain

$$\begin{aligned} &\|u'_m(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2 + 2\lambda \int_0^t \|u'_m(\xi)\|_H^2 d\xi \\ &= \|u_{1m}\|_H^2 + \mu(0)\|u_{0m}\|_V^2 + 2\int_0^t \frac{\mu'(\xi)}{\mu(\xi)}\mu(\xi)\|u_m(\xi)\|_V^2 d\xi. \end{aligned} \tag{3.10}$$

From (3.7), (3.8), (3.10) and Gronwall Inequality, we conclude that

$$\|u'_m(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2 + 2\lambda \int_0^t \|u'_m(\xi)\|_H^2 d\xi \leq R_1^2 \exp\Big(2\int_0^T \varphi_1(\xi) d\xi\Big), \tag{3.11}$$

for all $T \leq T_m$, where $R_1^2 = \|u_1\|_H^2 + \mu(0)\|u_0\|_V^2$ and $\varphi_1(\xi) = \frac{|\mu'(\xi)|}{\mu(\xi)}$. This estimate allow us to extend the approximate solution to the whole interval $[0, T]$ and (3.11) holds for all $T > 0$.

Estimate II: Differentiating (3.9) with respect to t and putting $w = u_m''(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_m''(t)\|_H^2 + \mu(t)\|u_m'(t)\|_V^2) + \mu'(t)((u_m(t), u_m''(t))) + \lambda\|u_m''(t)\|_H^2 \\ &= \frac{1}{2}\mu'(t)\|u_m'(t)\|_V^2. \end{aligned} \tag{3.12}$$

Taking $w = u_m''(t)$ in (3.9) we have

$$((u_m(t), u_m''(t))) = -\frac{1}{\mu(t)}[\|u_m''(t)\|_H^2 + \lambda(u_m'(t), u_m''(t))]. \tag{3.13}$$

Substituting (3.13) in (3.12) and integrating the resultant equation from 0 to t we obtain

$$\begin{aligned} &\|u_m''(t)\|_H^2 + \mu(t)\|u_m'(t)\|_V^2 + 2\lambda\int_0^t \|u_m''(\xi)\|_H^2 d\xi \\ &= \|u_m''(0)\|_H^2 + \mu(0)\|u_{1m}\|_V^2 + 2\lambda\int_0^t \frac{\mu'(\xi)}{\mu(\xi)}(u_m'(\xi), u_m''(\xi))d\xi \\ &\quad + \int_0^t \frac{\mu'(\xi)}{\mu(\xi)}[2\|u_m''(\xi)\|_H^2 + \mu(\xi)\|u_m'(\xi)\|_V^2]d\xi. \end{aligned} \tag{3.14}$$

From elementary and the Poincaré inequalities, we have

$$\begin{aligned} &2\lambda\int_0^t \frac{\mu'(\xi)}{\mu(\xi)}(u_m'(\xi), u_m''(\xi))d\xi \\ &\le \lambda\int_0^t \frac{|\mu'(\xi)|}{\mu(\xi)}\Big(\frac{\mu(\xi)}{m_0R^2}\|u_m'(\xi)\|_V^2 + \|u_m''(\xi)\|_H^2\Big)d\xi. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we obtain

$$\begin{aligned} &\|u_m''(t)\|_H^2 + \mu(t)\|u_m'(t)\|_V^2 + 2\lambda\int_0^t \|u_m''(\xi)\|_H^2 d\xi \\ &\le \|u_m''(0)\|_H^2 + \mu(0)\|u_{1m}\|_V^2 + \int_0^t \varphi_2(\xi)\Big(\|u_m''(\xi)\|_H^2 + \mu(\xi)\|u_m'(\xi)\|_V^2\Big)d\xi, \end{aligned} \tag{3.16}$$

where

$$\varphi_2(\xi) = \frac{|\mu'(\xi)|}{\mu(\xi)}\Big[2 + \lambda\Big(1 + \frac{1}{R^2m_0}\Big)\Big].$$

Now we are going to estimate $u_m''(0)$. Taking $t = 0$ and $w = u_m''(0)$ in the approximate equation (3.9) we obtain

$$\|u_m''(0)\|_H^2 = \mu(0)(\Delta u_{0m}, u_m''(0)) - \lambda(u_{1m}, u_m''(0)),$$

from here and the convergence (3.7) and (3.8) we conclude

$$\|u_m''(0)\|_H \le \mu(0)\|\Delta u_0\|_H + \lambda\|u_1\|_H. \tag{3.17}$$

The estimates (3.16), (3.17), the convergence (3.8) and Gronwall's inequality give us

$$\|u_m''(t)\|_H^2 + \mu(t)\|u_m'(t)\|_V^2 + 2\lambda\int_0^t \|u_m''(\xi)\|_H^2 d\xi \le R_2^2 \exp\Big(\int_0^t \varphi_2(\xi)d\xi\Big), \tag{3.18}$$

for all $t \in [0, T]$, where $R_2^2 = \mu(0)\|\Delta u_0\|_H + \lambda\|u_1\|_H + \mu(0)\|u_1\|_V^2$.

Estimate III: Differentiating (3.9) twice with respect to t and putting $w = u_m'''(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2) + 2\mu'(t)((u_m'(t), u_m'''(t))) \\ &+ \mu''(t)((u_m(t), u_m'''(t))) + \lambda\|u_m'''(t)\|_H^2 \\ &= \frac{\mu'(t)}{2}\|u_m''(t)\|_V^2. \end{aligned} \tag{3.19}$$

Taking $w = u_m'''(t)$ in (3.9) we have

$$((u_m(t), u_m'''(t))) = -\frac{1}{\mu(t)}[(u_m''(t), u_m'''(t)) + \lambda(u_m'(t), u_m'''(t))]. \tag{3.20}$$

Differentiating (3.9) with respect to t and putting $w = u_m'''(t)$ we infer

$$\begin{aligned} &((u_m'(t), u_m'''(t))) \\ &= -\frac{1}{\mu(t)}[\|u_m'''(t)\|_H^2 + \mu'(t)((u_m(t), u_m'''(t))) + \lambda(u_m''(t), u_m'''(t))]. \end{aligned} \tag{3.21}$$

Substituting (3.20) in (3.21) we have

$$\begin{aligned} ((u_m'(t), u_m'''(t))) = -\frac{1}{\mu(t)}\Big[&\|u_m'''(t)\|_H^2 + \lambda(u_m''(t), u_m'''(t)) \\ &- \frac{\mu'(t)}{\mu(t)}\Big((u_m''(t), u_m'''(t)) + \lambda(u_m'(t), u_m'''(t))\Big)\Big]. \end{aligned} \tag{3.22}$$

Using (3.20) and (3.22) in (3.19) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2) - 2\frac{\mu'(t)}{\mu(t)}\Big[\|u_m'''(t)\|_H^2 + \lambda(u_m''(t), u_m'''(t)) \\ &- \frac{\mu'(t)}{\mu(t)}\Big((u_m''(t), u_m'''(t)) + \lambda(u_m'(t), u_m'''(t))\Big)\Big] \\ &- \frac{\mu''(t)}{\mu(t)}\Big((u_m''(t), u_m'''(t)) + \lambda(u_m'(t), u_m'''(t))\Big) + \lambda\|u_m'''(t)\|_H^2 \\ &= \frac{\mu'(t)}{2}\|u_m''(t)\|_V^2. \end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned} &\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2 + 2\lambda\int_0^t \|u_m'''(\xi)\|_H^2 d\xi \\ &= \|u_m'''(0)\|_H^2 + \mu(0)\|u_m''(0)\|_V^2 + 4\int_0^t \frac{\mu'(\xi)}{\mu(\xi)}\|u_m'''(\xi)\|_H^2 d\xi \\ &+ \int_0^t \frac{\mu'(\xi)}{\mu(\xi)}\mu(\xi)\|u_m''(\xi)\|_V^2 d\xi - 4\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^2(u_m''(\xi), u_m'''(\xi))d\xi \\ &- 4\lambda\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^2(u_m'(\xi), u_m'''(\xi))d\xi + 4\lambda\int_0^t \frac{\mu'(\xi)}{\mu(\xi)}(u_m''(\xi), u_m'''(\xi))d\xi \\ &+ 2\int_0^t \frac{\mu''(\xi)}{\mu(\xi)}(u_m''(\xi), u_m'''(\xi))d\xi + 2\lambda\int_0^t \frac{\mu''(\xi)}{\mu(\xi)}(u_m'(\xi), u_m'''(\xi))d\xi. \end{aligned} \tag{3.23}$$

We observe that

$$\begin{aligned} &-4\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^2 (u''_m(\xi), u'''_m(\xi))d\xi \\ &\leq 2\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^2 \Big(\frac{\mu(\xi)\|u''_m(\xi)\|_V^2}{\mu(\xi)R^2} + \|u'''_m(\xi)\|_H^2\Big)d\xi; \end{aligned} \tag{3.24}$$

$$\begin{aligned} &-4\lambda\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^2 (u'_m(\xi), u'''_m(\xi))d\xi \\ &\leq \frac{\lambda}{2}\int_0^t \|u'_m(\xi)\|_H^2 d\xi + 8\lambda\int_0^t \Big(\frac{\mu'(\xi)}{\mu(\xi)}\Big)^4 \|u'''_m(\xi)\|_H^2 d\xi; \end{aligned} \tag{3.25}$$

$$\begin{aligned} &4\lambda\int_0^t \frac{\mu'(\xi)}{\mu(\xi)} (u''_m(\xi), u'''_m(\xi))d\xi \\ &\leq 2\lambda\int_0^t \frac{|\mu'(\xi)|}{\mu(\xi)} \Big(\frac{\mu(\xi)\|u''_m(\xi)\|_V^2}{\mu(\xi)R^2} + \|u'''_m(\xi)\|_H^2\Big)d\xi; \end{aligned} \tag{3.26}$$

$$\begin{aligned} &2\int_0^t \frac{\mu''(\xi)}{\mu(\xi)} (u''_m(\xi), u'''_m(\xi))d\xi \\ &\leq \int_0^t \frac{|\mu''(\xi)|}{\mu(\xi)} \Big(\frac{\mu(\xi)\|u''_m(\xi)\|_V^2}{\mu(\xi)R^2} + \|u'''_m(\xi)\|_H^2\Big)d\xi; \end{aligned} \tag{3.27}$$

$$\begin{aligned} &2\lambda\int_0^t \frac{\mu''(\xi)}{\mu(\xi)} (u'_m(\xi), u'''_m(\xi))d\xi \\ &\leq \frac{\lambda}{2}\int_0^t \|u'_m(\xi)\|_H^2 d\xi + 2\lambda\int_0^t \Big(\frac{\mu''(\xi)}{\mu(\xi)}\Big)^2 \|u'''_m(\xi)\|_H^2 d\xi. \end{aligned} \tag{3.28}$$

Adding (3.10) with (3.23) and using the estimates (3.24)–(3.28), we infer

$$\begin{aligned} &\|u'''_m(t)\|_H^2 + \mu(t)\|u''_m(t)\|_V^2 + \|u'_m(t)\|_H^2 \\ &+ \mu(t)\|u_m(t)\|_V^2 + 2\lambda\int_0^t \|u'''_m(\xi)\|_H^2 d\xi + 2\lambda\int_0^t \|u'_m(\xi)\|_H^2 d\xi \\ &\leq \|u'''_m(0)\|_H^2 + \mu(0)\|u''_m(0)\|_V^2 + \|u_{1m}\|_H^2 + \mu(0)\|u_{0m}\|_V^2 \\ &\quad + \int_0^t \Big[2(2+\lambda)\frac{|\mu'(\xi)|}{\mu(\xi)} + 2\frac{|\mu'(\xi)|^2}{\mu(\xi)^2} + \frac{|\mu''(\xi)|}{\mu(\xi)} + 2\lambda\frac{|\mu''(\xi)|^2}{\mu(\xi)^2} \\ &\quad + 8\lambda\frac{|\mu'(\xi)|^4}{\mu(\xi)^4}\Big]\|u'''_m(\xi)\|_H^2 d\xi \\ &\quad + \int_0^t \Big(\frac{|\mu'(\xi)|}{\mu(\xi)} + 2\frac{|\mu'(\xi)|^2}{R^2\mu(\xi)^3} + 2\frac{\lambda|\mu'(\xi)|}{R^2\mu(\xi)^2} + \frac{|\mu''(\xi)|}{R^2\mu(\xi)^2}\Big)\mu(\xi)\|u''_m(\xi)\|_V^2 d\xi \\ &\quad + 2\int_0^t \frac{|\mu'(\xi)|}{\mu(\xi)}\mu(\xi)\|u_m(\xi)\|_V^2 d\xi. \end{aligned}$$

We consider

$$\begin{aligned} \varphi_3(\xi) &= 2(2+\lambda)\frac{|\mu'(\xi)|}{\mu(\xi)} + 2\frac{|\mu'(\xi)|^2}{\mu(\xi)^2} + \frac{|\mu''(\xi)|}{\mu(\xi)} + 2\lambda\frac{|\mu''(\xi)|^2}{\mu(\xi)^2} \\ &\quad + 8\lambda\frac{|\mu'(\xi)|^4}{\mu(\xi)^4} + \frac{|\mu'(\xi)|}{\mu(\xi)} + 2\frac{|\mu'(\xi)|^2}{R^2\mu(\xi)^3} + 2\frac{\lambda|\mu'(\xi)|}{R^2\mu(\xi)^2} + \frac{|\mu''(\xi)|}{R^2\mu(\xi)^2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2 + \|u_m'(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2 \\ &+ 2\lambda\int_0^t \|u_m'''(\xi)\|_H^2 d\xi + 2\lambda\int_0^t \|u_m'(\xi)\|_H^2 d\xi \\ &\le \|u_m'''(0)\|_H^2 + \mu(0)\|u_m''(0)\|_V^2 + \|u_{1m}\|_H^2 + \mu(0)\|u_{0m}\|_V^2 \\ &\quad + \int_0^t \varphi_3(\xi)\Big(\|u_m'''(\xi)\|_H^2 + \mu(\xi)\|u_m''(\xi)\|_V^2 + \mu(\xi)\|u_m(\xi)\|_V^2\Big)d\xi. \end{aligned} \tag{3.29}$$

Now we estimate $\|u_m'''(0)\|_H$ and $\|u_m''(0)\|_V$. Differentiating (3.9) with respect to t and putting $t = 0$ and $w = u_m'''(0)$ we infer

$$\|u_m'''(0)\|_H^2 \le (\mu(0)\|\Delta u_{1m}\|_H + |\mu'(0)|\|\Delta u_{0m}\|_H + \lambda\|u_m''(0)\|_H)\|u_m'''(0)\|_H.$$

From this inequality, (3.7), (3.8) and (3.17) we obtain

$$\|u_m'''(0)\|_H \le \mu(0)\|\Delta u_1\|_H + |\mu'(0)|\|\Delta u_0\|_H + \lambda(\mu(0)\|\Delta u_0\|_H + \lambda\|u_1\|_H). \tag{3.30}$$

On the other hand, from the approximate equation we have

$$u_m''(0) - \mu(0)\Delta u_m(0) + \lambda u_m'(0) = 0 \quad \text{in } V_m.$$

Thus, using the convergence (3.7) and (3.8) we infer

$$\|u_m''(0)\|_V = \|\mu(0)\Delta u_m(0) - \lambda u_m'(0)\|_V \le \mu(0)\|\Delta u_0\|_V + \lambda\|u_1\|_V + \eta, \tag{3.31}$$

for some $\eta > 0$. From (3.29)–(3.31) we conclude that

$$\begin{aligned} &\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2 + \|u_m'(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2 \\ &+ 2\lambda\int_0^t \|u_m'''(\xi)\|_H^2 d\xi + 2\lambda\int_0^t \|u_m'(\xi)\|_H^2 d\xi \\ &\le R_3^2 + \int_0^t \varphi_3(\xi)\Big(\|u_m'''(\xi)\|_H^2 + \mu(\xi)\|u_m''(\xi)\|_V^2 + \mu(\xi)\|u_m(\xi)\|_V^2\Big)d\xi, \end{aligned}$$

where

$$\begin{aligned} R_3^2 &= \mu(0)\|\Delta u_1\|_H + |\mu'(0)|\|\Delta u_0\|_H + \lambda(\mu(0)\|\Delta u_0\|_H + \lambda\|u_1\|_H) \\ &\quad + \mu(0)^2\|\Delta u_0\|_V + \lambda\mu(0)\|u_1\|_V + \|u_1\|_H^2 + \mu(0)\|u_0\|_V^2 + \eta. \end{aligned}$$

The Gronwall Inequality allow us to infer that

$$\begin{aligned} &\|u_m'''(t)\|_H^2 + \mu(t)\|u_m''(t)\|_V^2 + \|u_m'(t)\|_H^2 + \mu(t)\|u_m(t)\|_V^2 \\ &\le R_3^2 \exp\Big(\int_0^T \varphi_3(\xi)d\xi\Big), \end{aligned} \tag{3.32}$$

for all $t \in [0, T]$.

Passage to the limit: Estimates (3.11), (3.18), and (3.32) yield a subsequence of $(u_m)_{m\in\mathbb{N}}$, which we still denote in the same way, and function u in the space $L^\infty_{\rm loc}(0,\infty;V)$ such that

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{\rm loc}(0,\infty;V), \\ u_m' &\overset{*}{\rightharpoonup} u' \quad \text{in } L^\infty_{\rm loc}(0,\infty;V), \\ u_m'' &\overset{*}{\rightharpoonup} u'' \quad \text{in } L^\infty_{\rm loc}(0,\infty;V), \\ u_m''' &\overset{*}{\rightharpoonup} u''' \quad \text{in } L^\infty_{\rm loc}(0,\infty;H). \end{aligned}$$

This convergence allow us to pass to limit in the approximate equation (3.9) and infer that (3.6) holds a.e. in $(0,T)$. Therefore, for a.e. $t\in[0,T]$, we have

$$-\Delta u(t)=-\frac{1}{\mu(t)}(u''(t)+\lambda u'(t)). \tag{3.33}$$

We observe the right hand side of (3.33) is in V which is a subset of $H^1(\mathbb{R}^n)$. As $u(t)\in V\subset H^1(\mathbb{R}^n)$ it follows, by elliptic regularity, that $u(t)\in H^3(\mathbb{R}^n)$, a.e. in $[0,T]$. Differentiating (3.33) we have

$$-\Delta u'(t)=-\frac{\mu(t)(u'''(t)+\lambda u''(t))-(u''(t)+\lambda u'(t))\mu'(t)}{(\mu(t))^2}, \tag{3.34}$$

As the right hand side of (3.34) is in $L^2(\mathbb{R}^n)$, we conclude, by elliptic regularity, that $u'\in L^\infty_{\rm loc}(0,\infty;W)$. The proof of (3.2), (3.3) and the uniqueness are standard. □

Proposition 3.2. *Suppose that* (3.4) *holds. Then for each $u_0\in W$ and $u_1\in V$ Problem* (3.1)–(3.3) *has a unique solution satifying*

$$u\in C^0([0,\infty);W)\cap C^1([0,\infty);V)\cap C^2([0,\infty);H). \tag{3.35}$$

Proof. Let $(u_0,u_1)\in W\times V$ be the initial data. As $W\cap H^3(\mathbb{R}^n)$ and W are dense in W and V, respectively, then there exists two sequences $(u_m^0)_{m\in\mathbb{N}}$ and $(u_m^1)_{m\in\mathbb{N}}$ in $W\cap H^3(\mathbb{R}^n)$ and W, respectively, such that

$$u_m^0\to u_0 \text{ in } W \quad\text{and}\quad u_m^1\to u_1 \text{ in } V,\text{ when } m\to\infty. \tag{3.36}$$

By Proposition 3.1, for each pair of initial data (u_m^0,u_m^1), there exists u_m solution of (3.1)–(3.3) in the class (3.5). Therefore

$$u_m''-\mu\Delta u_m+\lambda u_m'=0 \quad\text{in } L^\infty_{\rm loc}(0,\infty;V), \tag{3.37}$$
$$u_m'''-\mu'\Delta u_m-\mu\Delta u_m'+\lambda u_m''=0 \quad\text{in } L^\infty_{\rm loc}(0,\infty;H). \tag{3.38}$$

Let η,m be natural numbers. Define $v_m=u_\eta-u_m$, from (3.37) we have

$$v_m''-\mu\Delta v_m+\lambda v_m'=0 \quad\text{in } L^\infty_{\rm loc}(0,\infty;V). \tag{3.39}$$

Let $T>0$ be a real number arbitrarily fixed. Multiplying (3.39) by v_m' and integrating in $\mathbb{R}^n\times(0,T)$, we obtain

$$\|v_m'(t)\|_H^2+\mu(t)\|v_m(t)\|_V^2\le\|v_m'(0)\|_H^2+\mu(0)\|v_m(0)\|_V^2. \tag{3.40}$$

On the other hand, from (3.38) we obtain

$$v_m'''-\mu'\Delta v_m-\mu\Delta v_m'+\lambda v_m''=0 \quad\text{in } L^\infty_{\rm loc}(0,T;H). \tag{3.41}$$

Multiplying (3.41) by v_m'', integrating in $\mathbb{R}^n\times(0,T)$ and taking the same way of (3.16), we infer that

$$\begin{aligned}&\|v_m''(t)\|_H^2+\mu(t)\|v_m'(t)\|_V^2+2\lambda\int_0^t\|v_m''(\xi)\|_H^2d\xi\\&\le\|v_m''(0)\|_H^2+\mu(0)\|v_m'(0)\|_V^2+\int_0^t\varphi_2(\xi)\Big(\|v_m''(\xi)\|_H^2+\mu(\xi)\|v_m'(\xi)\|_V^2\Big)d\xi.\end{aligned}$$

This inequality, (3.39) and Gronwall's inequality allow us to infer

$$\begin{aligned}&\|v_m''(t)\|_H^2+\mu(t)\|v_m'(t)\|_V^2+2\lambda\int_0^t\|v_m''(\xi)\|_H^2d\xi\\&\le(2\mu(0)\|\Delta v_m(0)\|_H^2+2\lambda\|v_m'(0)\|_H^2+\mu(0)\|v_m'(0)\|_V^2)\exp\Big(\int_0^T\varphi_2(\xi)d\xi\Big),\end{aligned} \tag{3.42}$$

for all $t \in [0,T]$.

Therefore, for all $T > 0$, the convergence (3.36) and the estimates (3.40) and (3.42) give us that $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,T];V) \cap C^2([0,T];H)$. Thus, there exists $u \in C^1([0,T];V) \cap C^2([0,T];H)$ such that

$$u_m \to u \quad \text{in } C^1([0,T];V) \cap C^2([0,T];H).$$

The regularity $u \in C^0([0,T];W)$ is obtained by standard elliptic regularity argument (as the proof of Proposition 3.1). Passing to the limit in the equation (3.37) we conclude the proof. □

Remark 3.3. If $(u_0, u_1) \in W \times V$, then it is possible to get an existence result with the function μ less regular than (3.4). In fact, we can consider $\mu \in W^{1,1}_{\rm loc}(0,\infty)$ and to prove that there exists a function u in the class

$$u \in L^\infty_{\rm loc}(0,\infty;W),\;\; u' \in L^\infty_{\rm loc}(0,\infty;V), u'' \in L^\infty_{\rm loc}(0,\infty;H)$$

which is the unique solution of (3.1)–(3.3). The proof is analogous to estimate I and II of the Proposition 3.1. We will use this regularity in our next result.

Now, we prove the local existence result. We consider the assumption

$$M \in C^2(\mathbb{R}^+;\mathbb{R}^+); \quad M(s) \ge m_0 > 0, \quad \text{for all } s \in [0,\infty). \tag{3.43}$$

Theorem 3.4 (Local existence)**.** *Suppose that* (3.43) *holds. Then for each* $u_0 \in W$ *and* $u_1 \in V$ *there exists a value* $T_{\max} > 0$ *and a unique solution* $u : \mathbb{R}\times[0,T_{\max}) \to \mathbb{R}$ *of* (1.1) *satisfying*

$$u \in C^0([0,T_{\max});W) \cap C^1([0,T_{\max});V) \cap C^2([0,T_{\max});H)\,. \tag{3.44}$$

Proof. We will use the Banach contraction theorem. For each $T > 0$ and $\rho > 0$ we define the space

$$\begin{aligned} X_{\rho,T} = \big\{&u \in L^\infty(0,T;W);\; u' \in L^\infty(0,T;V);\; u'' \in L^\infty(0,T;H);\\ &\|u\|_{L^\infty(0,T;V)} + \|u'\|_{L^\infty(0,T;V)} \le \rho,\; u(0) = u_0,\; u'(0) = u_1\big\}\end{aligned}$$

endowed with the distance

$$d(u,v) = \|u - v\|_{L^\infty(0,T;V)} + \|u' - v'\|_{L^\infty(0,T;H)}.$$

We have that $X_{\rho,T}$ with $d(u,v)$ is a complete metric space. Given $v \in X_{\rho,T}$, we have that $\mu(t) := M(\|v(t)\|_V^2) \in W^{1,1}(0,T)$. Let z be the unique solution of

$$z'' - M(\|v(t)\|_V^2)\Delta z + \lambda z' = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{3.45}$$

$$z(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{3.46}$$

$$z'(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \tag{3.47}$$

Define $S : X_{\rho,T} \to \mathcal{H}$ by

$$S(v) = z,$$

where $\mathcal{H}$ is the set of solutions of (3.45)–(3.47) associated with v. Now we will proof that S maps $X_{\rho,T}$ into itself. In fact, we observe that

$$|\mu'(t)| = |2M'(\|v(t)\|_V^2)((v(t),v'(t)))| \le 2k\|v(t)\|_V\|v'(t)\|_V \le 2k\rho^2, \tag{3.48}$$

where $k = \max_{0\le s\le \rho^2} |M'(s)|$. Using the same arguments of (3.11) we have

$$m_0\|z_m(t)\|_V^2 \le R_1^2 \exp\Big(2\int_0^T \frac{|\mu'(\xi)|}{\mu(\xi)}d\xi\Big). \tag{3.49}$$

Combining (3.48) with (3.49) we obtain

$$m_0^{1/2}\|z_m(t)\|_V \le R_1 \exp\Big(\frac{2k\rho^2}{m_0}T\Big), \quad \text{for all } t\in[0,T]. \tag{3.50}$$

On the other hand, taking the same way of (3.18) we have

$$m_0\|z'(t)\|_V^2 \le R_2^2 \exp\Big(\int_0^T \frac{|\mu'(\xi)|}{\mu(\xi)}\Big[2+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]d\xi\Big), \tag{3.51}$$

for all $t\in[0,T]$. The estimates (3.48) and (3.51) allow us to infer

$$m_0^{1/2}\|z'(t)\|_V \le R_2 \exp\Big(\frac{k\rho^2 T}{m_0}\Big[2+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big), \tag{3.52}$$

for all $t\in[0,T]$. Since (3.50) and (3.52) hold, we have

$$\|z_m(t)\|_V + \|z'(t)\|_V \le 2\frac{R_1+R_2}{m_0^{1/2}} \exp\Big(\frac{k\rho^2 T}{m_0}\Big[3+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big),$$

for all $t\in[0,T]$. Choosing $\rho > 2\frac{R_1+R_2}{m_0^{1/2}}$ and $T < \ln\big(\frac{m_0^{1/2}\rho}{2(R_1+R_2)}\big)^{\frac{1}{\kappa}}$, where $\kappa = \frac{k\rho^2}{m_0}\Big[3+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]$, we conclude that

$$\|z_m(t)\|_V + \|z'(t)\|_V < \rho, \quad \text{for all } t\in[0,T],$$

therefore $S(X_{\rho,T})\subset X_{\rho,T}$.

Now we prove that S is a contraction. We consider $v_1, v_2 \in X_{\rho,T}$ and define $z_1 = S(v_1)$, $z_2 = S(v_2)$ and $\omega = z_1 - z_2$. Therefore,

$$z_1'' - M(\|v_1(t)\|_V^2)\Delta z_1 + \lambda z_1' = 0 \quad \text{in } \mathbb{R}^n\times(0,T), \tag{3.53}$$

$$z_2'' - M(\|v_2(t)\|_V^2)\Delta z_2 + \lambda z_2' = 0 \quad \text{in } \mathbb{R}^n\times(0,T), \tag{3.54}$$

$$\omega(x,0) = \omega'(x,0) = 0, \quad x\in\mathbb{R}^n. \tag{3.55}$$

Then equations (3.53) and (3.54) give us

$$\omega'' - M(\|v_1(t)\|_V^2)\Delta\omega + \lambda\omega' = [M(\|v_1(t)\|_V^2) - M(\|v_2(t)\|_V^2)]\Delta z_2 \tag{3.56}$$

in $\mathbb{R}^n\times(0,T)$. Multiplying (3.56) by ω' and integrating over $\mathbb{R}^n$ we obtain

$$\begin{aligned} &\frac12\frac{d}{dt}(\|\omega'(t)\|_H^2 + M(\|v_1(t)\|_V^2)\|\omega(t)\|_V^2) + \lambda\|\omega'(t)\|_H^2 \\ &= M'(\|v_1(t)\|_V^2)((v_1(t), v_1'(t)))\|\omega(t)\|_V^2 \\ &\quad + [M(\|v_1(t)\|_V^2) - M(\|v_2(t)\|_V^2)](\Delta z_2(t), \omega'(t)). \end{aligned} \tag{3.57}$$

Now, we are going to estimate the right hand side of (3.57). Since $v_1, v_2 \in X_{\rho,T}$ we have

$$M'(\|v_1(t)\|_V^2)((v_1(t), v_1'(t)))\|\omega(t)\|_V^2 \le k\rho^2\|\omega(t)\|_V^2. \tag{3.58}$$

From the mean value theorem, there exists $s^*\in\mathbb{R}^+$ between $\|v_1(t)\|_V^2$ and $\|v_2(t)\|_V^2$ such that

$$|M(\|v_1(t)\|_V^2) - M(\|v_2(t)\|_V^2)| \le |M'(s^*)|(\|v_1(t)\|_V^2 - \|v_2(t)\|_V^2).$$

As $v_1, v_2\in X_{\rho,T}$, we have that $s^*\le\rho^2$ and this implies that $|M'(s^*)|\le k$. Thus

$$|M(\|v_1(t)\|_V^2) - M(\|v_2(t)\|_V^2)| \le 2k\rho d(v_1,v_2). \tag{3.59}$$

As $M(\|v_2(t)\|_V^2)\Delta z_2(t) = z_2''(t) + \lambda z_2'(t)$, the estimate (3.11) and (3.18) give us

$$\begin{aligned} m_0^{1/2}\|\Delta z_2(t)\|_H &\le \|z_2''(t)\|_H + \lambda\|z_2'(t)\|_H \\ &\le 2(R_1+R_2)\exp\Big(\frac{k\rho^2 T}{m_0}\Big[2+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big). \end{aligned} \tag{3.60}$$

Integrating (3.57) from 0 to $t \le T$ and using (3.58)–(3.60) we obtain

$$\begin{aligned} &\frac{1}{2}(\|\omega'(t)\|_H^2 + m_0\|\omega(t)\|_V^2) + \lambda\int_0^t \|\omega'(\xi)\|_H^2 d\xi \\ &\le \frac{2k\rho^2}{m_0}\int_0^t \frac{m_0}{2}\|\omega(\xi)\|_V^2 d\xi + g(t), \end{aligned} \tag{3.61}$$

where

$$g(t) = 4k\rho d(v_1,v_2)(\frac{R_1+R_2}{m_0^{1/2}})\exp\Big(\frac{k\rho^2 T}{m_0}\Big[2+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big)\int_0^t \|\omega'(\xi)\|_H d\xi.$$

As g is increasing function, the Gronwall inequality and (3.61) allow us to infer

$$\begin{aligned} \frac{m_0}{2}\|\omega(t)\|_V^2 &\le 4k\rho d(v_1,v_2)(\frac{R_1+R_2}{m_0^{1/2}}) \\ &\quad\times \exp\Big(\frac{k\rho^2 T}{m_0}\Big[4+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big)\int_0^t \|\omega'(\xi)\|_H d\xi. \end{aligned} \tag{3.62}$$

Putting (3.62) in (3.61) and using the Brezis Lemma (see [4, page 157]) in the resultant equation we have

$$\|\omega'(t)\|_H \le k_1(T)Td(v_1,v_2) \tag{3.63}$$

where

$$\begin{aligned} k_1(T) &= 8k^2\rho^3 T(\frac{R_1+R_2}{m_0^{\frac{3}{2}}})\exp\Big(\frac{k\rho^2 T}{m_0}\Big[4+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big) \\ &\quad + 2k\rho(\frac{R_1+R_2}{m_0^{1/2}})\exp\Big(\frac{k\rho^2 T}{m_0}\Big[2+\lambda\Big(1+\frac{1}{R^2 m_0}\Big)\Big]\Big). \end{aligned}$$

Since (3.62) and (3.63) hold we obtain

$$\|\omega(t)\|_V \le \sqrt{\frac{2}{m_0}}k_1(T)Td(v_1,v_2). \tag{3.64}$$

Combining (3.63) and (3.64), we have

$$d(z_1,z_2) = \|\omega(t)\|_V + \|\omega'(t)\|_H \le \Big(1+\sqrt{\frac{2}{m_0}}\Big)k_1(T)Td(v_1,v_2), \tag{3.65}$$

choosing $T>0$ small enough we conclude that S is a contraction. This gives us the existence of a positive real number T_0 and a function $u:\mathbb{R}^n\times[0,T_0]\to R$ local solution of (1.1).

The next step will be to prove the existence of a maximal interval of existence. We consider the problem

$$U'' - M(\|U(t)\|_V^2)\Delta U + \lambda U' = 0 \quad \text{in } \mathbb{R}^n\times(0,T), \tag{3.66}$$

$$U(x,0) = u(x,T_0), \quad x\in\mathbb{R}^n, \tag{3.67}$$

$$U'(x,0) = u'(x,T_0), \quad x\in\mathbb{R}^n. \tag{3.68}$$

Therefore, the calculus above gives a real number $T_1 > 0$ and a unique solution, U, of (3.66)–(3.68) in the interval $[0, T_1]$. Define

$$V(t) = \begin{cases} u(t) & \text{if } 0 \le t \le T_0, \\ U(t - T_0) & \text{if } T_0 \le t \le T_0 + T_1, \end{cases}$$

then V is a solution of (1.1) in whole $[0, T_0 + T_1]$ with initial data u_0 and u_1.

On the other hand, if w_1 and w_2 are two solutions of (1.1) in any interval, $[0, T]$, of existence of solution, then defining $\overline{w} = w_1 - w_2$ and taking the same way of (3.53)–(3.65) we can infer that

$$\|\overline{w}(t)\|_V + \|\overline{w}'(t)\|_H \le C[\|\overline{w}(0)\|_V + \|\overline{w}'(0)\|_H] = 0,$$

which shows us that the local solution of (1.1) is unique.

Now, for each i, we define $I_i = [0, T_i] \subset \mathbb{R}$, where T_i is characterized has the positive real number such that $u_i : \mathbb{R}^n \times [0, T_i] \to \mathbb{R}$ is the local solution of (1.1). By the uniqueness proved above, we conclude that if $T_i < T_j$, then $u_i = u_j$ in $[0, T_i]$.

We will denote by J any index set. Define $\mathcal{C} = \{I_i;\ i \in J\} \cup \{\cup_{i \in J} I_i\}$ endowed with the order relation $A \preceq B \iff A \subset B$ or $A = B$. We observe that, if Θ is a subset of $\mathcal{C}$, which is totally ordered set with the order induced by $\mathcal{C}$, then the set $\Upsilon = \cup_{i \in J} I_i \in \mathcal{C}$ is an upper bound of Θ. Thus, by Zorn's Lemma there exists a maximal element, $I_{\max}$, of $\mathcal{C}$. By $\mathcal{C}$ definition this element is given by

$$I_{\max} = [0, T_{\max}] = \cup_{i \in J} [0, T_i].$$

Now, we conclude that the local solution has the regularity (3.44). Let $u \in X_{\rho,T}$ the local solution of (1.1) obtained above. We define $\mu(t) = M(\|u(t)\|_V^2)$ and consider v the unique solution of the linear problem

$$v'' - M(\|u(t)\|_V^2)\Delta v + \lambda v' = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \tag{3.69}$$

$$v(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{3.70}$$

$$v'(x, 0) = u_1(x), \quad x \in \mathbb{R}^n \tag{3.71}$$

given by Proposition 3.2. Then v as the regularity (3.35). Since the solution v is unique and u is a solution of (3.69)–(3.71), then $u = v$. Therefore u also has the regularity (3.35). □

To prove our global existence result we need the following additional assumption on M:

$$M'(s) \le \begin{cases} C_1 & \text{if } s \le 1, \\ C_2 s & \text{if } s > 1. \end{cases} \tag{3.72}$$

For each pair $(u_0, u_1) \in W \times V$ we set

$$a = 1 - \frac{6}{m_0}\|u_0\|_V^2 (C_1 + 3C_2),$$

$$b = \frac{8}{m_0}\left(C_1 + 6C_2\|u_0\|_V^2\right)\left(3\|u_1\|_V^2 + 2M(\|u_0\|_V^2)\|\Delta u_0\|_H^2\right),$$

$$c = \frac{32C_2}{m_0}\left(3\|u_1\|_V^2 + 2M(\|u_0\|_V^2)\|\Delta u_0\|_H^2\right)^2.$$

Theorem 3.5 (Global existence)**.** *Suppose that M satisfies* (3.43) *and* (3.72)*. Let $u_0 \in W$ be such that*

$$\|u_0\|_V^2 < \frac{m_0}{6(C_1 + 3C_2)}, \tag{3.73}$$

$u_1 \in V$, $\lambda > 0$ *and*

$$\lambda > \Big(\frac{b+\sqrt{b^2+4ac}}{2a}\Big)^{1/2}. \tag{3.74}$$

Then, (1.1) *has a unique solution with*

$$u \in C_b^0([0,\infty);W), \quad u' \in C_b^0([0,\infty);V), \quad u'' \in C_b^0([0,\infty);H)\,.$$

Remark 3.6. From (3.73) we conclude that $a > 0$. Define

$$\psi_0 = \frac{3}{2}\|u_1\|_V^2 + \frac{3\lambda^2}{8}\|u_0\|_V^2 + M(\|u_0\|_V^2)\|\Delta u_0\|_H^2.$$

It is not difficult to see that the inequality

$$\frac{8C_1}{\lambda}\psi_0 + \frac{64C_2}{\lambda^3}\psi_0^2 < \frac{\lambda m_0}{2} \tag{3.75}$$

is equivalent to

$$a\lambda^4 - b\lambda^2 - c > 0. \tag{3.76}$$

We know that (3.76) holds when (3.74) holds.

To prove our result of global existence we will start considering regular initial data u_0 and u_1, precisely,

$$u_0 \in W \cap H^3(\mathbb{R}^n) \quad \text{and} \quad u_1 \in W. \tag{3.77}$$

Therefore, Theorem 3.4 gives us the existence of a unique function u solution of (1.1). We consider $\mu(t) = M(\|u(t)\|_V^2)$, then $\mu \in W^{2,1}_{\rm loc}(0,\infty)$. Thus, the Proposition 3.1 gives us w, in the class (3.5), solution of (3.1)–(3.3) with μ defined above. But u also is a solution of (3.1)–(3.3). Then, by the uniqueness of solution, we conclude that $u = w$. Therefore u has the following regularity, which is gives by Proposition 3.1,

$$\begin{gathered} u \in L^\infty_{\rm loc}(0,\infty; W\cap H^3(\mathbb{R}^n)), \quad u' \in L^\infty_{\rm loc}(0,\infty;W), \\ u'' \in L^\infty_{\rm loc}(0,\infty;V), \quad u''' \in L^\infty_{\rm loc}(0,\infty;H). \end{gathered} \tag{3.78}$$

Proof of Theorem 3.5 with regular data. To extend the local solution given by Theorem 3.4 it is sufficient to prove that there exists a constant C such that

$$\|u'(t)\|_V^2 + \|\Delta u(t)\|_H^2 + \|u(t)\|_V^2 \le C, \tag{3.79}$$

for all $t \ge 0$. Multiplying (1.1) by $-\Delta u'$ and integrating over $\mathbb{R}^n$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u'(t)\|_V^2 + M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2) + \lambda\|u'(t)\|_V^2 \\ &= 2\|\Delta u(t)\|_H^2 M'(\|u(t)\|_V^2)((u(t),u'(t))). \end{aligned} \tag{3.80}$$

On the other hand, multiplying (1.1) by $-\Delta u$, integrating over $\mathbb{R}^n$ and multiplying the resultant equation by $\lambda/2$ we have

$$\frac{\lambda}{2}\Big[\frac{d}{dt}\Big(((u'(t),u(t))) + \frac{\lambda}{2}\|u(t)\|_V^2\Big) - \|u'(t)\|_V^2 + M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2\Big] = 0. \tag{3.81}$$

Combining (3.80) and (3.81) and using the assumption (3.72) (which implies that $M'(s) \le C_1 + C_2 s$, for all $s \ge 0$) we infer

$$\begin{aligned} &\frac{1}{2}\psi'(t) + \frac{\lambda}{2}\|u'(t)\|_V^2 + \frac{\lambda m_0}{2}\|\Delta u(t)\|_H^2 \\ &\le 2(C_1 + C_2\|u(t)\|_V^2)\|u(t)\|_V\|u'(t)\|_V\|\Delta u(t)\|_H^2, \end{aligned}$$

where

$$\psi(t)=\|u'(t)\|_V^2+M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2+\frac{\lambda}{2}((u'(t),u(t)))+\frac{\lambda^2}{4}\|u(t)\|_V^2.$$

From here,

$$\frac{1}{2}\psi'(t)+\frac{\lambda}{2}\|u'(t)\|_V^2+\Big(\frac{\lambda m_0}{2}-\varphi(t)\Big)\|\Delta u(t)\|_H^2\le 0, \tag{3.82}$$

where

$$\varphi(t)=2(C_1+C_2\|u(t)\|_V^2)\|u(t)\|_V\|u'(t)\|_V.$$

Now, we prove that

$$\varphi(t)<\frac{\lambda m_0}{2},\quad \text{for all } t\ge 0. \tag{3.83}$$

In fact, we observe the inequality

$$\frac{\lambda}{2}((u'(t),u(t)))\ge -\frac{1}{2}\|u'(t)\|_V^2-\frac{\lambda^2}{8}\|u(t)\|_V^2$$

implies

$$\psi(t)\ge\frac{1}{2}\|u'(t)\|_V^2+M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2+\frac{\lambda^2}{8}\|u(t)\|_V^2. \tag{3.84}$$

The estimate (3.84) gives us

$$\|u(t)\|_V^2\le\frac{8\psi(t)}{\lambda^2}\quad\text{and}\quad \|u'(t)\|_V^2\le 2\psi(t).$$

Using this estimate on φ definition, we have

$$\varphi(t)\le\frac{8C_1}{\lambda}\psi(t)+\frac{64C_2}{\lambda^3}\psi^2(t). \tag{3.85}$$

Since $\psi(0)\le\psi_0$ from (3.85) and (3.75) we obtain

$$\varphi(0)\le\frac{8C_1}{\lambda}\psi_0+\frac{64C_2}{\lambda^3}\psi_0^2<\frac{\lambda m_0}{2}.$$

Suppose that (3.83) is not true. As the function $t\mapsto\varphi(t)$ is continuous, there exists $t^*>0$ such that

$$\varphi(t)<\frac{\lambda m_0}{2},\quad \text{for all } t\in[0,t^*),\ \text{and } \varphi(t^*)=\frac{\lambda m_0}{2}. \tag{3.86}$$

Integrating (3.82) from 0 to t^* we obtain

$$\psi(t^*)<\psi_0. \tag{3.87}$$

From (3.75), (3.85) and (3.87) we conclude that $\varphi(t^*)<\frac{\lambda m_0}{2}$ which is a contraction with (3.86). Combining (3.82)–(3.84) we can conclude (3.79). □

Now, we prove the theorem with u_0 and u_1 less regular than (3.77).

Proof of Theorem 3.5. Let $u_0\in W$ and $u_1\in V$ be a couple of initial data. It is sufficient to prove that there exists a positive constant C such that

$$\|u'(t)\|_V^2+\|\Delta u(t)\|_H^2+\|u(t)\|_V^2\le C, \tag{3.88}$$

for all $t\ge 0$. As $W\cap H^3(\mathbb{R}^n)$ and W are dense in W and V, respectively, we can use the same arguments of the proof of the Proposition 3.2 and conclude that there exists a sequence $(u_m)_{m\in\mathbb{N}}$ of regular solutions (in the class (3.78)) such that

$$u_m\to u \text{ in } C^1([0,T];V)\cap C^2([0,T];H)\quad\text{and}\quad \Delta u_m\to\Delta u \text{ in } C^0([0,T];H).$$

This gives us

$$\|u_m'(t)\|_V^2 + \|\Delta u_m(t)\|_H^2 + \|u_m(t)\|_V^2 \to \|u'(t)\|_V^2 + \|\Delta u(t)\|_H^2 + \|u(t)\|_V^2, \quad (3.89)$$

as $m \to \infty$.

On the other hand, from the proof with regular data, for all $m \in \mathbb{N}$, we have

$$\|u_m'(t)\|_V^2 + \|\Delta u_m(t)\|_H^2 + \|u_m(t)\|_V^2 \le C. \quad (3.90)$$

Combining (3.89) with (3.90) we conclude (3.88). □

4. Exponential decay

To state our stability result we will use the know lemma due Nakao (see [29]).

Lemma 4.1 (Nakao). *Let $\varphi(t)$ be a bounded non negative function on $[0,\infty)$ satisfying*

$$\sup_{t\le\tau\le t+1} \varphi(\tau) \le C(\varphi(t) - \varphi(t+1)) + h(t)$$

where $C > 0$ is a constant and h is a non negative function satisfying $h(t) \le r_0 \exp(-s_0 t)$, for all $t \ge 0$, r_0, s_0 are positive constants. Then there exist positive constants r_1 and s_1 such that

$$\varphi(t) \le r_1 \exp(-s_1 t).$$

Let u the solution of (1.1) given by the Theorem 3.5. Define the weak energy by

$$E_w(t) = \frac{1}{2}\left(\|u'(t)\|_H^2 + \overline{M}(\|u(t)\|_V^2)\right) \quad (4.1)$$

where

$$\overline{M}(s) = \int_0^s M(\xi)d\xi.$$

Theorem 4.2 (Weak energy decay). *Under the assumptions of Theorem 3.5, suppose that $M'(s) \ge 0$, for all $s \ge 0$. There exist positive constants r_2 and s_2 such that*

$$E_w(t) \le r_2 \exp(-s_2 t), \quad \text{for all } t \ge 0. \quad (4.2)$$

Proof. Multiplying (3.6) by u' and integrating over $\mathbb{R}^n$ we have

$$E_w'(t) = -\lambda\|u'(t)\|_H^2 < 0, \quad (4.3)$$

thus E_w is a decreasing function. Integrating (4.2) from t to $t+1$ we obtain

$$\lambda \int_t^{t+1} \|u'(\xi)\|_H^2 d\xi = E_w(t) - E_w(t+1) := F^2(t). \quad (4.4)$$

By the mean value theorem for integrals, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\frac{\|u'(t_1)\|_H^2}{4} = \int_t^{t+\frac{1}{4}} \|u'(\xi)\|_H^2 d\xi \quad \text{and} \quad \frac{\|u'(t_2)\|_H^2}{4} = \int_{t+\frac{3}{4}}^{t+1} \|u'(\xi)\|_H^2 d\xi. \quad (4.5)$$

From (4.4) and (4.5) we obtain

$$\|u'(t_1)\|_H^2 + \|u'(t_2)\|_H^2 \le \frac{4}{\lambda}F^2(t). \quad (4.6)$$

From the definition (4.1) we infer

$$\|u(t)\|_H^2 \le \frac{2}{m_0 R^2} E_w(t). \quad (4.7)$$

On the other hand, multiplying (1.1) by u and integrating over $\mathbb{R}^n \times (t_1, t_2)$ we have

$$\begin{aligned}\int_{t_1}^{t_2} M(\|u(\xi)\|_V^2)\|u(\xi)\|_V^2 d\xi &= (u'(t_1), u(t_1)) - (u'(t_2), u(t_2)) \\ &\quad + \int_{t_1}^{t_2} \|u'(\xi)\|_H^2 d\xi - 2\lambda \int_{t_1}^{t_2} (u'(\xi), u(\xi)) d\xi.\end{aligned} \tag{4.8}$$

Now, we estimate each term of the right hand side of (4.8). Let $\varepsilon > 0$ be an arbitrary real number fixed. From (4.4), (4.6) and (4.7) we have

$$|(u'(t_i), u(t_i))| \le \frac{2}{\lambda\varepsilon} F^2(t) + \frac{\varepsilon}{m_0 R^2} E_w(t), \quad \text{for } i = 1, 2; \tag{4.9}$$

$$\int_{t_1}^{t_2} \|u'(\xi)\|_H^2 d\xi \le \frac{1}{\lambda} F^2(t); \tag{4.10}$$

$$2\lambda \int_{t_1}^{t_2} (u'(\xi), u(\xi)) d\xi \le \frac{1}{\varepsilon} F^2(t) + \frac{2\lambda\varepsilon}{m_0 R^2} E_w(t). \tag{4.11}$$

Combining (4.8)–(4.11) we obtain

$$\int_{t_1}^{t_2} M(\|u(\xi)\|_V^2)\|u(\xi)\|_V^2 d\xi \le \Big(\frac{4}{\lambda\varepsilon} + \frac{1}{\lambda} + \frac{1}{\varepsilon}\Big) F^2(t) + \frac{2\varepsilon}{m_0 R^2}(1+\lambda) E_w(t). \tag{4.12}$$

Since

$$E_w(t) \le \frac{1}{2}(\|u'(t)\|_H^2 + M(\|u(t)\|_V^2)\|u(t)\|_V^2),$$

(4.4) and (4.12) allow us to infer

$$\int_{t_1}^{t_2} E_w(\xi) d\xi \le \Big(\frac{4}{\lambda\varepsilon} + \frac{3}{2\lambda} + \frac{1}{\varepsilon}\Big) F^2(t) + \frac{2\varepsilon}{m_0 R^2}(1+\lambda) E_w(t).$$

From this and by the mean value theorem for integrals, there exists $t^* \in [t_1, t_2]$ such that

$$E_w(t^*) \le 2\int_{t_1}^{t_2} E_w(\xi) d\xi \le 2\Big(\frac{4}{\lambda\varepsilon} + \frac{3}{2\lambda} + \frac{1}{\varepsilon}\Big) F^2(t) + \frac{2\varepsilon}{m_0 R^2}(1+\lambda) E_w(t). \tag{4.13}$$

Integrating (4.3) from t to t^* and using (4.13), we have

$$E_w(t) \le 2\Big(\frac{8}{\lambda\varepsilon} + \frac{3}{2\lambda} + \frac{1}{\varepsilon} + \frac{1}{2}\Big) F^2(t) + \frac{2\varepsilon}{m_0 R^2}(1+\lambda) E_w(t),$$

taking $\varepsilon > 0$ small enough, we conclude that there exist a positive constant $C > 0$ such that

$$E_w(t) \le C F^2(t),$$

this and Nakao's Lemma give (4.2). $\square$

To prove our result of exponential decay of strong energy we start by considering (as in Theorem 3.5) regular initial data u_0 and u_1, precisely,

$$u_0 \in W \cap H^3(\mathbb{R}^n) \quad \text{and} \quad u_1 \in W. \tag{4.14}$$

Therefore, the solution u is in the class

$$\begin{gathered} u \in L^\infty_{\rm loc}(0,\infty; W \cap H^3(\mathbb{R}^n)), \quad u' \in L^\infty_{\rm loc}(0,\infty; W), \\ u'' \in L^\infty_{\rm loc}(0,\infty; V), \quad u''' \in L^\infty_{\rm loc}(0,\infty; H). \end{gathered} \tag{4.15}$$

Let u_0 and u_1 with the regularity (4.14) and u the solution of (1.1), given by Theorem 3.5, with the regularity (4.15). We define the strong energy associated to (1.1) by

$$E_s(t) = \frac{1}{2}(\|u'(t)\|_V^2 + M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2).$$

Now we can establish our second decay result:

Theorem 4.3 (Strong energy decay). *Under the assumptions of Theorem 4.2 suppose that*(4.14) *holds. Then there exist positive constants r_3 and s_3 such that*

$$E_s(t) \le r_3 \exp(-s_3 t), \quad \text{for all } t \ge 0. \tag{4.16}$$

Proof. Multiplying (1.1) by $-\Delta u'$ and integrating over $\mathbb{R}^n$ we obtain

$$\frac{1}{2}E_s'(t) + \lambda\|u'(t)\|_V^2 = 2\|\Delta u(t)\|_H^2 M'(\|u(t)\|_V^2)((u(t), u'(t))). \tag{4.17}$$

From here and using the assumption (3.72), we obtain

$$\frac{1}{2}E_s'(t)+\lambda\|u'(t)\|_V^2 \le 2(C_1+C_2\|u(t)\|_V^2)\|u(t)\|_V\|u'(t)\|_V\|\Delta u(t)\|_H^2 := I(t). \tag{4.18}$$

Integrating over $[t, t+1]$ we have

$$\lambda\int_t^{t+1}\|u'(\xi)\|_V^2 d\xi \le E_s(t) - E_s(t+1) + \sup_{t\le\tau\le t+1} I(\tau) := D^2(t). \tag{4.19}$$

This and by mean value theorem we obtain $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\|_V \le \frac{1}{\sqrt{\lambda}}D(t), \quad \text{for } i = 1, 2. \tag{4.20}$$

Moreover, the mean value theorem gives us a $t^* \in [t_1, t_2]$ such that

$$E_s(t^*) \le 2\int_{t_1}^{t_2} E_s(\xi)d\xi. \tag{4.21}$$

Multiplying the equation (1.1) by $-\Delta u$ and integrating over $\mathbb{R}^n$ we have

$$\frac{d}{dt}((u'(t), u(t))) + \lambda((u'(t), u(t))) - \|u'(t)\|_V^2 + M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2 = 0. \tag{4.22}$$

From (4.19) and (4.20) we obtain

$$((u'(t_i), u(t_i))) \le CD(t)\sup_{t\le\tau\le t+1}\|u(\tau)\|_V, \quad \text{for } i = 1, 2; \tag{4.23}$$

$$\lambda\int_{t_1}^{t_2}((u'(t), u(t)))dt \le CD^2(t) + \sup_{t\le\tau\le t+1}\|u(\tau)\|_V^2; \tag{4.24}$$

$$\int_{t_1}^{t_2}\|u'(t)\|_V^2 dt \le CD^2(t). \tag{4.25}$$

Integrating (4.22) over the interval $[t_1, t_2]$ and using (4.23)–(4.25), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} M(\|u(\xi)\|_V^2)\|\Delta u(\xi)\|_H^2 d\xi \\ &\le CD^2(t) + CD(t)\sup_{t\le\tau\le t+1}\|u(\tau)\|_V + \sup_{t\le\tau\le t+1}\|u(\tau)\|_V^2. \end{aligned} \tag{4.26}$$

Integrating (4.17) from t to t^* and observing (4.18) and (4.21), we have

$$\begin{aligned}E_s(t) &= E_s(t^*) + \lambda\int_t^{t^*} \|u'(\xi)\|_V^2 d\xi - 2\int_t^{t^*} \|\Delta u(\xi)\|_H^2 M'(\|u(\xi)\|_V^2)((u(\xi),u'(\xi))d\xi \\ &\le C\int_t^{t+1} \|u'(\xi)\|_V^2 d\xi + 2\int_{t_1}^{t_2} M(\|u(\xi)\|_V^2)\|\Delta u(\xi)\|_H^2 d\xi + \int_{t_1}^{t_2} I(\xi)d\xi.\end{aligned}$$

This, (4.19) and (4.26) give us

$$E_s(t) \le C(D^2(t) + D(t)\sup_{t\le\tau\le t+1}\|u(\tau)\|_V + \sup_{t\le\tau\le t+1}\|u(\tau)\|_V^2 + \sup_{t\le\tau\le t+1} I(\tau)).$$

From this inequality and as

$$\|u(t)\|_V^2 \le \frac{2}{m_0}E_w(t),$$

we have

$$E_s(t) \le C(D^2(t) + E_w(t) + \sup_{t\le\tau\le t+1} I(\tau)). \tag{4.27}$$

We observe that

$$\begin{aligned}CI(\tau) &\le \Big[C\Big(C_1 + \frac{2C_2}{m_0}E_w(0)\Big)E_s(0)\Big]^2\|u(\tau)\|_V^2 + \frac{1}{4}\|u'(\tau)\|_V^2 \\ &\le CE_w(t) + \frac{1}{4}E_s(t).\end{aligned} \tag{4.28}$$

Since

$$D^2(t) = E_s(t) - E_s(t+1) + \sup_{t\le\tau\le t+1} I(\tau)$$

we can combine (4.27) and (4.28) and we conclude that

$$E_s(t) \le C(E_s(t) - E_s(t+1) + E_w(t)),$$

this inequality, (4.2) and Nakao's lemma imply (4.16). □

Remark 4.4. Theorem 4.3 can be proved for less regular initial data. In fact, let $(u_0,u_1)\in W\times V$ be a couple of initial data. As $W\cap H^3(\mathbb{R}^n)$ and W are dense in W and V, respectively, we can use the same arguments of the proof of the Proposition 3.2 and conclude that there exists a sequence $(u_m)_{m\in\mathbb{N}}$ of regular solutions (in the class (4.15)) such that

$$u_m \to u \text{ in } C^1([0,T];V)\cap C^2([0,T];H) \quad\text{and}\quad \Delta u_m \to \Delta u \text{ in } C^0([0,T];H).$$

This gives us that

$$\begin{aligned}&\frac{1}{2}(\|u_m'(t)\|_V^2 + M(\|u_m(t)\|_V^2)\|\Delta u_m(t)\|_H^2) \\ &\to \frac{1}{2}(\|u'(t)\|_V^2 + M(\|u(t)\|_V^2)\|\Delta u(t)\|_H^2) := E_s(t),\end{aligned} \tag{4.29}$$

when $m\to\infty$. The convergence (4.29) and (4.16) allow us to infer that

$$E_s(t) \le r_3\exp(-s_3 t), \quad \text{for all } t\ge 0.$$

5. Viscoelastic dissipation

In this section we give an overview on the proof of viscoelastic damping case, i.e., we will describe results concerning (1.4) and we show what are the main differences when the proofs are compared with the ones give in previews sections. When we compare with the frictional case, the main problem is into estimate III of proposition 3.1. If we try to take the same way, it will be necessary to differentiate μ three times, but to prove the local existence result we do not have regularity enough on solution to impose some assumption on μ'''. Therefore, the strategy was change the estimate III. Below, we describe what was our way to overcome the difficulties.

Suppose that the assumption (3.4) holds and let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be differentiable function such that $g' \in L^2(0, \infty)$, $g(0) > 0$ and $c_0 := m_0 - \int_0^\infty g(s)ds > 0$. Suppose that there exists a differentiable function, l, such that

$$g'(t) \leq -l(t)g(t), \quad \text{for all } t \geq 0, \tag{5.1}$$

$$\Big|\frac{l'(t)}{l(t)}\Big| \leq k, \quad l(t) > 0, \quad l'(t) \leq 0, \quad \text{for all } t > 0. \tag{5.2}$$

Proposition 5.1. *Suppose* (3.4), (5.1) *and* (5.2) *hold. Then for each* $u_0 \in W \cap H^3(\mathbb{R}^n)$ *and* $u_1 \in W$ *there exists a unique function* u *satisfying*

$$u \in L^\infty_{\rm loc}(0, \infty; W \cap H^3(\mathbb{R}^n)),\ u' \in L^\infty_{\rm loc}(0, \infty; W),$$
$$u'' \in L^\infty_{\rm loc}(0, \infty; V),\ u''' \in L^\infty_{\rm loc}(0, \infty; H),$$

and that is a solution of

$$\begin{gathered} u'' - \mu(t)\Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u'(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered}$$

Proof. Let $(w_j)_{j\in\mathbb{N}}$ be an orthonormal bases in $W \cap H^3(\mathbb{R}^n)$. For each $m \in \mathbb{N}$, we denote U_m the m-dimensional subspaces spanned by the first m vectors of $(w_j)_{j\in\mathbb{N}}$. Let $T > 0$ be any fixed positive number. From Ordinary Differential Equations Theory for each $m \in \mathbb{N}$ we can find $0 < T_m \leq T$, $u_m : \mathbb{R}^n \times [0, T_m] \to \mathbb{R}$ of the form

$$u_m(x, t) = \sum_{j=1}^m \rho_{jm}(t)w_j(x),$$

satisfying the approximate problem

$$\begin{gathered} (u''_m(t), w_j) + \mu(t)((u_m(t), w_j)) - \int_0^t g(t-s)((u_m(s), w_j))ds = 0; \\ u_m(0) = \sum_{i=1}^m u_0^i w_i \to u_0 \text{ in } W \cap H^3(\mathbb{R}^n), \quad u'_m(0) = \sum_{i=1}^m u_1^i w_i \to u_1 \text{ in } W. \end{gathered} \tag{5.3}$$

Estimate I and II: Taking the same way of estimate I and II of Proposition 3.1 and making usual calculus it is possible to prove that

$$\|u'_m(t)\|_H^2 + c_0\|u_m(t)\|_V^2 \leq R_4 \exp\Big(\frac{1}{2c_0}\int_0^t |\mu'(\xi)|\ d\xi\Big),$$

for all $t \in [0,T]$, where $R_4 = \|u_1\|_H^2 + \mu(0)\|u_0\|_V^2$, and

$$\|u_m''(t)\|_H^2 + \|u_m'(t)\|_V^2 + \|u_m(t)\|_V^2 \le R_5 \exp\Big(\int_0^t \phi_1(\xi)\, d\xi\Big), \tag{5.4}$$

for all $t \in [0,T]$, where ϕ_1 is a function that depends only of μ, μ', μ'' and g, and the constant R_5 depends only on the initial data, $\mu(0)$, c_0 and g.

Estimate III: From (5.3) we have

$$((u_m''(t), w)) + \mu(t)((u_m(t), w)) - \int_0^t g(t-s)((u_m(s), w))ds = 0$$

for all $w \in V$ and a.e. in $(0,T)$. Denoting by $\mathcal{D}'(Q)$, where $Q = \mathbb{R}^n \times (0,T)$, the space of distribution, we obtain

$$\Big\langle -\mu(\cdot)\Delta u_m(\cdot) + \int_0^{\cdot} g(\cdot - s)\Delta u_m(s), \theta\Big\rangle_{\mathcal{D}'(Q)\times\mathcal{D}(Q)} = \int_0^T \int_{\mathbb{R}^n} u_m''\theta\, dx\, dt,$$

for all $\theta \in \mathcal{D}(Q)$. As $u_m'' \in L^2(Q)$, we obtain

$$\Delta\Big(-\mu(t)u_m(t) + \int_0^t g(t-s)u_m(s)\, ds\Big) \in L^2(\mathbb{R}^n) \tag{5.5}$$

a.e. in $(0,T)$. Therefore,

$$u_m'' - \mu(t)\Delta u_m + \int_0^t g(t-s)\Delta u_m(s)\, ds = 0, \tag{5.6}$$

a.e. in $\mathbb{R}^n \times (0,T)$. For each $t \in (0,T)$ fixed, we consider the elliptic operator, $A_{(t)}$, defined by

$$A_{(t)}(u_m(t)) = \Delta\Big(-\mu(t)u_m(t) + \int_0^t g(t-s)u_m(s)\, ds\Big).$$

Then, from (5.5) and (5.6), for each $t \in (0,T)$ and $m \in \mathbb{N}$, we obtain

$$A_{(t)}(u_m(t)) = u_m''(t) \in L^2(\mathbb{R}^n).$$

From this and using elliptic regularity, we conclude that $u_m(t) \in H^2(\mathbb{R}^n)$ and

$$\|u_m(t)\|_{H^2(\mathbb{R}^n)} \le \|u_m''(t)\|_{L^2(\mathbb{R}^n)} \tag{5.7}$$

a.e. in $(0,T)$. Since $u \in H$, then $\hat{u}(\xi) = 0$ a.e. in $\|\xi\| \le R$, thus (5.4) and (5.7) allow us to conclude that

$$\|u_m(t)\|_W^2 \le R_5 \exp\Big(\int_0^t \phi_1(\xi)\, d\xi\Big), \tag{5.8}$$

for all $t \in [0,T]$.

Estimate IV: Now, we can take the derivative of (5.3) twice. In fact, we can use the same arguments used in estimate III of Proposition 3.1. It will generate the term

$$\int_0^t \mu''(\xi)((u_m(\xi), u_m'''(\xi)))d\xi$$

which can be estimate by as follows

$$\begin{aligned}\int_0^t \mu''(\xi)((u_m(\xi), u_m'''(\xi)))d\xi &= -\int_0^t \mu''(\xi)(\Delta u_m(\xi), u_m'''(\xi))d\xi \\ &\le C\int_0^t \|\Delta u_m(\xi)\|_H \|u_m'''(\xi)\|_H d\xi\end{aligned}$$

$$\leq C(\varepsilon) + \varepsilon \int_0^t \|u_m'''(\xi)\|_H^2 d\xi,$$

here ε is a positive constant which will be choose posteriorly. We observe that in last inequality we used the estimate (5.8). Therefore, choosing $\varepsilon > 0$ small enough, we can conclude that

$$\|u_m'''(t)\|_H^2 + \|u_m''(t)\|_V^2 + \|u_m'(t)\|_V^2 + \|u_m(t)\|_V^2 \leq R_6 \exp\Big(\int_0^t \phi_2(\xi)\ d\xi\Big),$$

for all $t \in [0, T]$, where ϕ_2 is a function that depends only on μ, μ', μ'' and g, and the constant R_6 depends only on the initial data, $\mu(0)$, c_0 and g. These estimate are sufficient for concluding Proposition 5.1. □

Remark 5.2. (a) Proposition 5.1 allows us to prove an existence result to linear problem analogous to Proposition 3.2. (b) It is possible to get a local existence result analogous to Theorem 3.4. For the proof it is necessary change the metric space by

$$\begin{aligned} X_{\rho,T} = \Big\{ & u \in L^\infty(0,T;W);\ u' \in L^\infty(0,T;V);\ u'' \in L^\infty(0,T;H); \\ & \|u\|_{L^\infty(0,T;W)} + \|u'\|_{L^\infty(0,T;V)} + \|u''\|_{L^\infty(0,T;H)} \leq \rho, \\ & u(0) = u_0,\ u'(0) = u_1 \Big\} \end{aligned}$$

endowed with the distance

$$d(u,v) = \|u - v\|_{L^\infty(0,T;V)} + \|u' - v'\|_{L^\infty(0,T;H)}.$$

(c) It is not difficult to prove that $\|u(t)\|_V + \|u'(t)\|_H \leq C$ a.e. $t > 0$. This allows to extend the local solution as an element of $\{v \in C([0,\infty);V) \cap C^1([0,\infty);H)\}$ and define the energy by

$$\begin{aligned} E_{\rm mem}(t) = & \frac{1}{2}\|u'(t)\|_H^2 + \frac{1}{2}\overline{M}(\|u(t)\|_V^2) - \frac{1}{2}\Big(\int_0^t g(s)ds\Big)\|\nabla u(t)\|_H^2 \\ & + \frac{1}{2}\int_0^t g(t-s)\|u(t) - u(s)\|_H^2 ds, \end{aligned}$$

for all $t \geq 0$. Using the same methodology of [22] it is possible to get general decay rates to the problem, i.e., $E_{\rm mem}(t) \leq c_1 \exp(-c_2 \int_{t_0}^t l(s)ds)$, for all $t \geq t_0$.

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