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A METHOD FOR SOLVING ILL-POSED ROBIN-CAUCHY PROBLEMS FOR SECOND-ORDER ELLIPTIC EQUATIONS IN MULTI-DIMENSIONAL CYLINDRICAL DOMAINS

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Dedicated to Professor Tynysbek Kalmenov on his 70th birthday

Abstract. In this article we consider the Robin-Cauchy problem for multidimensional elliptic equations in a cylindrical domain. The method of spectral expansion in eigenfunctions of the Robin-Cauchy problem for equations with deviating argument establishes a criterion of the strong solvability of the considered Robin-Cauchy problem. It is shown that the ill-posedness of the Robin-Cauchy problem is equivalent to the existence of an isolated point of the continuous spectrum for a self-adjoint operator with the deviating argument.

1. Introduction

As it is known, the solution of the Cauchy problem for the Laplace equation is unique but unstable. First of all it should be noted that the existence and uniqueness of its solution is essentially guaranteed by the universal Cauchy-Kovalevskaja theorem, which holds for elliptic problems. However, the existence of the solution is guaranteed only in a small data. Traditionally the ill-posedness of the elliptic Cauchy problem is determined in relation to its equivalence to Fredholm integral equations of the first kind. The problem of solving the operator equation of the first kind can not be correct, since the operator which is inverse to completely continuous operator is not continuous.

The Cauchy problem for the Laplace equation is one of the main examples of ill-posed problems. One can pick up the harmonic functions with arbitrarily small Cauchy data on a piece of the domain boundary, which will be arbitrarily large in the domain (the famous example of Hadamard) [5]. For the formulation of the problem to be correct, it is necessary to restrict the class of solutions. The stability of a two dimensional problem in the class of bounded solutions firstly was proved by Carleman [1].

From Carleman's results immediately follow estimations characterizing this stability. In the mentioned work Carleman established a formula for determining an complex variable analytic function from the data only on part of the arc. However, this formula is unstable and therefore can not be directly used as an efficient

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method. The first results related to the construction of an efficient algorithm for solving the problem, best to our knowledge, are published simultaneously in works Pucci [16] and Lavrent'ev [12]. Estimates characterizing the stability of a spatial problem in the class of bounded solutions, were first obtained by M.M. Lavrent'ev [12] for harmonic functions, given in a straight cylinder and vanishing on the generators. The Cauchy data were given on the base of the cylinder. Just after, similar estimates were obtained by Mergelyan [14] for the functions within a sphere and by Lavrent'ev [13] for an arbitrary spatial domain with sufficiently smooth boundary. Around the same time, Landis [11] obtained estimates characterizing the stability of spatial problem for an arbitrary elliptic equation.

The above results laid the foundation for the theory of ill-posed Cauchy problems for elliptic equations. By now this theory has deep development both in the plane, and for the spatial cases, and also for general elliptic equations of high order, etc. Methods of regularization and solutions of ill-posed problems have been proposed in [3, 4, 6, 17, 18, 19]. In these works the concept of conditional correctness of such problems is introduced and algorithms for constructing their solutions are proposed.

In contrast to the presented results, in this paper a new criterion of well-posedness (ill-posedness) of initial boundary value problem for a general second order elliptic equation is proved. The principal difference of our work from the work of other authors is the application of spectral problems for equations with deviating argument in the study of ill posed boundary value problems. The present work is an extension of results [7]-[9] on the case of more general elliptic operators in a multidimensional cylindrical domain.

2. Formulation of the problem and main results

Let $D = \Omega \times (0,1)$ be a cylinder and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary S. In D we consider a mixed Robin-Cauchy problem for elliptic equations

$$\begin{aligned} Lu &\equiv u_{yy}(x,t) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)(x,y) + a(x)u(x,y) \\ &= f(x,y), \quad (x,y) \in D, \end{aligned} \tag{2.1}$$

with the Robin condition

$$\sum_{i,j=1}^{n} \nu_i \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)(x,y) + b(x)u(x,y) = 0, \quad x \in S,\ y \in [0,1], \tag{2.2}$$

and Cauchy conditions

$$u(0,x) = u_y(0,x) = 0, \quad x \in \Omega \cup S. \tag{2.3}$$

Here $a_{ij}(x), a(x)$ and $b(x)$ are given bounded measurable functions satisfying the following conditions:

$$\begin{gathered} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c\sum_{i=1}^{n}\xi_i^2, \quad c \text{ is a positive constant} \\ a_{ij}(x) = a_{ji}(x), \quad a(x), b(x) \geq 0, \end{gathered} \tag{2.4}$$

and $\nu = (\nu_1, \dots, \nu_n)$ denotes the outer unit normal on the boundary S.

Definition 2.1. The function $u \in L^2(D)$ will be called a strong solution of the Robin-Cauchy problem (2.1)-(2.3), if there exists a sequence of functions $u_n \in C^2(\bar{D})$ satisfying conditions (2.2) and (2.3), such that u_n and Lu_n converge in the norm $L^2(D)$ respectively to u and f.

In the future, the following eigenvalue problem for an elliptic equation with deviating argument will play an important role. Find numerical values of λ (eigenvalues), under which the problem for a differential equation with deviating argument

$$\begin{aligned} Lu &\equiv u_{yy}(x,y) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)(x,y) + a(x)u(x,y) \\ &= \lambda u(x,1-y), \quad (x,y) \in D, \end{aligned} \tag{2.5}$$

has nonzero solutions (eigenfunctions) satisfying conditions (2.2) and (2.3). Obviously, the equivalent representation of equation (2.5) has the form

$$LPu(x,y) = \lambda u(x,y), \quad \text{in } D,$$

where $Pu(x,y) = u(x,1-y)$ is a unitary operator.

We consider the spectral problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u_k}{\partial x_j}\Big)(x) + a(x)u_k(x) = \mu_k u_k(x), \quad x \in \Omega, \tag{2.6}$$

$$\sum_{i,j=1}^{n} \nu_i \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u_k}{\partial x_j}\Big)(x) + b(x)u_k(x) = 0, \quad x \in S. \tag{2.7}$$

It is known [2], that problem (2.6)–(2.7) with the condition (2.4) is self-adjoint and non-negative definite operator in $L^2(\Omega)$ and it has a discrete spectrum. All eigenvalues of the problem (2.6)–(2.7) are discrete and non-negative, and the system of eigenfunctions form a complete orthonormal system in $L^2(\Omega)$.

By μ_k we denote all eigenvalues (numbered in decreasing order) and by $u_k(x), k \in \mathbb{N}$ denote a complete system of all orthonormal eigenfunctions of the problem (2.6)-(2.7) in $L^2(\Omega)$.

Theorem 2.2. *The spectral Robin-Cauchy problem* (2.5), (2.2), (2.3) *has a complete orthonormal system of eigenfunctions*

$$u_{km}(x,y) = u_k(x) \cdot v_{km}(y), \tag{2.8}$$

where $k, m \in N$, $v_{km}(y)$ are non-zero solutions of the problem

$$v''_{km}(y) - \mu_k v_{km}(y) = \lambda_{km} v_{km}(1-y), \quad 0 < y < 1, \tag{2.9}$$

$$v_{km}(0) = v'_{km}(0) = 0, \tag{2.10}$$

and λ_{km} are eigenvalues of problem (2.5), (2.2), (2.3). *In addition for large k the smallest eigenvalue λ_{k1} has the asymptotic behavior*

$$\lambda_{k1} = 4\mu_k \exp(-\sqrt{\mu_k})(1 + o(1)). \tag{2.11}$$

Theorem 2.3. *A strong solution of the Robin-Cauchy problem* (2.1)–(2.3) *exists if and only if $f(x,\, y)$ satisfies the inequality*

$$\sum_{k=1}^{\infty} \big|\frac{\tilde{f}_{k1}}{\lambda_{k1}}\big|^2 < \infty, \tag{2.12}$$

where $\tilde{f}_{km} = (f(x, 1-y), u_{km}(x, y))$.

If condition (2.12) *holds, then a solution of* (2.1)–(2.3) *can be written as*

$$u(x,y) = \sum_{k=1}^{\infty} \frac{\tilde{f}_{k1}}{\lambda_{k1}} u_{k1}(x,\, y) + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x,y). \tag{2.13}$$

By $\tilde{L}^2(D)$ we denote a subspace of $L^2(D)$, spanned by the eigenvectors

$$\{u_{k1}(x,y)\}_{k=p+1}^{\infty},$$

$p \in \mathbb{N}$ and by $\hat{L}^2(D)$ we denote its orthogonal complement

$$L^2(D) = \tilde{L}^2(D) \oplus \hat{L}^2(D).$$

Theorem 2.4. *For any* $f \in \hat{L}^2(D)$ *a solution of the problem* (2.1)–(2.3) *exists, is unique and belongs to* $\hat{L}^2(D)$. *This solution is stable and has the form*

$$u(x,y) = \sum_{k=1}^{p} \frac{\tilde{f}_{k1}}{\lambda_{k1}} u_{k1}(x,\, y) + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x,y). \tag{2.14}$$

3. Auxiliary statements

In this section we present some auxiliary results to prove the main results.

Lemma 3.1. *For each fixed value of the index* k *the spectral problem* (2.9)-(2.10) *has a complete orthonormal in* $L^2(0,1)$ *system of eigenfunctions* $v_{km}(y)$, $m \in \mathbb{N}$, *corresponding to the eigenvalues* λ_{km}. *These eigenvalues* λ_{km} *are roots of the equation*

$$\begin{aligned} &\sqrt{\mu_k - \lambda} \cosh \frac{\sqrt{\mu_k + \lambda}}{2} \cosh \frac{\sqrt{\mu_k - \lambda}}{2} \\ &- \sqrt{\mu_k + \lambda} \sinh \frac{\sqrt{\mu_k + \lambda}}{2} \sinh \frac{\sqrt{\mu_k - \lambda}}{2} = 0. \end{aligned} \tag{3.1}$$

Proof. Indeed, applying an inverse operator L_C^{-1} to the Cauchy eigenvalue problem (2.9)–(2.10) we arrive at the operator equation

$$v_{km}(y) = \lambda L_C^{-1} P v_{km}(y),$$

where $Pf(y) = f(1-y)$, and a function $\phi(y) = L_C^{-1} f(y)$ is the solution of the Cauchy problem

$$\phi''(y) - \mu_k \phi(y) = f(y),\ \phi(0) = \phi'(0) = 0, \quad \forall f \in L^2(0,1).$$

Then for the operator L_C^{-1} we have the representation

$$L_C^{-1} f(y) = \frac{1}{\sqrt{\mu_k}} \int_0^y f(\xi) \sinh \sqrt{\mu_k}(y - \xi) d\xi, \quad \forall f \in L^2(0,1). \tag{3.2}$$

Therefore, the adjoint to L_C^{-1} operator has the form

$$(L_C^{-1})^* f(y) = \frac{1}{\sqrt{\mu_k}} \int_y^1 f(\xi) \sinh \sqrt{\mu_k}(\xi - y) d\xi, \quad \forall f \in L^2(0,1). \tag{3.3}$$

Taking into account representation (3.2) and (3.3), it is easy to make sure that

$$L_C^{-1} P f = P (L_C^{-1})^* f.$$

Then the chain of equalities

$$L_C^{-1} P f = P (L_C^{-1})^* f = P^* (L_C^{-1})^* f = (L_C^{-1} P)^* f,\ \forall \quad f \in L^2(0,1),$$

allows us to conclude that the operator $L_C^{-1}P$ is completely continuous self-adjoint Hilbert-Schmidt operator [10]. Therefore for each $k \in \mathbb{N}$, the spectral problem (2.9)–(2.10) has a complete orthonormal system of functions $v_{km}(y)$, $m \in \mathbb{N}$ in $L^2(0,1)$.

We are looking for eigenfunctions of problem (2.5), (2.2), (2.3) by means of the Fourier method of separation of variables in the form

$$u_k(x,y) = u_k(x)v(y),$$

where $k \in \mathbb{N}$. Therefore, to determine the unknown function $v(y)$ we get the spectral problem (2.9), (2.10). It is easy to show that the general solution of equation (2.9) has the form

$$v(y) = c_1 \quad \cosh\sqrt{\mu_k+\lambda}\big(y-\frac{1}{2}\big) + c_2 \quad \sinh\sqrt{\mu_k-\lambda}\big(y-\frac{1}{2}\big),$$

where c_1 and c_2 are some constants. Using the initial conditions (2.9), we arrive at the system of linear homogeneous equations concerning these constants. As we know, this system has a nontrivial solution if the determinant of the system

$$\Delta(\lambda) = \det\begin{pmatrix} \cosh\frac{\sqrt{\mu_k+\lambda}}{2} & \sinh\frac{\sqrt{\mu_k-\lambda}}{2} \\ \sqrt{\mu_k+\lambda}\sinh\frac{\sqrt{\mu_k+\lambda}}{2} & \sqrt{\mu_k-\lambda}\cosh\frac{\sqrt{\mu_k-\lambda}}{2} \end{pmatrix}$$

is zero. Thus, for determining the parameter λ we get (3.1). The proof is complete. □

Let

$$\varpi_k(\lambda) = \ln\Big(\coth\frac{\sqrt{\mu_k+\lambda}}{2}\Big) + \ln\Big(\coth\frac{\sqrt{\mu_k-\lambda}}{2}\Big) - \frac{1}{2}\ln\Big(\frac{\mu_k+\lambda}{\mu_k-\lambda}\Big). \tag{3.4}$$

Lemma 3.2. *There exists a number λ_0 such that for all*

$$0 < \lambda < \lambda_0 < \frac{\mu_k}{4\mu_k+\theta}, \quad k \in \mathbb{N},\ \theta \in (0,\,1)\,,$$

the following statements are true:

1. *the function $\varpi_k'(\lambda)$ is of a fixed sign;*
2. *for the function $\varpi_k''(\lambda)$,*

$$\|\lambda\mu_k\varpi_k''(\lambda)| < 1, \quad k > 1.$$

Proof. By Lemma 3.1 we have the real eigenvalues of (2.9)-(2.10), that is, real roots λ_{km} of equation (3.1). It is easy to verify that $\lambda_{km} > 0$.

Indeed, let us write the asymptotic behavior of the smallest eigenvalues λ_{km} at $k \to \infty$. After a nontrivial transformation of equation (3.1), we have

$$\frac{\sqrt{\mu_k+\lambda}}{\sqrt{\mu_k-\lambda}} = \coth\frac{\sqrt{\mu_k+\lambda}}{2}\coth\frac{\sqrt{\mu_k-\lambda}}{2}. \tag{3.5}$$

Assuming $|\lambda| < 1$ and taking the logarithm of both sides of (3.5), we obtain (3.4). By calculating the derivative $\varpi_k(\lambda)$, we get

$$\varpi_k'(0) = -\frac{1}{\mu_k}.$$

Then the required boundary of monotonicity of $\varpi_k(\lambda)$ can be determined from the relation

$$\varpi_k'(\lambda_0) = \varpi_k'(0) + \varpi_k''(\theta\lambda_0)\lambda_0 < 0.$$

Here $0 < \lambda_0 < 1$ and $\theta \in (0,1)$ are arbitrary numbers. Thus, for determining λ_0 we have the condition

$$\lambda_0 \mu_k \varpi_k''(\theta\lambda_0) < 1. \tag{3.6}$$

We write explicitly the second derivative of $\varpi_k(\lambda)$:

$$\begin{aligned}\varpi_k''(\lambda) &= \frac{\cosh\sqrt{\mu_k+\lambda}}{4(\mu_k+\lambda)\sinh^2\sqrt{\mu_k+\lambda}} + \frac{\cosh\sqrt{\mu_k-\lambda}}{4(\mu_k-\lambda)\sinh^2\sqrt{\mu_k-\lambda}} \\ &\quad + \frac{1}{4\sqrt{(\mu_k+\lambda)^3}\sinh\sqrt{\mu_k+\lambda}} + \frac{1}{4\sqrt{(\mu_k-\lambda)^3}\sinh\sqrt{\mu_k-\lambda}} \\ &\quad - \frac{2\lambda\mu_k}{(\mu_k^2-\lambda^2)^2}.\end{aligned}$$

As

$$\frac{2\lambda_0\theta\mu_k}{(\mu_k^2-(\lambda_0\theta)^2)^2} \geq -\frac{1}{(\mu_k+\lambda_0\theta)^2}$$

and

$$\frac{\cosh\sqrt{\mu_k\pm\lambda_0\theta}}{\sinh^2\sqrt{\mu_k\pm\lambda_0\theta}} \leq \frac{1}{\cosh\sqrt{\mu_k\pm\lambda_0\theta}-1},$$

the inequality

$$\varpi_k''(\lambda_0\theta) \leq \frac{1}{(\mu_k-\lambda_0\theta)} \frac{2+(1-\ exp(-\sqrt{\mu_k-\lambda_0\theta}))^2}{(1-\exp(-\sqrt{\mu_k-\lambda_0\theta}))^2}$$

is true. Hence

$$\varpi_k''(\lambda_0\theta) < \frac{1}{(\mu_k-\lambda_0\theta)} \frac{3-2\exp(-\sqrt{\mu_k-\lambda_0\theta})+\exp(-2\sqrt{\mu_k-\lambda_0\theta})}{(1-\exp(-\sqrt{\mu_k-\lambda_0\theta}))^2}. \tag{3.7}$$

Further, for large values k, from (3.7) we obtain the validity of the inequality

$$\varpi_k''(\lambda_0\theta) \leq \frac{4}{\mu_k-\lambda_0\theta}.$$

Applying the condition (3.6) to the last inequality, we obtain the desired estimate for λ_0:

$$\lambda_0 < \frac{\mu_k}{4\mu_k+\theta}, \quad \mu_k > 1,\ 0<\theta<1.$$

The proof is complete. □

Consider now the question of an asymptotic behavior of the eigenvalues of problem (2.9)–(2.10) for large k.

Lemma 3.3. *An asymptotic behavior of eigenvalues of the problem* (2.9)-(2.10), *not exceeding* λ_0, *for the large values of* k *has the form* (2.11).

Proof. According to Lemma 3.2 the monotonic function $\varpi_k(\lambda)$ in the interval $(0,\lambda_0)$ can have only one zero. By the Taylor formula we have

$$\varpi_k(\lambda) = \varpi_k(0) + \frac{\varpi_k'(0)}{1!}\lambda + \frac{\varpi_k''(\theta\lambda)}{2!}\lambda^2 < 0\,,\ 0<\theta<1.$$

Substituting the calculated values of the function ϖ_k and its derivative ϖ_k', we get

$$\varpi_k(\lambda) = 2\ln\Big(\coth\frac{\sqrt{\mu_k}}{2}\Big) - \frac{\lambda}{\mu_k} + \varpi_k''(\theta\lambda)\frac{\lambda^2}{2}.$$

Then the zero of the linear part of the function

$$\mu_k\varpi_k(\lambda) = 2\mu_k \ln\Big(\coth\frac{\sqrt{\mu_k}}{2}\Big) - \lambda + \frac{\mu_k\lambda^2}{2}\varpi_k''(\theta\lambda)$$

will be

$$\lambda_{k1} = 2\mu_k \ln\Big(\frac{1+\exp(-\sqrt{\mu_k})}{1-\exp(-\sqrt{\mu_k})}\Big).$$

For sufficiently large values $k \in \mathbb{N}$, considering the asymptotic formulas, λ_{k1} can be written as

$$\lambda_{k1} = 4\mu_k \exp(-\sqrt{\mu_k})(1+o(1)).$$

Taking into account the result of Lemma 3.2 on a circle $|\lambda| = 4\mu_k \exp(-\sqrt{\mu_k})(1+\varepsilon)$, where ε is a greatly small positive number, for sufficiently large $k \geq k_0(\varepsilon)$ it is easy to check the validity of the inequality

$$\begin{aligned} &\Big|\varpi_k''(\theta\lambda)\mu_k\lambda^2\Big|_{|\lambda|=4\mu_k\exp(-\sqrt{\mu_k})(1+\varepsilon)} \\ &\leq C\Big|2\mu_k \ln(\frac{1+\exp(-\sqrt{\mu_k})}{1-\exp(-\sqrt{\mu_k})}) - \lambda\Big|_{|\lambda|=4\mu_k\exp(-\sqrt{\mu_k})(1+\varepsilon)}. \end{aligned}$$

Then, by Rouche's theorem [20], we have that the quantity of zeros of $\mu_k\varpi_k(\lambda)$ and its linear part coincide and are inside the circle $|\lambda| = 4\mu_k \exp(-\sqrt{\mu_k})(1+\varepsilon)$. Consequently, the function $\mu_k\varpi_k(\lambda)$ for $0 < \lambda < \lambda_0$ has one zero, the asymptotic behavior is given by formula (2.11). the proof is complete. □

4. Proof the main results

Theorem 2.2. By $u_k(x), k \in \mathbb{N}$ we denote a complete system of orthonormal eigenfunctions of the problem (2.6)-(2.7) in $L^2(\Omega)$. By Lemma 3.1, for each fixed value of the k the spectral problem (2.9)–(2.10) has complete orthonormal system of eigenfunctions $v_{km}(t), \quad m = 1, 2, \dots$ in $L^2(0,1)$. Then the system (2.8) forms a complete orthogonal system in $L^2(D)$. Consequently, problem (2.5), (2.3) does not have other eigenvalues and eigenfunctions. the proof is complte. □

Theorem 2.3. Let $u \in C^2(D)$ be a solution of problem (2.1)–(2.3). Then, by the completeness and orthonormality of eigenfunctions $u_{km}(x,t)$ of problem (2.5), (2.2), (2.3), the function $u(x,t)$ in $L^2(D)$ can be expanded in a series [15]

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_{km}u_{km}(x,t), \tag{4.1}$$

where a_{km} are the Fourier coefficients of the system. Rewriting equation (2.1) in the form

$$\begin{aligned} LPu &= P(u_{yy}(x,y) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})(x,y) + a(x)u(x,y)) \\ &= Pf(x,y), \end{aligned} \tag{4.2}$$

and substituting the solution of form (4.1) in equation (4.2) according to representation

$$P(\frac{\partial^2 u_{km}}{\partial y^2}(x,y) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})(x,y) + a(x)u(x,y)) = \lambda_{km}u_{km}(x,y),$$

we have

$$a_{km} = \frac{\tilde{f}_{km}}{\lambda_{km}},$$

where $\tilde{f}_{km} = (f(x,\, 1-y), u_{km}(x,\, y))$.

Thus for solutions $u(x,y)$ we obtain the following explicit representation

$$u(x,y) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x,y). \tag{4.3}$$

Note that the representation (4.3) remains true for any strong solution of problem (2.1)-(2.3). We have obtained this representation under the assumption that the solution of the Robin-Cauchy problem (2.1)-(2.3) exists.

The question naturally arises, for what subset of the functions $f \in L^2(D)$ there exists a strong solution?

To answer this question, we represent the formula (4.3) in the form (2.13) from which, by Parseval's equality, it follows

$$\|\mathrm{u}\|^2 = \sum_{k=1}^{\infty} \Big|\frac{\tilde{f}_{k1}}{\lambda_{k1}}\Big|^2 + \sum_{k=1}^{\infty}\sum_{m=2}^{\infty} \Big|\frac{\tilde{f}_{km}}{\lambda_{km}}\Big|^2. \tag{4.4}$$

By Lemma 3.3 we have $\lambda_{km} \geq \frac{1}{4}$, $m > 1$. Therefore, the right-hand side of equality (4.4) is bounded only for those $f(x,\, y)$ for which the weighted norm (2.12) is bounded. This fact completes the proof. $\square$

Theorem 2.4. Obviously the operator L is invariant in $\hat{L}^2(D)$. By Theorem 2.3, for any $f \in \hat{L}^2(D)$ there exists a unique solution of problem (2.1)–(2.3) and it can be represented in the form (2.14). Therefore, determined infinite-dimensional space $\hat{L}^2(D)$ is the space of correctness of the Robin-Cauchy problem (2.1)-(2.3). The proof is complete. $\square$

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