SELF-SIMILAR SOLUTIONS FOR A SUPERDIFFUSIVE HEAT EQUATION WITH GRADIENT NONLINEARITY

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ABSTRACT. This article studies the existence, stability, self-similarity and symmetries of solutions for a superdiffusive heat equation with superlinear and gradient nonlinear terms with initial data in new homogeneous Besov-Morrey type spaces. Unlike in previous works on such time-fractional partial differential equations of order $\alpha \in (1,2)$, we take non null initial velocities into consideration, where new difficulties arise from. We overcome them by developing an appropriate decomposition of the two-parametric Mittag-Leffler function to obtain Mikhlin-type estimates and obtain our existence theorem.

1. Introduction

Let $\Delta_x = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator, $u : \mathbb{R}^{1+N} \to \mathbb{R}$, and ∂_t^α be the Caputo's fractional derivative of order $1 < \alpha < 2$ (see subsection 2.2). In this article, we study the equation

$$\partial_t^\alpha u = \Delta_x u + \kappa_1 |\nabla_x u|^q + \kappa_2 |u|^{\rho-1} u, \quad \kappa_1 \neq 0, \ \kappa_2 \in \mathbb{R}, \tag{1.1}$$

subject to the initial data

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \tag{1.2}$$

where $q > 1$ and $\rho > 1$. Note that the rescaled function $u_\gamma(t,x) := \gamma^{\frac{2}{\rho-1}} u(\gamma^{\frac{2}{\alpha}} t, \gamma x)$ solves (1.1) with initial data

$$\varphi_\gamma(x) = \gamma^{\frac{2}{\rho-1}} \varphi(\gamma x), \quad \psi_\gamma(x) = \gamma^{\frac{2}{\rho-1}+\frac{2}{\alpha}} \psi(\gamma x), \tag{1.3}$$

provided that $q = \frac{2\rho}{\rho+1}$ and $u(t,x)$ solves (1.1)-(1.2). Hence, we obtain a *scaling map* of solutions,

$$u(t,x) \mapsto u_\gamma(t,x), \quad \text{for all } \gamma > 0, \tag{1.4}$$

and solutions invariant by (1.4) will be called *self-similar solutions*, that is,

$$u(t,x) = u_\gamma(t,x). \tag{1.5}$$

In the study of self-similar solutions, the natural candidates to be initial data are the homogeneous functions,

$$\varphi(\gamma x) = \gamma^{-\frac{2}{\rho-1}} \varphi(x), \quad \psi(\gamma x) = \gamma^{-\frac{2}{\rho-1}-\frac{2}{\alpha}} \psi(x).$$

In this work we are interested in existence of self-similar solutions to (1.1)-(1.2). For this purpose, we study (1.1)-(1.2) through its integral formulation

$$u(t,x) = G_{\alpha,1}(t)\varphi(x) + G_{\alpha,2}(t)\psi(x) + \mathcal{N}_\alpha(u)(t,x), \tag{1.6}$$

where

$$\widehat{G_{\alpha,j}(t)}f(\xi) = t^{j-1}E_{\alpha,j}(-4\pi^2 t^\alpha|\xi|^2)\widehat{f}(\xi), \quad j = 1, 2, f \in \mathcal{S}'(\mathbb{R}^N), \tag{1.7}$$

$$\mathcal{N}_\alpha(u) = \int_0^t G_{\alpha,1}(t-s)\int_0^s r_\alpha(s-\tau)\left(\kappa_2|u|^{\rho-1}u + \kappa_1|\nabla_x u|^q\right)d\tau ds, \tag{1.8}$$

$$r_\alpha(t) = t^{\alpha-2}/\Gamma(\alpha-1). \tag{1.9}$$

Hereafter a solution u will be understood as a distribution $u(t,\cdot)$ satisfying (1.6), for each $t > 0$.

The presence of the gradient requires suitable estimates in certain Sobolev-Morrey spaces $\mathcal{M}_{r,\mu}^1$ and this motivated us to study the problem in the space X_β of all bounded continuous functions $u : (0,\infty) \to \mathcal{M}_{r,\mu}$ endowed with the norm

$$\|u\|_{X_\beta} = \sup_{t>0} t^{\frac{\alpha}{2}+\beta}\|u(t)\|_{\mathcal{M}_{r,\mu}^1} + \sup_{t>0} t^\beta\|u(t)\|_{\mathcal{M}_{r,\mu}}, \tag{1.10}$$

where β, r and μ will be chosen later (see (3.1)). Assuming $\psi \not\equiv 0$ brings new difficulties because we need to obtain suitable estimates for two-parametric Mittag-Leffler function $E_{\alpha,2}(4\pi^2 t^\alpha|\xi|^2)$. More precisely, we develop an appropriate decomposition for Mittag-Leffler function to obtain a suitable estimate (see (4.10) and (4.14)) which enables us to introduce the space

$$\mathcal{I} = \{(\varphi,\psi) \in \mathcal{S}' \times \mathcal{S}'; (\varphi,\psi) \in D(\alpha,\beta) \times \tilde{D}(\alpha,\beta)\}, \tag{1.11}$$

where

$$D(\alpha,\beta) := \{\varphi \in \mathcal{S}' : G_{\alpha,1}(t)\varphi \in X_\beta\}, \quad \widetilde{D}(\alpha,\beta) := \{\psi \in \mathcal{S} : G_{\alpha,2}(t)\psi \in X_\beta\},$$

for all $t > 0$. Hence, applying Lemma 5.1 we obtain (see Remark 3.2-(B)) that $\mathcal{M}_{p,\mu} \times \mathcal{M}_{p,\mu}^{-2/\alpha} \subseteq D(\alpha,\beta) \times \widetilde{D}(\alpha,\beta)$. It is remarkable that the investigation of self-similarity and symmetries for (1.1)-(1.2) allows us to deal with following prototype functions

$$\varphi(x) = \epsilon_1|x|^{-\frac{2}{\rho-1}}, \quad \psi(x) = \epsilon_2|x|^{-\frac{2}{\rho-1}-\frac{2}{\alpha}},$$

which belong to $D(\alpha,\beta) \times \widetilde{D}(\alpha,\beta)$ but can be arbitrarily large in the space $L^2(\mathbb{R}^N) \times \dot{H}^{\frac{2}{\alpha}}(\mathbb{R}^N)$. See Remark 3.4-(A).

Our symmetry result, roughly speaking, says that if the initial data φ and ψ are invariant on the orthogonal group acting on $\mathbb{R}^N$ so the solution is. In particular, we show the existence of radial self-similar solutions (see Remark 3.6-(A)).

We point out that our results hold for $\alpha = 1$ and $\psi = 0$ and, in this case, the upper bound $(\gamma_2 - \gamma_1) + \frac{N-\mu}{p_1} - \frac{N-\mu}{p_2}$ in Lemma 5.1 can be removed. On the other hand, for $1 < \alpha < 2$ the Mikhlin theorem yields more restrictive constraints to Lemma 5.1 than the usual estimates for the heat semigroup in such a way that Theorem 3.1 cannot come near to $\alpha = 2$.

Now, let us to review some works. Fujita [5] remarked that the linear counterpart of equation (1.1) has similarities with wave and heat equations and presents certain qualitative properties which qualifies it as a reasonable interpolation between these equations. When $\kappa_2 = 0$, $\alpha = 1$ and $\psi = 0$, (1.1)-(1.2) turns into the viscous Hamilton-Jacobi equation. Using scaling technique, Ben-Artzi et al [3] found the number $r_c = \frac{N(q-1)}{2-q}$ and showed that it is a critical exponent for existence of

solutions in L^r. In particular, the problem is well-posed when $r \geq r_c$ and $1 < q < 2$. In Remark 3.2-(C) we provide an existence result for this problem in Morrey spaces $\mathcal{M}_{p, N-\frac{N}{r_c}p}$ which are strictly larger than the Lebesgue spaces, namely,

$$L^r(\mathbb{R}^N) \subsetneq \mathcal{M}_{p,\mu}(\mathbb{R}^N) \subsetneq D(1,\beta), \tag{1.12}$$

provided that $\frac{N}{r} = \frac{N-\mu}{p}$ and $p < r$. Our existence result is then compatible with [3, Theorem 2.1], in view of $1 < p \leq r_c \leq r$. In particular, the initial data taken in Theorem 3.1 is larger than those considered in [3].

Recently several authors have addressed the study of global existence, self-similarity, asymptotic self-similarity and radial symmetry of solutions for the semi-linear heat equation with gradient nonlinear terms. See e.g. [4, 7, 17, 18, 19]. In [17] it is assumed that φ belongs to homogeneous Besov space $\dot{B}_{r_1,\infty}^{-\beta_1}$ and

$$\|\varphi\|_{\dot{B}_{r_1,\infty}^{-\beta_1}} = \sup_{t>0} t^{\beta_1/2}\|e^{t\Delta}\varphi\|_{L^{r_1}(\mathbb{R}^N)} \leq \epsilon, \quad \beta_1 = \frac{2}{\rho-1} - \frac{N}{r_1}.$$

By employing the Gagliardo-Nirenberg inequality, the authors studied the existence and asymptotic behavior of global mild solutions. On the other hand, our functional approach enables us to control the gradient estimates without making use of the Gagliardo-Nirenberg inequality and allows us to deal with a larger class of functions space for initial data.

Let us now review some works concerning to (1.1)-(1.2) with $\psi = 0$ and $\kappa_1 = 0$. In [8] the authors established $L^p - L^q$ estimates for $\{G_{\alpha,1}(t)\}_{t\geq 0}$ and showed a blowup alternative and local well-posedness in $L^q(\mathbb{R}^N)$-framework for any $\varphi \in L^q(\mathbb{R}^N)$, where $q \geq \frac{N\alpha(\rho+1)}{2}$. Using Mikhlin-Hormander's type theorem on Morrey spaces, de Almeida and Ferreira [1] studied self-similarity, symmetry, antisymmetry and positivity of global solutions with small data $\varphi \in \mathcal{M}_{p,\mu}$, $\mu = N - \frac{2p}{\rho-1}$. In [2], the authors established existence, self-similarity, symmetries and asymptotic behavior of solutions in Besov-Morrey spaces $\mathcal{N}_{p,\mu,\infty}^\sigma$ and provided a maximal class of existence in the sense that there is no known results in $X \supsetneq \mathcal{N}_{p,\mu,\infty}^\sigma$. Indded, we notice that

$$\mathcal{M}_{p,\lambda} \subsetneq \mathcal{N}_{p,\mu,\infty}^\sigma \quad \text{and} \quad \dot{B}_{r,\infty}^k \subset \mathcal{N}_{p,\mu,\infty}^\sigma, \tag{1.13}$$

where $\frac{N-\lambda}{p} = -\sigma + \frac{N-\mu}{p} = -k + \frac{N}{r}$, $\sigma = \frac{N-\mu}{p} - \frac{2}{\rho-1}$, $k = \frac{N}{r} - \frac{2}{\rho-1}$ and $1 \leq p < r$. All spaces in (1.12) and (1.13) are invariant by scaling.

We still observe that problem (1.1)–(1.2) can be studied with a Fourier multiplier $\sigma(D)$ in place of Δ_x, where $|\sigma(\xi)| \leq C|\xi|^k$ due to estimates (4.10) and (4.14) into Propositions 4.2 and 4.3. Example of such an operator is the Riesz potential $(-\Delta_x)^{k/2}f = \mathcal{F}^{-1}|\xi|^k\mathcal{F}f$, where $\mathcal{F}$ denote the Fourier transform in $\mathcal{S}'$.

This manuscript is organized as follows. Some basic properties of the Sobolev-Morrey spaces and Mittag-Leffler functions are reviewed in Section 2. We state and make some remarks on our results in Section 3 and their proofs are performed in Section 6. Sections 4 and 5 are reserved to a careful study of the several estimates which are crucial to yield our results.

2. Preliminaries

In this section we review some well-known properties of the Morrey spaces and Sobolev-Morrey spaces, more details can be found in [9, 10, 12, 13, 15]. Also, we

obtain an integral equation which is formally equivalent to (1.1)-(1.2) in the lines
of [11].

2.1. **Sobolev-Morrey spaces.** Let $Q_r(x_0)$ be the open ball in $\mathbb{R}^N$ centered at
x_0 and with radius $r > 0$. Given two parameters $1 \leq p < \infty$ and $0 \leq \mu < N$,
the Morrey space $\mathcal{M}_{p,\mu} = \mathcal{M}_{p,\mu}(\mathbb{R}^N)$ is defined to be the set of all functions $f \in L^p(Q_r(x_0))$ such that

$$\|f\|_{\mathcal{M}_{p,\mu}} := \sup_{x_0 \in \mathbb{R}^n, \, r > 0} r^{-\frac{\mu}{p}} \|f\|_{L^p(Q_r(x_0))} < \infty,$$

which is a Banach space endowed with this norm. For $s \in \mathbb{R}$ and $1 \leq p < \infty$,
the homogeneous Sobolev-Morrey space $\mathcal{M}_{p,\mu}^s = (-\Delta_x)^{-s/2}\mathcal{M}_{p,\mu}$ is the Banach
space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ modulo polynomials $\mathcal{P}$ with N
variables. If $s < \frac{N-\mu}{p}$ and $p > 1$, from [9, Theorem 1.1] or [13], it holds that

$$\|f\|_{\mathcal{M}_{p,\mu}} \sim \Big\| \Big(\sum_{\nu \in \mathbb{Z}} |\mathcal{F}^{-1}\psi_\nu(\xi)\mathcal{F}f|^2 \Big)^{1/2} \Big\|_{\mathcal{M}_{p,\mu}},$$

where $\sim$ denotes norm equivalence and $\{\psi_\nu\}_{\nu \in \mathbb{Z}}$ is a homogeneous Littlewood-Paley
resolution of unity, that is,

$$\psi_\nu(\xi) = \phi_\nu(\xi) - \phi_{\nu-1}(\xi), \quad \phi_\nu(\xi) = \phi_0(\xi/2^\nu),$$

for $\phi_0 \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_0 = 1$ on the ball $Q_1(0)$ and $\operatorname{supp}\phi_0 \subset Q_2(0)$. In
particular, using (2.1) and that $|\xi|^s \sim 2^{s\nu}$ on the $\operatorname{supp}\psi_\nu(\xi) \subset \{\xi \in \mathbb{R}^N : 2^{\nu-1} < |\xi| < 2^{\nu+1}\}$, we obtain

$$
\begin{aligned}
\Big\| &\Big(\sum_{\nu \in \mathbb{Z}} |2^{s\nu}\mathcal{F}^{-1}\psi_\nu(\xi)\mathcal{F}f|^2 \Big)^{1/2} \Big\|_{\mathcal{M}_{p,\mu}} \\
&\sim \Big\| \Big(\sum_{\nu \in \mathbb{Z}} |\mathcal{F}^{-1}\psi_\nu(\xi)|\xi|^s\mathcal{F}f|^2 \Big)^{1/2} \Big\|_{\mathcal{M}_{p,\mu}} \\
&= \Big\| \Big(\sum_{\nu \in \mathbb{Z}} |2^{\nu N}\check{\psi}(2^\nu \cdot) * (|\xi|^s\widehat{f})^\vee|^2 \Big)^{1/2} \Big\|_{\mathcal{M}_{p,\mu}} \\
&\sim \|(|\cdot|^s\widehat{f})^\vee\|_{\mathcal{M}_{p,\mu}}.
\end{aligned}
\tag{2.1}
$$

Given $f \in \mathcal{M}_{p,\mu}^s$, the quantity (2.1) define two equivalent norms on Sobolev-Morrey
space, namely,

$$
\begin{aligned}
\|f\|_{\mathcal{M}_{p,\mu}^s} &= \|(|\cdot|^s\widehat{f})^\vee\|_{\mathcal{M}_{p,\mu}}, \\
\|f\|_{\mathcal{M}_{p,\mu}^s} &= \Big\| \Big(\sum_{\nu \in \mathbb{Z}} |2^{s\nu}\mathcal{F}^{-1}\psi_\nu(\xi)\mathcal{F}f|^2 \Big)^{1/2} \Big\|_{\mathcal{M}_{p,\mu}}.
\end{aligned}
\tag{2.2}
$$

It follows from Littlewood-Paley decomposition of the Lebesgue space $L^p(\mathbb{R}^N)$ and
homogeneous Sobolev space $H_p^s(\mathbb{R}^N)$ that $\mathcal{M}_{p,0} = L^p(\mathbb{R}^N)$ and $\mathcal{M}_{p,0}^s = \dot{H}_p^s(\mathbb{R}^N)$,
respectively. Also, Morrey and Sobolev-Morrey spaces present the following scaling

$$\|f(\gamma\cdot)\|_{\mathcal{M}_{p,\mu}} = \gamma^{-\frac{N-\mu}{p}}\|f\|_{\mathcal{M}_{p,\mu}} \quad \text{and} \quad \|f(\gamma\cdot)\|_{\mathcal{M}_{p,\mu}^s} = \gamma^{s-\frac{N-\mu}{p}}\|f\|_{\mathcal{M}_{p,\mu}^s},$$

where the exponents s and $s - \frac{N-\mu}{p}$ are called *scaling index* and *regularity index*,
respectively.

Lemma 2.1. *Suppose that $s \in \mathbb{R}$, $1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \mu_i < N$, $i = 1, 2, 3$.*

(i) *(Inclusion)* If $\frac{N-\mu_1}{p_1} = \frac{N-\mu_2}{p_2}$ and $p_1 \leq p_2$,

$$\mathcal{M}_{p_2,\mu_2} \subset \mathcal{M}_{p_1,\mu_1}. \tag{2.3}$$

(ii) *(Sobolev-type embedding)* Let $p_1 \leq p_2$,

$$\mathcal{M}_{p_1,\mu}^s \subset \mathcal{M}_{p_2,\mu}^{s-(\frac{N-\mu}{p_1}-\frac{N-\mu}{p_2})}. \tag{2.4}$$

(iii) *(Höder inequality)* Let $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$ and $\frac{\mu_3}{p_3} = \frac{\mu_2}{p_2} + \frac{\mu_1}{p_1}$. If $f_j \in \mathcal{M}_{p_j,\mu_j}$ with $j = 1,2$, then $f_1 f_2 \in \mathcal{M}_{p_3,\mu_3}$ and

$$\|f_1 f_2\|_{p_3,\mu_3} \leq \|f_1\|_{p_1,\mu_1} \|f_2\|_{p_2,\mu_2}. \tag{2.5}$$

Finally, notice that the following homogeneous functions, of degree $-d$ and $s-d$, belong to Morrey and Sobolev-Morrey spaces, respectively:

$$\rho_0(x) = Y_k(x)|x|^{-d-k} \in \mathcal{M}_{p,\mu} \quad \text{and} \quad \rho_s(x) = Y_k(x)|x|^{s-d-k} \in \mathcal{M}_{p,\mu}^s, \tag{2.6}$$

where $Y_k(x) \in L^p(\mathbb{S}^{N-1})$ is a harmonic homogeneous polynomial of degree k, $\mu = N - dp$, $0 < d - s < N$ and $1 < p < N/d$. Indeed, using [20, Theorem 4.1] we obtain $\widehat{\rho}_s(\xi) = \gamma_{k,s} Y_k(\xi)|\xi|^{d-s-k-N}$ provided $0 < d-s < N$, where $\gamma_{k,s}$ is a positive constant. It follows from (2.2) that

$$\|\rho_s\|_{\mathcal{M}_{p,\mu}^s} = \Big\|\Big(\sum_{\nu=-\infty}^{+\infty} |2^{s\nu} \mathcal{F}^{-1}\psi_\nu(\xi)\gamma_{k,s}Y_k(\xi)|\xi|^{d-s-k-N}|^2 \Big)^{1/2}\Big\|_{\mathcal{M}_{p,\mu}}$$

$$\sim \Big\|\Big(\sum_{\nu=-\infty}^{+\infty} |\mathcal{F}^{-1}\psi_\nu(\xi)|\xi|^s \gamma_{k,s}Y_k(\xi)|\xi|^{d-s-k-N}|^2 \Big)^{1/2}\Big\|_{\mathcal{M}_{p,\mu}}$$

$$= \|\rho_0\|_{\mathcal{M}_{p,\mu}},$$

which is finite. In fact, polar coordinates in $\mathbb{R}^N$ and homogeneity of $Y_k(x) \in L^p(\mathbb{S}^{N-1})$ yield

$$\|\rho_0\|_{L^p(Q_r)}^p = \int_{\mathbb{S}^{N-1}} |Y_k(x')|^p \int_0^r t^{N-dp-1} dt \, d\sigma(x') = \|Y_k\|_{L^p(\mathbb{S}^{N-1})}^p \, r^\mu,$$

where $\mu = N - dp$, $1 < p < N/d$.

2.2. Duhamel formula.

We consider the partial fractional differential equation

$$\partial_t^\alpha u(t,x) = \Delta_x u(t,x) - f(t,x), \quad x \in \mathbb{R}^N, \, t > 0,$$

$$u(t,x)\big|_{t=0} = \varphi(x), \quad \frac{\partial}{\partial t}u(t,x)\big|_{t=0} = \psi(x), \tag{2.7}$$

for $\alpha \in (1,2)$ and ∂_t^α stands for partial fractional derivative given by

$$\partial_t^\alpha f(t,x) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial_s^m f(s,x)}{(t-s)^{\alpha+1-m}} ds, \quad m-1 < \alpha \leq m, \, m \in \mathbb{N}.$$

Formally, applying the Fourier transform in (2.7), we obtain the fractional ordinary differential equation

$$\partial_t^\alpha \widehat{u}(t,\xi) + 4\pi^2 |\xi|^2 \widehat{u}(t,\xi) = \widehat{f}(t,\xi),$$

$$\widehat{u}(t,\xi)|_{t=0} = \widehat{\varphi}(\xi), \quad \partial_t \widehat{u}(t,\xi)\big|_{t=0} = \widehat{\psi}(\xi)$$

which, by [11, Example 4.10], is equivalent to

$$\widehat{u}(t,\xi) = E_{\alpha,1}(-4\pi^2 t^\alpha |\xi|^2)\widehat{\varphi}(\xi) + t E_{\alpha,2}(-4\pi^2 t^\alpha |\xi|^2)\widehat{\psi}(\xi)$$
$$+ \int_0^t E_{\alpha,1}(-4\pi^2(t-s)^\alpha |\xi|^2) \int_0^s r_\alpha(s-\tau)\widehat{f}(\tau,\xi)d\tau ds, \tag{2.8}$$

where $E_{\alpha,\beta}(z)$ denotes the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{and} \quad E_\alpha(z) := E_{\alpha,1}(z), \text{ for all } \alpha, \beta > 0. \tag{2.9}$$

Hence, in original variables, we have

$$u(t,x) = G_{\alpha,1}(t)\varphi(x) + G_{\alpha,2}(t)\psi(x) + \mathcal{N}_\alpha(u)(t,x),$$

where $\widehat{G_{\alpha,j}(t)}f(\xi)$ is defined by (1.7) and $\mathcal{N}_\alpha$ is defined by (1.8).

Note that $G_{2,2}(t)$ is the wave group $\left(\frac{\sin(4\pi^2 t|\xi|)}{4\pi^2 t|\xi|}\right)^\vee$, $G_{2,1}(t) = \left(\cos(4\pi^2 t|\xi|)\right)^\vee$ and $G_{1,1}(t) = (e^{-4\pi^2 t|\xi|^2})^\vee$ is the heat semigroup.

3. Functional setting and theorems

Before starting our theorems, let $\beta > 0$ and $0 \leq \mu < N$ be such that

$$\beta = \frac{\alpha}{2}\Big(\frac{N-\mu}{p} - \frac{N-\mu}{r}\Big) \quad \text{and} \quad \mu = N - \frac{2p}{\rho-1}, \tag{3.1}$$

which make $\|\cdot\|_{X_\beta}$ invariant by *scaling map* (1.4).

3.1. **Existence of solutions.** Given a Banach space Y, we will denote $B_Y(\varepsilon)$ a closed ball of radius ε centered at the origin of the space Y. Our existence and stability result is stated as follows.

Theorem 3.1. *Let $N \geq 2$, $1 < \alpha < 2$, $q = \frac{2\rho}{\rho+1}$, and $0 \leq \mu = N - \frac{2p}{\rho-1}$, for $p > 1$. Suppose that $\frac{N-\mu}{p} - \frac{N-\mu}{r} < 2$, $r > \rho > 1 + \alpha$,*

$$\frac{p}{r} < \frac{1}{\alpha} - \frac{1}{2}, \quad \frac{\alpha}{2-\alpha} < q < \frac{2}{\alpha}, \quad \Big(1 - \frac{p}{r}\Big) < \frac{\rho-1}{\alpha}\Big(\frac{1}{q} - \frac{\alpha}{2}\Big). \tag{3.2}$$

(i) (Global solution) There exist $\varepsilon > 0$ such that if $\|\varphi\|_{D(\alpha,\beta)} + \|\psi\|_{\widetilde{D}(\alpha,\beta)} \leq \varepsilon$, then problem (1.1)-(1.2) has a unique global-in-time mild solution $u \in B_{X_\beta}(2\varepsilon)$ satisfying

$$\|u(t,\cdot)\|_{\mathcal{M}_{r,\mu}} \leq Ct^{-\beta} \quad \text{and} \quad \|\nabla_x u(t,\cdot)\|_{\mathcal{M}_{r,\mu}} \leq Ct^{-\beta-\alpha/2}. \tag{3.3}$$

(ii) (Stability in X_β) The solution u in Theorem 3.1(i) is stable with respect to the initial data φ and ψ, that is, the data-map solution $(\varphi, \psi) \mapsto u$ is locally Lipschitz continuous from $D(\alpha,\beta) \times \widetilde{D}(\alpha,\beta)$ into X_β:

$$\|u - \tilde{u}\|_{X_\beta} \leq C\big(\|\varphi - \tilde{\varphi}\|_{D(\alpha,\beta)} + \|\psi - \tilde{\psi}\|_{\widetilde{D}(\alpha,\beta)}\big), \tag{3.4}$$

where u and $\tilde{u}$ are solutions of (1.1) with initial values (φ, ψ) and $(\tilde{\varphi}, \tilde{\psi})$, respectively.

Remark 3.2. Let us compare our theorem with some previous results.

(A) If $\psi = 0$, we may take $\varphi \in \mathcal{M}_{p,\mu}$ in Theorem 3.1-(i) with smallness on $\|\varphi\|_{\mathcal{M}_{p,\mu}}$.

(B) Theorem 3.1-(i) holds for $\alpha = 1$ and $\psi = 0$. Hence, the space $D(1, \beta)$ strictly includes the space $\mathcal{N}(\varphi)$ taken in [17]. Indeed, let $r < r_1$ and $\mu_2 = 0$ in Lemma 2.1-(i) to get

$$\|\varphi\|_{D(1,\beta)} = \sup_{t>0} t^\beta \|e^{t\Delta}\varphi\|_{\mathcal{M}_{r,\mu}} + \sup_{t>0} t^{\frac{1}{2}+\beta} \|e^{t\Delta}\varphi\|_{\mathcal{M}^1_{r,\mu}},$$

$$\leq \sup_{t>0} t^\beta \|e^{t\Delta}\varphi\|_{L^{r_1}} + \sup_{t>0} t^{\frac{1}{2}+\beta} \|e^{t\Delta}\varphi\|_{\dot{H}^1_{r_1}} = \|\varphi\|_{\mathcal{N}(\varphi)}.$$

On the one hand (see [14, (2.56)]), homogeneous Besov-Morrey spaces can be defined by

$$\mathcal{N}^{-2s}_{r,\mu,\infty} = \{ f \in \mathcal{S}' : \|f\|_{\mathcal{N}^{-2s}_{r,\mu,\infty}} = \sup_{t>0} t^{-s} \|e^{t\Delta} f\|_{\mathcal{M}_{r,\mu}} < \infty \}, \quad s > 0.$$

Hence, the space $D(1, \beta)$ is a kind of Besov-Morrey spaces. On the other hand, when $\alpha \neq 1$ the norms $\|\varphi\|_{D(\alpha,\beta)} = \|G_{\alpha,1}(t)\varphi\|_{X_\beta}$ and $\|\psi\|_{\widetilde{D}(\alpha,\beta)} = \|G_{\alpha,2}(t)\psi\|_{X_\beta}$ satisfy

$$\|\varphi\|_{D(\alpha,\beta)} \leq C\|\varphi\|_{\mathcal{M}_{p,\mu}} \quad \text{and} \quad \|\psi\|_{\widetilde{D}(\alpha,\beta)} \leq C\|\psi\|_{\mathcal{M}^{-2/\alpha}_{p,\mu}}$$

in view of Lemma 5.1. So $\mathcal{M}_{p,\mu} \subset D(\alpha, \beta)$ and $\mathcal{M}^{-2/\alpha}_{p,\mu} \subset \widetilde{D}(\alpha, \beta)$.

(C) (Viscous Hamilton-Jacobi) Let $\kappa_2 = 0$, $\psi = 0$ in (1.1)-(1.2), $\mu = N - \frac{q-1}{2-q}p$ and $\|\varphi\|_{D(\alpha,\beta)}$ small enough. Using the proof of Theorem 3.1, the problem (1.1)-(1.2) has a unique solution $u \in C((0, \infty); \mathcal{M}_{r,\mu}) \cap C((0, \infty); \mathcal{M}^1_{r,\mu})$ such that

$$\sup_{t>0} t^{\frac{(N-\mu)}{2}\alpha(\frac{1}{p}-\frac{1}{r})} \|u(t)\|_{\mathcal{M}_{r,\mu}} \leq C,$$

$$\sup_{t>0} t^{\frac{\alpha}{2}+\frac{(N-\mu)}{2}\alpha(\frac{1}{p}-\frac{1}{r})} \|\nabla u(t)\|_{\mathcal{M}_{r,\mu}} \leq C,$$

under the assumptions in Theorem 3.1 with the change $\rho = \frac{2}{2-q}$. In other words, we obtain a version of Theorem 2.1 and Proposition 2.3 of [3] when $1 < \alpha < 2$. If $\alpha = 1$, the assumption $\frac{N-\mu}{p} - \frac{N-\mu}{r} < 2$ is not necessary because of the smoothing effect of the heat semigroup in $\mathcal{M}_{p,\mu}$ (see e.g. [10]).

3.2. **Self-similar solutions.** As we commented before, a necessary condition for initial data to produce self-similar solutions is homogeneity and simplest candidates are the radial functions

$$\varphi(x) = \varepsilon_1 |x|^{-\frac{2}{\rho-1}} \quad \text{and} \quad \psi(x) = \varepsilon_2 |x|^{-\frac{2}{\rho-1}-\frac{2}{\alpha}}. \tag{3.5}$$

Hence, we need $D(\alpha, \beta)$ and $\widetilde{D}(\alpha, \beta)$ to satisfy

$$\|\psi_\gamma\|_{\widetilde{D}(\alpha,\beta)} = \|\psi\|_{\widetilde{D}(\alpha,\beta)} \quad \text{and} \quad \|\varphi_\gamma\|_{D(\alpha,\beta)} = \|\varphi\|_{D(\alpha,\beta)}, \tag{3.6}$$

and it comes from the scaling invariance of X_β. Moreover, (2.6) and Remark 3.2-(B) permit us to take the *singular functions* (3.5) as initial data, since $\varphi \in \mathcal{M}_{p,\mu} \subset D(\alpha, \beta)$ and $\psi \in \mathcal{M}^{-2/\alpha}_{p,\mu} \subset \widetilde{D}(\alpha, \beta)$ provided $\mu = N - \frac{2p}{\rho-1}$, $\rho > \max\{1 + \frac{2}{N}, 1 + \frac{2\alpha}{\alpha N-2}\}$ and $1 < p < r$.

Theorem 3.3 (Self-similarity)**.** *Under the assumptions of Theorem 3.1, let φ and ψ be homogeneous functions of degree $-\frac{2}{\rho-1}$ and $-\frac{2}{\rho-1} - \frac{2}{\alpha}$, respectively. Then the solution u of Theorem 3.1-(i) is self-similar.*

Remark 3.4. Let us remark some consequences of this theorem.

(A) (Infinity energy data) In Theorem 3.3 we can build singular initial data (ψ, φ) which can be arbitrarily large in $L^2(\mathbb{R}^N) \times \dot{H}^{2/\alpha}(\mathbb{R}^N)$, provided that $\frac{2}{\alpha} + \frac{2}{\rho-1} < \frac{N}{2}$ and $1 < p < \frac{N(\rho-1)}{2}$. Indeed, let $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ and $\psi \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ be given by (3.5). Using $\widehat{\varphi}(\xi) = \gamma_{0,0}\varepsilon_1|\xi|^{\frac{2}{\rho-1}-N}$, we see that φ and ψ are arbitrarily large in $\dot{H}^{2/\alpha}$ and L^2 in view of

$$\|\psi\|^2_{L^2(\mathbb{R}^N)} = \varepsilon_2^2 \int_{\mathbb{R}^N} |x|^{-\frac{4}{\rho-1}-\frac{4}{\alpha}} dx$$
$$= \varepsilon_2^2 \lim_{\omega_2 \to \infty} \int_0^{\omega_2} \int_{\mathbb{S}^{N-1}} r^{-\frac{4}{\rho-1}-\frac{4}{\alpha}} r^{N-1} d\sigma dr$$
$$= C \lim_{\omega_2 \to \infty} \omega_2^{-\frac{4}{\rho-1}-\frac{4}{\alpha}+N} = +\infty$$

and

$$\|\varphi\|^2_{\dot{H}^{\frac{2}{\alpha}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{4/\alpha} |\widehat{\varphi}(\xi)|^2 d\xi = \gamma_{0,0}^2 \varepsilon_1^2 \int_{\mathbb{R}^N} |\xi|^{4/\alpha+\frac{4}{\rho-1}-2N} d\xi$$
$$= C \lim_{\omega_1 \to 0} \int_{\omega_1}^{\infty} \int_{\mathbb{S}^{N-1}} r^{4/\alpha+\frac{4}{\rho-1}-N-1} d\sigma dr$$
$$= C \lim_{\omega_1 \to 0} \omega_1^{\frac{4}{\alpha}+\frac{4}{\rho-1}-N} = +\infty.$$

Then, even the initial data φ and ψ are in the Morrey spaces $\mathcal{M}_{p,\mu}$ and $\mathcal{M}_{p,\mu}^{-2/\alpha}$, respectively, they may be arbitrarily large in $\dot{H}^{2/\alpha}(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$.

(B) Inspired by [16], we use a Littlewood-Paley decomposition of the Sobolev-Morrey spaces (see subsection 2.1) to build general singular functions for Theorem 3.3. In fact, let $Y_{k_1}(x)$, $Y_{k_2}(x)$ be homogeneous harmonic polynomials of degree k_1 and k_2, respectively. Consider $S(\varphi, \psi)$ the set of functions $(\varphi, \psi) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ such that

$$\varphi(x) = \epsilon_1 \frac{Y_{k_1}(x)}{|x|^{\frac{2}{\rho-1}+k_1}} \quad \text{and} \quad \psi(x) = \epsilon_2 \frac{Y_{k_2}(x)}{|x|^{\frac{2}{\rho-1}+\frac{2}{\alpha}+k_2}}.$$

By (2.6), the set $S(\varphi, \psi)$ gives us a class of data for existence of self-similar solutions for (1.1)-(1.2).

3.3. **Symmetries.** This subsection concerns with symmetries of solutions obtained in Theorems 3.1 and 3.3. It is straightforward to check that $E_{\alpha,1}(4\pi^2 t^\alpha |\xi|^2)$ and $tE_{\alpha,2}(4\pi^2 t^\alpha |\xi|^2)$ are invariant by the set $\mathcal{O}(N)$ of all rotations in $\mathbb{R}^N$. It follows that $G_{\alpha,1}(t)$ and $G_{\alpha,2}(t)$ are $\mathcal{O}(N)-$ invariant. Hence, it is natural to ask whether or not the solutions of the above theorems present symmetry properties under certain qualitative conditions on the initial data.

Let $\mathcal{A}$ be a subset of $\mathcal{O}(N)$. A function h is said symmetric under action $\mathcal{A}$ when $h(x) = h(T(x))$ for all $T \in \mathcal{A}$. If $h(x) = -h(T(x))$, the function h is said antisymmetric under the action of $\mathcal{A}$.

Theorem 3.5. *Let the hypotheses of Theorem 3.1 be satisfied. The solution $u(\cdot, t)$ is symmetric for all $t > 0$, whenever φ and ψ are symmetric under action $\mathcal{A}$.*

Remark 3.6. A radially symmetric solution is a self-similar solution, if the profile ω depends only on $r = |x|$, that is, there is a function $\mathcal{U}$ such that $u(t,x) = t^{-\frac{\alpha}{\rho-1}}\mathcal{U}(|x|/t^{\frac{\alpha}{2}})$, $t > 0$.

(A) Let $\mathcal{A} = \mathcal{O}(N)$ in Theorem 3.5. If φ and ψ are radial and homogeneous functions of degree $-\frac{2}{\rho-1}$ and $-\frac{2}{\rho-1} - \frac{2}{\alpha}$, respectively (see Remark 3.4), then Theorems 3.1, 3.3 and 3.5 imply that (1.1)-(1.2) have a unique self-similar solution $u \in X_\beta$ which is radially symmetric in $\mathbb{R}^N$.

(B) Unlike the case $\kappa_1 = 0$, antisymmetry does not hold in general, for $\kappa_1 \neq 0$.

4. Technical estimates

In this section we prove some Mikhlin-type estimates for Mittag-Leffler functions. In spite of the fact that these estimates are necessary in the proof of Lemma 5.1, they are of independent interest. We start the section with a suitable decomposition of $E_{\alpha,\beta}(z)$.

4.1. Decompositions of $E_{\alpha,\beta}(z)$.

Proposition 4.1. *Let* $z \in \mathbb{C}$ *be such that* $\mathcal{R}e(z) > 0$ *and define*

$$\omega_{\alpha,\beta}(z) = \frac{z^{\frac{1-\beta}{\alpha}}}{\alpha} \Big[\exp\big(a_\alpha(z) + \frac{1-\beta}{\alpha}\pi i\big) + \exp\big(b_\alpha(z) - \frac{1-\beta}{\alpha}\pi i\big) \Big],$$

$$l_{\alpha,\beta}(z) = \int_0^\infty \mathcal{H}_{\alpha,\beta}(s) e^{-z^{1/\alpha} s^{1/\alpha}} z^{\frac{1}{\alpha}(1-\beta)} ds,$$

where

$$\mathcal{H}_{\alpha,\beta}(s) = \frac{1}{\alpha\pi} \frac{\sin[(\alpha-\beta)\pi] - s\sin(\beta\pi)}{s^2 + 2s\cos(\alpha\pi) + 1} s^{\frac{1-\beta}{\alpha}}, \tag{4.1}$$

$$a_\alpha(z) = z^{1/\alpha} e^{\frac{\pi i}{\alpha}}, \quad b_\alpha(z) = z^{1/\alpha} e^{-\frac{\pi i}{\alpha}}.$$

Suppose that $1 < \alpha < 2$ *and* $1 \leq \beta \leq 2$, *then*

$$E_{\alpha,\beta}(-z) = \omega_{\alpha,\beta}(z) + l_{\alpha,\beta}(z). \tag{4.2}$$

Proof. Recall that Mittag-Leffler function can be written as

$$E_{\alpha,\beta}(-z) = \frac{1}{2\pi i} \int_{Ha} \frac{t^{\alpha-\beta} e^t}{t^\alpha + z} dt, \tag{4.3}$$

where Ha is the Hankel path, i.e. a path starts and ends at $-\infty$ and encircles the disk $|t| \leq |z|^{1/\alpha}$ positively. The integrand $\Phi(t) = \frac{t^{\alpha-\beta} e^t}{t^\alpha + z}$ of (4.3) has two poles $a_\alpha(z)$ and $b_\alpha(z)$, because $1 < \alpha < 2$. Proceeding as in [5, Lemma 1.1], the residues theorem yields

$$2\pi i E_{\alpha,\beta}(-z) = \int_\infty^R \Phi(te^{-\pi i}) d(te^{-\pi i}) + 2\pi i \left(\text{Res}(\Phi, a_\alpha(z)) + \text{Res}(\Phi, b_\alpha(z))\right)$$

$$- \int_\epsilon^R \Phi(te^{-\pi i}) d(te^{-\pi i}) - \int_R^\epsilon \Phi(te^{\pi i}) d(te^{\pi i})$$

$$- \int_{-\pi}^\pi \Phi(\epsilon e^{\theta i}) d(\epsilon e^{\theta i}) + \int_R^\infty \Phi(te^{\pi i}) d(te^{\pi i})$$

$$=: I_1(R) + 2\pi i \left(\text{Res}(\Phi, a_\alpha(z)) + \text{Res}(\Phi, b_\alpha(z))\right) - I_2(\epsilon, R)$$

$$- I_3(R, \epsilon) - I_4(\epsilon) + I_5(R).$$

We first get

$$\lim_{R \to \infty} I_1(R) = \lim_{\epsilon \to 0^+} I_4(\epsilon) = \lim_{R \to \infty} I_5(R) = 0.$$

An easy computation yields

$$l_{\alpha,\beta}(z) = -\frac{1}{2\pi i} \lim_{R\to\infty, \epsilon\to 0^+} I_2(\epsilon, R) + I_3(R, \epsilon).$$

Indeed,

$$\frac{1}{2\pi i} \lim_{\epsilon\to 0^+, R\to\infty} \left[I_2(\epsilon, R) + I_3(R, \epsilon)\right]$$

$$= \frac{1}{2\pi i} \left(\int_0^\infty \frac{e^{-t} t^{\alpha-\beta} e^{(\alpha-\beta)\pi i}}{t^\alpha e^{\alpha\pi i} + z} dt - \int_0^\infty \frac{e^{-t} t^{\alpha-\beta} e^{-(\alpha-\beta)\pi i}}{t^\alpha e^{-\alpha\pi i} + z} dt \right)$$

$$= -\frac{1}{2\pi i} 2i \int_0^\infty e^{-t} t^{\alpha-\beta} \frac{z \sin[(\alpha-\beta)\pi] - t^\alpha \sin(\beta\pi)}{t^{2\alpha} + 2t^\alpha z \cos(\alpha\pi) + z^2} dt \qquad (4.4)$$

$$= -\int_0^\infty \mathcal{H}_{\alpha,\beta}(s) \exp\left(-z^{1/\alpha} s^{1/\alpha}\right) z^{\frac{1}{\alpha}(1-\beta)} ds, \qquad (4.5)$$

$$= -l_{\alpha,\beta}(z),$$

where the change $t \mapsto z^{1/\alpha} s^{1/\alpha}$ was used from (4.4) to (4.5). Also, we obtain

$$\text{Res}(\Phi, a_\alpha(z)) = \frac{z^{\frac{1-\beta}{\alpha}}}{\alpha} \exp(a_\alpha(z) + \pi i(1-\beta)/\alpha),$$

$$\text{Res}(\Phi, b_\alpha(z)) = \frac{z^{\frac{1-\beta}{\alpha}}}{\alpha} \exp(b_\alpha(z) - \pi i(1-\beta)/\alpha).$$

These give us the desired decomposition. $\qquad\qquad\square$

In particular, for $\beta = 1$ and $\beta = 2$ in Proposition 4.1, we have the decompositions

$$E_{\alpha,1}(-z) = \omega_{\alpha,1}(z) + l_{\alpha,1}(z) \qquad (4.6)$$

in [5, Lemma 1.1], and

$$E_{\alpha,2}(-z) = \omega_{\alpha,2}(z) + l_{\alpha,2}(z), \qquad (4.7)$$

in [6, Lemma 1.2-(IV)]. Notice that $\omega_{\alpha,1}(z)$ oscillates with frequency $\sin(\pi/\alpha)$ and amplitude decaying exponentially with rate $|\cos(\pi/\alpha)|$, in view of

$$\omega_{\alpha,1}(z) = \frac{2}{\alpha} \exp(z^{1/\alpha} \cos(\pi/\alpha)) \cos(z^{1/\alpha} \sin(\pi/\alpha)).$$

On the other hand, the function $l_{\alpha,1}(z)$ exhibits the relaxation phenomena of $E_{\alpha,1}(-z)$, namely,

$$l_{\alpha,1}(z) = \int_0^\infty \mathcal{H}_{\alpha,1}(s) \exp(-s^{1/\alpha} z^{1/\alpha}) ds = \int_0^\infty \exp(-s^{1/\alpha} z^{1/\alpha}) d\mu_\alpha(s),$$

where

$$\mathcal{H}_{\alpha,1}(s) = \frac{\sin(\alpha\pi)}{\alpha\pi} \frac{1}{s^2 + 2s\cos(\alpha\pi) + 1}$$

and $d\mu_\alpha(s) = \mathcal{H}_{\alpha,1}(s) ds$ is a finite measure in $\mathbb{R}_+$ such that $\mu_\alpha(\mathbb{R}_+) = 2 - \frac{2}{\alpha}$. Furthermore, when $\beta = \alpha$, the decomposition (4.2) is useful to show that the map $G_{\alpha,\beta}(\cdot)$, $\beta = 1, 2$, is differentiable for $t > 0$. Indeed, see (5.4) and (5.6) below.

4.2. **Mikhlin estimates for $E_{\alpha,\beta}(-\sigma(\xi))$.** We provide estimates for $E_{\alpha,1}(\sigma(\xi))$, $E_{\alpha,2}(\sigma(\xi))$ and $E_{\alpha,\alpha}(\sigma(\xi))$, where $\sigma \in C^\infty(\mathbb{R}^N\backslash\{0\}; (-\infty, 0))$ is the symbol of the Fourier multiplier

$$\sigma(D)f = \mathcal{F}^{-1}\sigma(\xi)\mathcal{F}f(\xi), \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Consider the change $z \mapsto \sigma(\xi)$ into (4.6) and write it as follows:

$$E_{\alpha,1}(\sigma(\xi)) = \omega_{\alpha,1}(-\sigma(\xi)) + l_{\alpha,1}(-\sigma(\xi)). \tag{4.8}$$

Proposition 4.2. *Let $\sigma(\xi) \in C^\infty(\mathbb{R}^N\backslash\{0\})$ be a function homogeneous of degree $k > 0$ and such that*

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[\sigma(\xi)]\Big| \le A|\xi|^{k-|\gamma|}, \quad \xi \ne 0 \tag{4.9}$$

for all multi-index $\gamma \in (\mathbb{N} \cup \{0\})^N$ with $|\gamma| \le [N/2] + 1$. Let $1 < \alpha < 2$ and $0 \le \delta < k$, there exists $C > 0$ such that

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta E_{\alpha,1}(\sigma(\xi))]\Big| \le CA|\xi|^{-|\gamma|}, \quad \xi \ne 0. \tag{4.10}$$

Proof. Taking (4.9) into account we obtain

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[-\sigma(\xi)]^l\Big| \le CA\,|\xi|^{-|\gamma|}|\xi|^{kl}, \quad \text{for all } l \in \mathbb{R}. \tag{4.11}$$

Hence, the γth-order derivative of the parcel $|\xi|^\delta\omega_{\alpha,1}(\sigma(\xi))$ can be estimated by

$$\begin{aligned}
&\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta\omega_{\alpha,1}(-\sigma(\xi))]\Big|\\
&= \Big|\frac{\partial^\gamma}{\partial\xi^\gamma}\big[|\xi|^\delta\exp(e^{\frac{i\pi}{\alpha}}(-\sigma(\xi))^{1/\alpha}) + |\xi|^\delta\exp(e^{-\frac{i\pi}{\alpha}}(-\sigma(\xi))^{1/\alpha})\big]\Big|\\
&\le C|\xi|^{-|\gamma|}\big[c_0|\xi|^\delta + c_1|\xi|^{\delta+\frac{k}{\alpha}} + \cdots + c_{|\gamma|}|\xi|^{\delta+\frac{|\gamma|k}{\alpha}}\big]e^{\cos(\frac{\pi}{\alpha})(-\sigma(\xi))^{1/\alpha}}\\
&\le CA|\xi|^{-|\gamma|}.
\end{aligned} \tag{4.12}$$

To estimate $l_{\alpha,1}(\sigma(\xi))$, recall that

$$l_{\alpha,1}(\sigma(\xi)) = \int_0^\infty \mathcal{H}_{\alpha,1}(s)\exp(-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha})ds.$$

Using the homogeneity $\sigma(\lambda\xi) = \lambda^k\sigma(\xi)$, we have

$$\begin{aligned}
&\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta e^{-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha}}]\Big|\\
&\le C|\xi|^{-|\gamma|}\big[c_0|\xi|^\delta + c_1|\xi|^{\delta+\frac{k}{\alpha}}s^{1/\alpha} + \cdots + c_{|\gamma|}|\xi|^{\delta+\frac{|\gamma|k}{\alpha}}s^{\frac{|\gamma|}{\alpha}}\big]e^{-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha}}\\
&= Cs^{-\frac{\delta}{k}}|\xi|^{-|\gamma|}\big[c_0|s^{\frac{1}{k}}\xi|^\delta + c_1|s^{\frac{1}{k}}\xi|^{\delta+\frac{k}{\alpha}} + \dots\\
&\quad + c_{|\gamma|}|s^{\frac{1}{k}}\xi|^{\delta+\frac{|\gamma|k}{\alpha}}\big]e^{-[-\sigma(s^{1/k}\xi)]^{1/\alpha}}\\
&\le CAs^{-\frac{\delta}{k}}|\xi|^{-|\gamma|}.
\end{aligned} \tag{4.13}$$

Then

$$\begin{aligned}
&\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta l_{\alpha,1}(-\sigma(\xi))]\Big|\\
&= \frac{\sin(\alpha\pi)}{\alpha\pi}\int_0^\infty \frac{1}{s^2 + 2s\cos(\alpha\pi) + 1}\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta e^{-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha}}]\Big|ds
\end{aligned}$$

$$\leq CA\Big(\frac{\sin(\alpha\pi)}{\alpha\pi}\int_0^\infty \frac{s^{-\frac{\delta}{k}}}{s^2+2s\cos(\alpha\pi)+1}ds\Big)|\xi|^{-|\gamma|}$$
$$\leq CA\,|\xi|^{-|\gamma|},$$

because $0 \leq \delta < k$. These estimates prove the proposition. $\square$

In general, we obtain the following proposition for the two-parametric Mittag-Leffler function.

Proposition 4.3. *Let $\sigma(\xi) \in C^\infty(\mathbb{R}^N\backslash\{0\})$ be a homogeneous function of degree $k > 0$ satisfying (4.9), for all multi-index $\gamma \in (\mathbb{N}\cup\{0\})^N$ with $|\gamma| \leq [N/2]+1$. Then, there exists a positive constant C (independent of δ and k) such that*

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta E_{\alpha,\beta}(\sigma(\xi))]\Big| \leq CA|\xi|^{-|\gamma|}, \quad \xi \neq 0 \tag{4.14}$$

provided that $1 < \alpha < 2$ and $k(\frac{\beta}{\alpha}-\frac{1}{\alpha}) \leq \delta < k$.

Proof. The proof is similar the proof of Proposition 4.2. Indeed, proceeding as in (4.12), it follows that

$$\Big|\frac{\partial^{\gamma_1}}{\partial\xi^{\gamma_1}}h_{\alpha,\beta}(\xi)\Big|$$
$$:= \Big|\frac{\partial^{\gamma_1}}{\partial\xi^{\gamma_1}}\Big[|\xi|^\delta \exp\big(a_\alpha(\sigma(\xi))+\frac{1-\beta}{\alpha}\pi i\big)$$
$$+ |\xi|^\delta \exp\big(b_\alpha(\sigma(\xi))-\frac{1-\beta}{\alpha}\pi i\big)\Big]\Big| \tag{4.15}$$
$$\leq CA|\xi|^{-|\gamma_1|}\big[c_0|\xi|^\delta + c_1|\xi|^{\delta+\frac{k}{\alpha}}+\cdots+c_{|\gamma_1|}|\xi|^{\delta+\frac{|\gamma_1|k}{\alpha}}\big]e^{\cos(\pi/\alpha)(-\sigma(\xi))^{1/\alpha}},$$

for all multi-index γ_1. Hence, Leibniz's rule, (4.11) and (4.15) give us

$$\Big|\frac{\partial^{\gamma_1}}{\partial\xi^\gamma}[|\xi|^\delta\omega_{\alpha,\beta}(-\sigma(\xi))]\Big|$$
$$\leq \sum_{\gamma_1\leq\gamma}\binom{\gamma}{\gamma_1}\Big|\frac{\partial^{\gamma_1}}{\partial\xi^{\gamma_1}}[\sigma(\xi)]^{\frac{1-\beta}{\alpha}}\Big|\,\Big|\frac{\partial^{\gamma-\gamma_1}}{\partial\xi^{\gamma-\gamma_1}}[h_{\alpha,\beta}(\xi)]\Big|$$
$$\leq CA|\xi|^{-|\gamma_1|-|\gamma-\gamma_1|}\big[c_0|\xi|^{\delta+k\left(\frac{1}{\alpha}-\frac{\beta}{\alpha}\right)}+\cdots+c|\xi|^{\delta+k\left(\frac{1}{\alpha}-\frac{\beta}{\alpha}\right)+\frac{|\gamma-\gamma_1|k}{\alpha}}\big]$$
$$\times e^{\cos(\pi/\alpha)(-\sigma(\xi))^{1/\alpha}}$$
$$\leq CA|\xi|^{-|\gamma|},$$

in view of $\delta + k(\frac{1}{\alpha}-\frac{\beta}{\alpha}) \geq 0$. Also, using (4.11) and (4.13), the Leibniz's rule yields

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}\big[\big((-\sigma(\xi))^{\frac{1-\beta}{\alpha}}\big)\big(|\xi|^\delta\exp(-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha})\big)\big]\Big| \leq CA|\xi|^{-|\gamma|}s^{-\frac{\delta}{k}+\frac{\beta}{\alpha}-\frac{1}{\alpha}}.$$

Hence, we estimate

$$\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}[|\xi|^\delta l_{\alpha,\beta}(-\sigma(\xi))]\Big| \leq \int_0^\infty |\mathcal{H}_{\alpha,\beta}(s)|\Big|\frac{\partial^\gamma}{\partial\xi^\gamma}\big[(-\sigma(\xi))^{\frac{1-\beta}{\alpha}}|\xi|^\delta\exp(-s^{1/\alpha}(-\sigma(\xi))^{1/\alpha})\big]\Big|ds$$
$$= CA(I+II)|\xi|^{-|\gamma|}$$
$$\leq C|\xi|^{-|\gamma|},$$

where the integrals I and II are defined by (see (4.1))

$$I = \frac{\sin[(\alpha - \beta)\pi]}{\alpha\pi} \int_0^\infty \frac{s^{-\frac{\delta}{k}}}{s^2 + 2s\cos(\alpha\pi) + 1} ds, \qquad (4.16)$$

$$II = -\frac{\sin(\beta\pi)}{\alpha\pi} \int_0^\infty \frac{s^{1-\frac{\delta}{k}}}{s^2 + 2s\cos(\alpha\pi) + 1} ds. \qquad (4.17)$$

Those integrals are finite in view of $\delta < k$. This completes the proof of the proposition. $\qquad\square$

5. Sobolev-Morrey estimates

In this section we obtain fundamental estimates which will be important to prove Theorem 3.1.

5.1. **Linear estimates.** Here, we present some estimates of the Mittag-Leffler operators $\{G_{\alpha,\beta}(t)\}_{t\geq 0}$ in Sobolev-Morrey spaces. Indeed, based on Propositions 4.2 and 4.3 with the homogeneous symbol $\sigma(\xi) = -4\pi^2|\xi|^2$ of degree 2, the following lemma can be proved by proceeding as in [2, Lemma 3.1-(i)].

Lemma 5.1. *Let* $\gamma_1 \leq \gamma_2 \in \mathbb{R}$, $1 < p_1 \leq p_2 < \infty$, $0 \leq \mu < N$, $1 < \alpha < 2$ *and* $\lambda = (\gamma_2 - \gamma_1) + \frac{N-\mu}{p_1} - \frac{N-\mu}{p_2}$. *There is a constant* C *such that*

$$\|G_{\alpha,1}(t)f\|_{\mathcal{M}_{p_2,\mu}^{\gamma_2}} \leq Ct^{-\frac{\alpha}{2}\lambda}\|f\|_{\mathcal{M}_{p_1,\mu}^{\gamma_1}}, \quad \text{if } \lambda < 2, \qquad (5.1)$$

$$\|G_{\alpha,2}(t)f\|_{\mathcal{M}_{p_2,\mu}^{\gamma_2}} \leq Ct^{-\frac{\alpha}{2}\lambda}\|f\|_{\mathcal{M}_{p_1,\mu}^{\gamma_1-\frac{2}{\alpha}}}, \quad \text{if } \lambda + \frac{2}{\alpha} < 2, \qquad (5.2)$$

$$\|G_{\alpha,\alpha}(t)f\|_{\mathcal{M}_{p_2,\mu}^{\gamma_2}} \leq Ct^{\alpha-1-\frac{\alpha}{2}\lambda}\|f\|_{\mathcal{M}_{p_1,\mu}^{\gamma_1}}, \quad \text{if } \left(2 - \frac{2}{\alpha}\right) < \lambda < 2, \qquad (5.3)$$

for all $f \in \mathcal{S}'(\mathbb{R}^N)$.

We finish this subsection by noticing that $\{\partial_t G_{\alpha,1}(t)\}_{t\geq 0}$ and $\{\partial_t G_{\alpha,2}(t)\}_{t\geq 0}$ are bounded in Morrey spaces. Indeed, a straightforward computation gives us

$$\frac{d}{dt} E_{\alpha,1}(-4\pi^2|\xi|^2 t^\alpha) = -4\pi^2|\xi|^2 \left[t^{\alpha-1} E_{\alpha,\alpha}(-4\pi^2|\xi|^2 t^\alpha) \right],$$

for $t > 0$ and $\xi \neq 0$. It follows from Lemma 5.1-(iii) that

$$\|\partial_t G_{\alpha,1}(t)f\|_{\mathcal{M}_{p_2,\mu}} \leq C\|G_{\alpha,\alpha}(t)f\|_{\mathcal{M}_{p_2,\mu}^2} \leq Ct^{-\frac{\alpha}{2}\left(\frac{N-\mu}{p_1} - \frac{N-\mu}{p_2}\right)-1}\|f\|_{\mathcal{M}_{p_1,\mu}}. \quad (5.4)$$

Using

$$tE_{\alpha,2}(-4\pi^2|\xi|^2 t^\alpha) = \int_0^t E_{\alpha,1}(-4\pi^2|\xi|^2 s^\alpha) ds, \qquad (5.5)$$

Lemma 5.1-(i) yields

$$\|\partial_t G_{\alpha,2}(t)f\|_{\mathcal{M}_{p_2,\mu}} = \|G_{\alpha,1}(t)f\|_{\mathcal{M}_{p_2,\mu}} \leq Ct^{-\frac{\alpha}{2}\left(\frac{N-\mu}{p_1} - \frac{N-\mu}{p_2}\right)}\|f\|_{\mathcal{M}_{p_1,\mu}}. \qquad (5.6)$$

5.2. **Nonlinear estimates.** This subsection is devoted to estimate the nonlinear term $\mathcal{N}_\alpha(u)$ on the functional space X_β. Firstly, let us denote $\mathbf{B}(\nu, \eta)$ by *special beta function* $\mathbf{B}(\nu, \eta) = \int_0^1 (1 - t)^{\nu-1} t^{\eta-1} dt$ which is finite, for all $\eta, \nu > 0$. Let $k_1, k_2, k_3 < 1$, for $t > 0$ and $s > 0$ the changes of variable $\tau \mapsto \tau s$ and $s \mapsto st$ give us

$$
\begin{aligned}
I(t) &= \int_0^t (t - s)^{-k_1} \int_0^s (s - \tau)^{-k_2} \tau^{-k_3} d\tau ds \\
&= \mathbf{B}(1 - k_2, 1 - k_3) \int_0^t (t - s)^{-k_1} s^{-k_2-k_3+1} ds \\
&= \mathbf{B}(1 - k_2, 1 - k_3) \mathbf{B}(1 - k_1, 2 - k_2 - k_3) t^{2-k_1-k_2-k_3}.
\end{aligned}
\tag{5.7}
$$

We freely use (5.7) in the next proof.

Lemma 5.2. *Under assumptions of Theorem 3.1, there is a positive constant $K = K(\kappa_1, \kappa_2)$ such that*

$$
\|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(v)\|_{X_\beta} \leq K \|u - v\|_{X_\beta} \left[\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1} + \|u\|_{X_\beta}^{q-1} + \|v\|_{X_\beta}^{q-1} \right]. \tag{5.8}
$$

Proof. Recall $\mathcal{N}_\alpha(u)$ and rewrite it as follows:

$$
\begin{aligned}
\mathcal{N}_\alpha(u)(t) &= \int_0^t G_{\alpha,1}(t - s) \int_0^s r_\alpha(s - \tau) \left(\kappa_2 |u|^{\rho-1} u + \kappa_1 |\nabla_x u|^q \right) d\tau ds \\
&=: \mathcal{N}_\alpha^1(u)(t) + \mathcal{N}_\alpha^2(u)(t).
\end{aligned}
\tag{5.9}
$$

The proof is divided in three steps.

First step: Estimates for $\mathcal{N}_\alpha^1(u)$. In (5.1), let $(\gamma_1, \gamma_2, p_1, p_2) = (0, 1, r/\rho, r)$ and $1 < \rho < r$ to obtain

$$
\begin{aligned}
&\|\mathcal{N}_\alpha^1(u)(t) - \mathcal{N}_\alpha^1(v)(t)\|_{\mathcal{M}_{r,\mu}^1} \\
&\leq C \int_0^t (t - s)^{-\lambda_1} \int_0^s r_\alpha(s - \tau) \|f(u) - f(v)\|_{\mathcal{M}_{r/\rho,\mu}} d\tau ds,
\end{aligned}
$$

where $f(u)(\tau) = \kappa_1 |u(\tau)|^{\rho-1} u(\tau)$ and $\lambda_1 = \frac{\alpha}{2} + \frac{\alpha}{2} \left(\frac{N-\mu}{r/\rho} - \frac{N-\mu}{r} \right)$. Using that

$$
\left| |a|^{\rho-1} a - |b|^{\rho-1} b \right| \leq C |a - b| \left(|a|^{\rho-1} + |b|^{\rho-1} \right), \quad \text{for all } \rho > 1 \tag{5.10}
$$

and $\frac{\rho}{r} = \frac{1}{r} + \frac{\rho-1}{r}$, the Hölder inequality (2.5) yields

$$
\|\mathcal{N}_\alpha^1(u)(t) - \mathcal{N}_\alpha^1(v)(t)\|_{\mathcal{M}_{r,\mu}^1} \leq C |\kappa_2| \int_0^t (t - s)^{-\lambda_1} \theta(s) ds, \tag{5.11}
$$

where $\theta(s)$ is given by

$$
\begin{aligned}
\theta(s) &= \int_0^s (s - \tau)^{\alpha-2} \|u(\tau) - v(\tau)\|_{\mathcal{M}_{r,\mu}} \left(\|u(\tau)\|_{\mathcal{M}_{r,\mu}}^{\rho-1} + \|v(\tau)\|_{\mathcal{M}_{r,\mu}}^{\rho-1} \right) d\tau \\
&\leq C \int_0^s (s - \tau)^{\alpha-2} \tau^{-\rho\beta} \tau^\beta \|u(\tau) - v(\tau)\|_{\mathcal{M}_{r,\mu}} \times \\
&\quad \times \tau^{\beta(\rho-1)} \left(\|u(\tau)\|_{\mathcal{M}_{r,\mu}}^{\rho-1} + \|v(\tau)\|_{\mathcal{M}_{r,\mu}}^{\rho-1} \right) d\tau \\
&\leq C \int_0^s (s - \tau)^{\alpha-2} \tau^{-\rho\beta} d\tau \|u - v\|_{X_\beta} \left(\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1} \right).
\end{aligned}
\tag{5.12}
$$

Notice that $\alpha(\rho - 1) \frac{N-\mu}{2r} = \alpha - (\rho - 1)\beta$ yields

$$
-\lambda_1 + \alpha - \rho\beta = -\frac{\alpha}{2} + (\rho - 1)\beta - \alpha + \alpha - \rho\beta = -\frac{\alpha}{2} - \beta.
$$

It follows that (5.11) can be bounded by

$$\|\mathcal{N}_\alpha^1(u)(t) - \mathcal{N}_\alpha^1(v)(t)\|_{\mathcal{M}_{r,\mu}^1} \leq C|\kappa_2|I_1(t) \|u-v\|_{X_\beta} \big(\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1}\big)$$
$$\leq C|\kappa_2|t^{-\beta-\frac{\alpha}{2}} \|u-v\|_{X_\beta} \big(\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1}\big), \tag{5.13}$$

because the integral $I_1(t)$ (see (5.7)) satisfies

$$I_1(t) = \int_0^t (t-s)^{-\lambda_1}\Big(\int_0^s (s-\tau)^{\alpha-2}\tau^{-\rho\beta}d\tau\Big)ds = Ct^{-\lambda_1+\alpha-\rho\beta} = Ct^{-\beta-\frac{\alpha}{2}}.$$

Proceeding in a similar fashion, we obtain

$$\|\mathcal{N}_\alpha^1(u)(t) - \mathcal{N}_\alpha^1(v)(t)\|_{\mathcal{M}_{r,\mu}} \leq C|\kappa_2|J_1(t) \|u-v\|_{X_\beta} \big(\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1}\big)$$
$$\leq C|\kappa_2|t^{-\beta} \|u-v\|_{X_\beta} \big(\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1}\big), \tag{5.14}$$

where $J_1(t)$ is given by

$$J_1(t) = \int_0^t (t-s)^{-\vartheta_1} \int_0^s (s-\tau)^{\alpha-2}\tau^{-\rho\beta}d\tau ds,$$

and $\vartheta_1 = \frac{\alpha}{2}\Big(\frac{N-\mu}{r/\rho} - \frac{N-\mu}{r}\Big)$. In view of

$$-\vartheta_1 + \alpha - \rho\beta = \frac{\alpha}{2}\big(\frac{N-\mu}{r/\rho} - \frac{N-\mu}{r}\big) + \alpha - \rho\beta = (\rho-1)\beta - \alpha + \alpha - \rho\beta = -\beta,$$

one has $J_1(t) = Ct^{-\beta}$.

The convergence of $I_1(t)$ and $J_1(t)$ follows from (3.2) for $\lambda_1 = \frac{\alpha}{2} + \vartheta_1 < 1$ and because $\frac{p}{r} < \big(\frac{1}{\alpha} - \frac{1}{2}\big)$ is equivalent to $\vartheta_1 = \alpha\frac{p}{r} < 1 - \frac{\alpha}{2}$. Further, $\big(1 - \frac{p}{r}\big) < \frac{\rho-1}{\alpha}\big(\frac{1}{q} - \frac{\alpha}{2}\big) = \frac{\rho-1}{\alpha\rho}\big(\frac{\rho+1}{2} - \frac{\alpha\rho}{2}\big) < \frac{\rho-1}{\alpha\rho}$ leads to $\rho\beta < 1$.

Second step: Estimates for $\mathcal{N}_\alpha^2(u)$. Using inequality (5.1) with $(\gamma_1, \gamma_2, p_1, p_2) = (0, 1, r/q, r)$, in view of $1 < q < \rho < r$, we obtain

$$\|\mathcal{N}_\alpha^2(u)(t) - \mathcal{N}_\alpha^2(v)(t)\|_{\mathcal{M}_{r,\mu}^1}$$
$$\leq C\int_0^t (t-s)^{-\lambda_2}\int_0^s r_\alpha(s-\tau)\|g(u) - g(v)\|_{\mathcal{M}_{r/q,\mu}}d\tau ds,$$

where $g(f)(\tau) = \kappa_2|\nabla_x f(\tau)|^q$ and $\lambda_2 = \frac{\alpha}{2} + \frac{\alpha}{2}\Big(q\frac{N-\mu}{r} - \frac{N-\mu}{r}\Big)$. It follows that

$$\| |\nabla_x u(t)|^q - |\nabla_x v(t)|^q\|_{\mathcal{M}_{r,\mu}} \leq C\|u(t) - v(t)\|_{\mathcal{M}_{r,\mu}^1}\big(\|u(t)\|_{\mathcal{M}_{r,\mu}^1}^{q-1} + \|v(t)\|_{\mathcal{M}_{r,\mu}^1}^{q-1}\big).$$

Hence,

$$\|\mathcal{N}_\alpha^2(u)(t) - \mathcal{N}_\alpha^2(v)(t)\|_{\mathcal{M}_{r,\mu}^1} \leq C|\kappa_1|\int_0^t (t-s)^{-\lambda_2}\tilde{\theta}(s)ds$$

where $\tilde{\theta}(s)$ is bounded as

$$\tilde{\theta}(s) = C\int_0^s (s-\tau)^{\alpha-2}\|u(\tau) - v(\tau)\|_{\mathcal{M}_{r,\mu}^1}\Big(\|u(\tau)\|_{\mathcal{M}_{r,\mu}^1}^{q-1} + \|v(\tau)\|_{\mathcal{M}_{r,\mu}^1}^{q-1}\Big)d\tau$$
$$\leq C\int_0^s (s-\tau)^{\alpha-2}\tau^{-q(\beta+\frac{\alpha}{2})}d\tau\|u-v\|_{X_\beta}\Big(\|u\|_{X_\beta}^{q-1} + \|v\|_{X_\beta}^{q-1}\Big). \tag{5.15}$$

Hence, we estimate

$$\big\|\mathcal{N}_\alpha^2(u)(t) - \mathcal{N}_\alpha^2(v)(t)\big\|_{\mathcal{M}_{r,\mu}^1} \leq C|\kappa_1|I_2(t) \|u-v\|_{X_\beta}\Big(\|u\|_{X_\beta}^{q-1} + \|v\|_{X_\beta}^{q-1}\Big)$$

$$\leq C|\kappa_1|t^{-\beta-\frac{\alpha}{2}}\,\|u-v\|_{X_\beta}\left(\|u\|_{X_\beta}^{q-1}+\|v\|_{X_\beta}^{q-1}\right),$$

where $I_2(t)$ is given by

$$\int_0^t (t-s)^{-\lambda_2}\int_0^s(s-\tau)^{\alpha-2}\tau^{-q(\beta+\frac{\alpha}{2})}d\tau ds = Ct^{-\lambda_2+\alpha-q(\beta+\alpha/2)} = Ct^{-\beta-\frac{\alpha}{2}}.$$

Indeed, in view of $q = \frac{2\rho}{\rho+1}$ and

$$(q-1)\beta = \alpha\frac{q-1}{\rho-1} - \alpha(q-1)\frac{N-\mu}{2r} = \frac{\alpha}{2}(2-q) - \alpha(q-1)\frac{N-\mu}{2r}, \qquad (5.16)$$

we obtain

$$\begin{aligned}
-\lambda_2 + \alpha - q(\beta+\alpha/2) &= -\alpha(q-1)\frac{N-\mu}{2r} - \frac{\alpha}{2} + \alpha - q(\beta+\alpha/2)\\
&= (q-1)\beta - \frac{\alpha}{2}(2-q) - \frac{\alpha}{2} + \alpha - q(\beta+\alpha/2)\\
&= -\frac{\alpha}{2} - \beta.
\end{aligned}$$

It remains to obtain estimates for $\sup_{0<t<T}t^\beta\|\mathcal{N}_\alpha^2(u)(t) - \mathcal{N}_\alpha^2(v)(t)\|_{\mathcal{M}_{r,\mu}}$. Proceeding as before, one has

$$\begin{aligned}
&\|\mathcal{N}_\alpha^2(u)(t) - \mathcal{N}_\alpha^2(v)(t)\|_{\mathcal{M}_{r,\mu}}\\
&\leq C|\kappa_1|J_2(t)\sup_{0<t<T}t^{\beta+\frac{\alpha}{2}}\|\nabla_x u(t) - \nabla_x v(t)\|_{\mathcal{M}_{r,\mu}}\\
&\quad\times \sup_{0<t<T}t^{(\beta+\frac{\alpha}{2})(q-1)}\left(\|\nabla_x u(t)\|_{\mathcal{M}_{r,\mu}}^{q-1} + \|\nabla_x v(t)\|_{\mathcal{M}_{r,\mu}}^{q-1}\right)\\
&\leq C|\kappa_1|J_2(t)\,\|u-v\|_{X_\beta}\left(\|u\|_{X_\beta}^{q-1}+\|v\|_{X_\beta}^{q-1}\right)\\
&\leq C|\kappa_1|t^{-\beta}\,\|u-v\|_{X_\beta}\left(\|u\|_{X_\beta}^{q-1}+\|v\|_{X_\beta}^{q-1}\right),
\end{aligned}$$

where $J_2(t)$ is given by

$$\int_0^t(t-s)^{-\vartheta_2}\int_0^s(s-\tau)^{\alpha-2}\tau^{-q(\beta+\frac{\alpha}{2})}d\tau ds = Ct^{-\vartheta_2+\alpha-q(\beta+\alpha/2)} = Ct^{-\beta}.$$

In fact, the inequality (5.1) with $(\gamma_1,\gamma_2,p_1,p_2) = (0,0,r/q,r)$ implies that $\vartheta_2 = \frac{\alpha}{2}\left(q\frac{N-\mu}{r} - \frac{N-\mu}{r}\right) = \alpha(q-1)\frac{N-\mu}{2r}$, which by (5.16) give us

$$-\vartheta_2 + \alpha - q(\beta + \frac{\alpha}{2}) = (q-1)\beta - \frac{\alpha}{2}(2-q) + \alpha - q(\beta + \frac{\alpha}{2}) = -\beta. \qquad (5.17)$$

The convergence of the beta functions $I_2(t), J_2(t)$ follows by our hypotheses in (3.2), because $\left(1-\frac{p}{r}\right) < \frac{\rho-1}{\alpha}\left(\frac{1}{q} - \frac{\alpha}{2}\right)$ is equivalent to $q(\beta + \alpha/2) < 1$ and $\frac{p}{r} < \left(\frac{1}{\alpha} - \frac{1}{2}\right)$ and $q = \frac{2\rho}{\rho+1}$ yields to $\vartheta_2 = \frac{\alpha}{\rho+1}\frac{p}{r} < \left(1 - \frac{\alpha}{2}\right)$ which is equivalent $\lambda_2 = \frac{\alpha}{2} + \vartheta_2 < 1$.

Third step: The two steps above lead to

$$\begin{aligned}
&\|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(v)\|_{X_\beta}\\
&\leq \|\mathcal{N}_\alpha^1(u) - \mathcal{N}_\alpha^1(v)\|_{X_\beta} + \|\mathcal{N}_\alpha^2(u) - \mathcal{N}_\alpha^2(v)\|_{X_\beta}\\
&\leq K(\kappa_1,\kappa_2)\|u-v\|_{X_\beta}\left[\left(\|u\|_{X_\beta}^{\rho-1}+\|v\|_{X_\beta}^{\rho-1}\right) + \left(\|u\|_{X_\beta}^{q-1}+\|v\|_{X_\beta}^{q-1}\right)\right].
\end{aligned}$$

This completes the proof.   □

6. PROOFS OF THEOREMS

Now, we put all estimates of Section 5 together to prove our theorems.

6.1. **Proof of Theorem 3.1. Part (i):** Notice that

$$\|G_{\alpha,1}(t)\varphi\|_{X_\beta} + \|G_{\alpha,2}(t)\psi\|_{X_\beta} = \|\varphi\|_{D(\alpha,\beta)} + \|\psi\|_{\widetilde{D}(\alpha,\beta)} \le \varepsilon, \tag{6.1}$$

where $\varepsilon > 0$ will be chosen such that

$$\left(2^\rho \varepsilon^{\rho-1} + 2^q \varepsilon^{q-1}\right) < \frac{1}{2K}. \tag{6.2}$$

Consider the complete metric $d_B(\cdot,\cdot)$ defined by $d_B(u,v) = \|u - v\|_{X_\beta}$ on the ball $B_{X_\beta}(2\varepsilon)$ and let $\Lambda : B_{X_\beta}(2\varepsilon) \to B_{X_\beta}(2\varepsilon)$ be the operator

$$\Lambda(u)(t) = G_{\alpha,1}(t)\varphi + G_{\alpha,2}(t)\psi + \mathcal{N}_\alpha(u)(t).$$

We would like to show that $\Lambda(B_{X_\beta}(2\varepsilon)) \subset B_{X_\beta}(2\varepsilon)$ and Λ is a contraction on metric space $(B_{X_\beta}(2\varepsilon), d_B)$. In fact, the continuity of $G_{\alpha,j}(\cdot) : (0,\infty) \to \mathcal{M}_{r,\mu}$, $j = 1,2$, was proved at the final of subsection 5.1, and the regularization property of the convolution imply that $(\Lambda u) : (0,\infty) \to \mathcal{M}_{r,\mu}$ is continuous, whenever $u \in X_\beta$. Further, from Lemma 5.2 we obtain

$$\|\Lambda(u) - \Lambda(v)\|_{X_\beta} = \|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(v)\|_{X_\beta}$$
$$\le K\|u - v\|_{X_\beta} \left[\|u\|_{X_\beta}^{\rho-1} + \|v\|_{X_\beta}^{\rho-1} + \|u\|_{X_\beta}^{q-1} + \|v\|_{X_\beta}^{q-1}\right] \tag{6.3}$$
$$\le K\left(2^\rho \varepsilon^{\rho-1} + 2^q \varepsilon^{q-1}\right) \|u - v\|_{X_\beta}$$

for all $u, v \in B_{X_\beta}(2\varepsilon)$. Now, (6.1) and (6.3) give us

$$\|\Lambda(u)\|_{X_\beta} \le \|G_{\alpha,1}(t)\varphi + G_{\alpha,2}(t)\psi\|_{X_\beta} + \|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(0)\|_{X_\beta}$$
$$\le \varepsilon + K\left(2^\rho \varepsilon^{\rho-1} + 2^q \varepsilon^{q-1}\right) 2\varepsilon < 2\varepsilon, \tag{6.4}$$

in view of (6.2) and provided that $u \in B_{X_\beta}(2\varepsilon)$. Hence, $\Lambda(B_{X_\beta}(2\varepsilon)) \subset B_{X_\beta}(2\varepsilon)$ and Λ is a contraction in $B_{X_\beta}(2\varepsilon)$. From the Banach fixed theorem there is a *mild solution* $u \in X_\beta$ for (1.1)-(1.2) which is unique in the ball $B_{X_\beta}(2\varepsilon)$.

Part (ii): Let u and $\tilde{u}$ be two *mild solutions* in $B_{X_\beta}(2\varepsilon)$, obtained in the Part (i), subject to initial data (φ, ψ) and $(\tilde{\varphi}, \tilde{\psi})$, respectively. Then

$$\|u - \tilde{u}\|_{X_\beta} \le \|G_{\alpha,1}(\varphi - \tilde{\varphi})\|_{X_\beta} + \|G_{\alpha,2}(\psi - \tilde{\psi})\|_{X_\beta} + \|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(\tilde{u})\|_{X_\beta}$$
$$\le \|\varphi - \tilde{\varphi}\|_{D(\alpha,\beta)} + \|\psi - \tilde{\psi}\|_{\widetilde{D}(\alpha,\beta)} + K\left(2^\rho \varepsilon^{\rho-1} + 2^q \varepsilon^{q-1}\right) \|u - \tilde{u}\|_{X_\beta},$$

which yields the Lipschitz continuity

$$\|u - \tilde{u}\|_{X_\beta} \le \frac{1}{\left(1 - \frac{1}{4K}\right)} \left(\|\varphi - \tilde{\varphi}\|_{D(\alpha,\beta)} + \|\psi - \tilde{\psi}\|_{\widetilde{D}(\alpha,\beta)}\right). \tag{6.5}$$

This completes the proof.

6.2. **Proof of Theorem 3.3.** Let $\delta_\lambda f(x) = f(\lambda x)$ and notice that $\widehat{\delta_\lambda f}(\xi) = \lambda^{-N}\widehat{f}(\xi/\lambda)$. Recall that $G_{\alpha,2}(t)\psi$ and $G_{\alpha,1}(t)\varphi$ are given by

$$G_{\alpha,2}(t)\psi := t\, k_{\alpha,2}(t,\cdot) * \psi \quad \text{and} \quad G_{\alpha,1}(t)\varphi := k_{\alpha,1}(t,\cdot) * \varphi,$$

where $\widehat{k}_{\alpha,2}(t,\xi) = tE_{\alpha,2}(-4\pi^2 t^\alpha |\xi|^2)$ and $\widehat{k}_{\alpha,1}(t,\xi) = E_{\alpha,1}(-4\pi^2 t^\alpha |\xi|^2)$. Notice that

$$[\delta_\gamma G_{\alpha,2}(\gamma^{\frac{2}{\alpha}}t)\psi]^\wedge(\xi) = \gamma^{-N}\gamma^{\frac{2}{\alpha}}t\widehat{k}_{\alpha,2}(\gamma^{\frac{2}{\alpha}}t,\xi/\gamma)\widehat{\psi}(\xi/\gamma)$$

$$= \gamma^{\frac{2}{\alpha}} [t E_{\alpha,2}(-4\pi^2 t^\alpha |\xi|^2)] \gamma^{-N} \widehat{\psi}(\xi/\gamma)$$

$$= \gamma^{-\frac{2}{\rho-1}} [G_{\alpha,2}(t)\psi]^\wedge(\xi),$$

in view of $\delta_\gamma \psi(x) = \gamma^{-\frac{2}{\rho-1} - \frac{2}{\alpha}} \psi(x)$. Hence,

$$[G_{\alpha,2}(t)\psi]_\gamma = G_{\alpha,2}(t)\psi_\gamma \quad \text{and} \quad [G_{\alpha,1}(t)\varphi]_\gamma = G_{\alpha,1}(t)\varphi_\gamma.$$

We can easily check that $\mathcal{N}_\alpha(u_\gamma) = [\mathcal{N}_\alpha(u)]_\gamma$, for all $\gamma > 0$. Here, we denoted $[\mathcal{N}_\alpha(u)]_\gamma(t,x) = \gamma^{\frac{2}{\rho-1}} \mathcal{N}_\alpha(u)(\gamma^{\frac{2}{\alpha}} t, \gamma x)$. Therefore,

$$u_\gamma(t) = [G_{\alpha,1}(t)\varphi]_\gamma + [G_{\alpha,2}(t)\psi]_\gamma + [\mathcal{N}_\alpha(u)]_\gamma = G_{\alpha,1}(t)\varphi_\gamma + G_{\alpha,2}(t)\psi_\gamma + \mathcal{N}_\alpha(u_\gamma)$$

is a mild solution $u_\gamma \in X_\beta$ of (1.1)-(1.2). From $\|u_\gamma\|_{X_\beta} = \|u\|_{X_\beta}$ and the uniqueness proved in Theorem 3.1-(i), we have

$$u(t,x) = u_\gamma(t,x), \quad \text{a.e. } x \in \mathbb{R}^N \text{ and for all } \gamma, t > 0.$$

This completes the proof.

6.3. **Proof of Theorem 3.5.** Let $G_{\alpha,2}(t)\psi := k_{\alpha,2}(t,\cdot) * \psi$ and $G_{\alpha,1}(t)\varphi := k_{\alpha,1}(t,\cdot) * \varphi$, be as above. For $T \in \mathcal{A}$, we have

$$k_{\alpha,1}(t, T(x)) = \int_{\mathbb{R}^N} e^{2\pi i \langle T(x), \xi \rangle} E_{\alpha,1}(-4\pi^2 t^\alpha |\xi|^2)) d\xi$$

$$= \int_{\mathbb{R}^N} e^{2\pi i \langle T(x), T(\xi) \rangle} E_{\alpha,1}(-4\pi^2 t^\alpha |T(\xi)|^2)) |\det T| d\xi$$

$$= \int_{\mathbb{R}^N} e^{2\pi i \langle x, \xi \rangle} E_{\alpha,1}(-4\pi^2 t^\alpha |\xi|^2)) d\xi = k_{\alpha,1}(t,x),$$

where we used the change of variable $\xi \mapsto T(\xi)$ and the fact that $|\det T| = 1$. In a similar fashion one has

$$k_{\alpha,2}(t, T(x)) = k_{\alpha,2}(t,x).$$

It follows that

$$u_1(t, T(x)) := \int_{\mathbb{R}^N} k_{\alpha,1}(t, T(x) - y)\varphi(y) dy + \int_{\mathbb{R}^N} k_{\alpha,2}(t, T(x) - y)\psi(y) dy$$

$$= \int_{\mathbb{R}^N} k_{\alpha,1}(t, T(x - z))\varphi(Tz) dz + \int_{\mathbb{R}^N} k_{\alpha,2}(t, T(x - z))\psi(Tz) dz$$

$$= -\int_{\mathbb{R}^N} k_{\alpha,1}(t, x - z)\varphi(z) dz - \int_{\mathbb{R}^N} k_{\alpha,2}(t, x - z)\psi(z) dz$$

$$= u_1(t, x)$$

when φ and ψ are symmetric under action $\mathcal{A}$. Let

$$\theta(s, x) = \int_0^s r_\alpha(s - t) \kappa_2 |u(t,x)|^{\rho-1} u(t,x) - \kappa_1 |\nabla_x u(t,x)|^q$$

and notice that

$$\theta(s, Tx) = \int_0^s r_\alpha(s - t) \left(\kappa_2 |u(t, Tx)|^{\rho-1} u(t, Tx) - \kappa_1 |\nabla_x u(t, Tx)|^q \right) d\tau$$

$$= \int_0^s r_\alpha(s - t) \left(\kappa_2 |u(t,x)|^{\rho-1} u(t,x) - \kappa_1 |T \nabla_x u(t, T(x))|^q \right) d\tau$$

$$= \theta(s, x),$$

that is, $\theta(t, \cdot)$ is symmetric whenever $u(t, \cdot)$ is. Hence,

$$\mathcal{N}_\alpha(u)(t, x) = \int_0^t \int_{\mathbb{R}^N} k_{\alpha,1}(t - s, x - y)\theta(s, y)dyds$$

is symmetric if $u(t, \cdot)$ is, for each $t > 0$. From now on, employing an induction argument in the Picard's sequence

$$u_1(t, x) = G_{\alpha,2}(t)\psi + G_{\alpha,1}(t)\varphi$$
$$u_k(t, x) = u_1(t, x) + \mathcal{N}_\alpha(u_{k-1})(t, x), \; k = 2, 3, \dots$$

one can prove that (u_k) is symmetric. It follows that $u(t, x)$ is symmetric, for all $t > 0$. The proof is complete.

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