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CONTROLLABILLITY OF SECOND-ORDER SOBOLEV-TYPE NEUTRAL IMPULSIVE INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

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Abstract. In this article, we prove sufficient conditions for the controllability of second-order Sobolev-type nonlinear neutral impulsive integrodifferential systems in Banach spaces. The results are obtained by using strongly continuous cosine families of operators and the fixed point approach. An application is provided to illustrate the theory.

1. Introduction

The field of differential equations is very rich and contains a large variety of species. However, there is one basic feature common to all problems defined by a differential equation: the equation relates a function to its derivatives in such a way that the function itself can be determined. In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state, then, generally, one is considering either ordinary or partial differential equations. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system.

A dynamical system may evolve through an observable quantity rather than the state of the system, a general class of evolutionary equations is defined. This class includes standard ordinary and partial differential equations as well as functional differential equations of retarded and neutral type. In this way, the theory serves as a unification of these classical problems. Dynamical systems theory holds the supreme position among all mathematical disciplines as it provides the foundation for unlocking many of the mysteries in nature and the universe which involve the evolution of time. The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting, and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations

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act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention.

A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system [9, 11]. The theory of impulsive differential equations [9, 17, 18] has seen considerable development by the monographs of Bainov and Simeonov [2]. Sobolev type equation appears in variety of physical problems such as flow of fluid through rocks, thermodynamics, propagation of long waves of small amplitude and shear in second order fluid and so on [1, 7]. Balachandran and Dauer [3] provide some sufficient conditions for controllability of integer functional evolution equations of Sobolev type by the theory of semigroup theory via the techniques of fixed point theorem [5, 16, 20].

The concept of impulsive control and its mathematical foundation called impulsive differential equations, or differential equations with impulse effects, or differential equations with discontinuous right hand sides have a long history. In fact, in mechanical systems impulsive phenomena had been studied for a long time under different names such as: mechanical systems with impacts. The study of impulsive control systems (control systems with impulse effects) has also a long history that can be traced back to the beginning of modern control theory. Many impulsive control methods were successfully developed under the framework of optimal control and were occasionally called impulse control.

Controllability is an important property of a control system, and the controllability property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control. A state x is controllable at time t if for some finite time t there exists an input $u(t)$ that transfers the state $x(t)$ from x to the origin at time t. That is a system is called controllable at time t if every state x in the state-space is controllable. It means a system with internal state vector x is called controllable if and only if the system states can be changed by changing the system input. The concept of controllability plays a major role in finite dimensional control theory, so that it is natural to try to generalize it to infinite dimensional system. The nonlinear system of controllability represented by differential equations in a finite dimensional space is discussed many authors by means of fixed point approach [12, 19]. Second order nonlinear differential and integrodifferential equations arise in problems connected with many other physical phenomena. So it is quite significant to study controllability problem for such systems in Banach spaces [4, 14, 15]. An abstract linear second order differential equations are related to strongly continuous cosine families of bounded linear operators [21, 22, 23].

From the above literature, it should be noted that there are several contributions on the existence and controllability of differential equations, existence and controllability of integrodifferential equations with and without randomness using one or more parameter families. Till now, the exact controllability of second order Sobolev-type neutral impulsive integrodifferential systems untreated in the literature.

Motivated by this fact, in this article, we make an attempt to fill this gap by studying controllability of second order Sobolev-type neutral impulsive integrodifferential systems in Banach spaces.

2. Preliminaries

Consider the nonlinear impulsive neutral integrodifferential systems with Sobolev type of the form

$$\begin{aligned}
&\frac{d}{dt}\big[(Bx(t))' + f(t, x(t), x'(t))\big] \\
&= Ax(t) + \int_0^t \mathscr{D}(t-s)x(s)ds + Gu(t) + g(t, x(t), x'(t)) \\
&\quad + \int_0^t k(t, s, x(s), x'(s))ds, \quad t \in \mathcal{I},\ t \neq t_k,
\end{aligned} \tag{2.1}$$

$$x(0) = x_0 \quad x'(0) = y_0 \tag{2.2}$$

$$\Delta x(t_k) = I_k\big(x(t_k), x'(t_k)\big), \quad \Delta x'(t_k) = J_k\big(x(t_k), x'(t_k)\big), \quad k = 1, 2, \dots, m, \tag{2.3}$$

where the state $x(\cdot)$ takes the values in the Banach space X, $x_0, y_0 \in X$, A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X, the interval $\mathcal{I} = [0, b]$, G is a bounded linear operator from U to X and the control function $u(\cdot)$ is given in $\mathscr{L}^2(\mathcal{I}, U)$, a Banach space of admissible control functions with U as a Banach space. B is a linear operator with domain and range contained in a Banach space X. $\mathscr{D}(t-s)$ is closed operator on X with dense domain X which is independent of t and the nonlinear operators $f, g : \mathcal{I} \times X \times X \to X$, $k : \mathcal{I}^2 \times X \times X \to X$ and $\mathcal{I}_k, J_k : X \times X \to X$, $k = 1, 2, \cdots$, are given appropriate functions and the symbol $\Delta x(t)$ represent the jump of the function x at t, which is defined by $\Delta x(t) = x(t^+) - x(t^-)$.

Through out this paper, X is a Banach spaces endowed with the norm $\|\cdot\|$. In what follows, we put $t = 0$, $t_{n+1} = b$ and we denote by $\mathcal{PC}$ the space formed by the functions $u : \mathcal{I} \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exist for all $i = 1, 2, \dots, m$. It is clear that $\mathcal{PC}$, endowed with the norm $\|x\|_{\mathcal{PC}} := \sup_{t \in \mathcal{I}} \|x(t)\|$, is a Banach space. Similarly, $\mathcal{PC}'$ will be the space of the functions $x(\cdot) \in \mathcal{PC}$ such that $\dot{x()}$ is continuously differentiable on I, t_i, $i = 1, 2, \dots, n$ and the derivatives

$$u'_R(t) = \lim_{s \to 0} \frac{u(t+s) - u(t^+)}{s}, \quad u'_L(t) = \lim_{s \to 0} \frac{u(t+s) - u(t^-)}{s}$$

are continuous on $[0, b[$ and $]0, b]$, respectively. Next, for $x \in \mathcal{PC}'$, we represent, by $u'(t)$, the left derivative at $t \in]0, b]$ and, by $u'(0)$, the right derivative at zero. It easy to see that $\mathcal{PC}'$, provided with the norm

$$\|u\|_{\mathcal{PC}'} := \|u\|_{\mathcal{PC}} + \|u'\|_{\mathcal{PC}}$$

is a Banach space.

The operator-valued function $\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$

defined on $D(A)\times E$. From this, it follows that $AS(t) : E \to X$ is bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$ for each $x \in E$. Furthermore, if $x : [0,\infty[\to X$ is locally integrable, then $z(t) := \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function, which is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s)\begin{bmatrix} 0 \\ x(s)\end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)ds \\ \int_0^t C(t-s)x(s)ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

To prove our main theorem we assume certain conditions on the operators A and B. Let X and Y be Banach spaces with norm $|\cdot|$ and $\|\cdot\|$ respectively. The operators $A : \mathcal{D}(A) \subset X \to Y$ and $B : \mathcal{D}(A) \subset X \to Y$ satisfy the following hypothesis:

(1) A and B are closed linear operators,
(2) $\mathcal{D}(B) \subset \mathcal{D}(A)$ and B is bijective,
(3) $B^{-1} : Y \to \mathcal{D}(B)$ is continuous.

These hypothesis and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$. Let $\mathbb{B}_r = \{x \in X : \|x\| \le r\}$ for some $r \ge 1$.

Definition 2.1. A one parameter family $\{C(t), t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathcal{I}$;
(ii) $C(0) = I$;
(iii) $C(t)x$ is continuous in t on $\mathcal{I}$, for each $x \in X$.

Define the associated sine family $S(t), t \in \mathcal{I}$ by

$$S(t)x := \int_0^t C(s)xds, \quad x \in X,\ t \in \mathcal{I}$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{I}\}$ is the operator $A : X \to X$, defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) := \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$.

Define $E := \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$. We assume

(A1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X.

To establish our main theorem, we need the following lemmas.

Lemma 2.2. *Let* (A1) *hold. Then*

(i) *there exist constants* $M \ge 1$ *and* $\omega \ge 0$ *such that* $\|C(t)\| \le Me^{\omega|t|}$ *and*

$$\|S(t) - S(t^*)\| \le M\Big|\int_t^{t^*} e^{\omega|s|}ds\Big|, \quad \text{for } t, t^* \in \mathcal{I};$$

(ii) $S(t)X \subset E$ *and* $S(t)E \subset D(A)$, *for* $t \in \mathcal{I}$;
(iii) $\frac{d}{dt}C(t)x = AS(t)x$, *for* $x \in E$ *and* $t \in \mathcal{I}$;
(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$, *for* $x \in D(A)$ *and* $t \in \mathcal{I}$.

Lemma 2.3 ([23]). *Let* (A1) *hold and* $v : \mathcal{R} \to X$ *be such that* v *is continuous and let* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then* q *is twice continuously differentiable and, for* $t \in \mathcal{I}$: $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ *and*

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

First we study the following Sobolev type neutral impulsive integrodifferential system

$$\begin{aligned} \frac{d}{dt}\big[(Bx(t))' + f(t,x(t))\big] &= Ax(t) + \int_0^t \mathscr{D}(t-s)x(s)ds + Gu(t) + g(t,x(t)) \\ &\quad + \int_0^t k(t,s,x(s))ds, \quad t \in (0,b],\ t \neq t_k, \end{aligned} \tag{2.4}$$

$$x(0) = x_0, \quad x'(0) = y_0 \tag{2.5}$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad \Delta x'(t_k) = J_k(x_{t_k}), \quad k = 1,2,\dots,m. \tag{2.6}$$

Definition 2.4. A continuous solution $x(\cdot)$ of the integral equation

$$\begin{aligned} x(t) &= B^{-1}S(t)[By_0 + f(0,x(0))] + B^{-1}C(t)Bx_0 \\ &\quad - \int_0^t B^{-1}C(t-s)f(s,x(s))ds \\ &\quad + \int_0^t B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &\quad + \int_0^t B^{-1}S(t-s)Gu(s)ds + \int_0^t B^{-1}S(t-s)\Big[g(s,x(s)) \\ &\quad + \int_0^s k(s,\tau,x(\tau))d\tau\Big]ds + \sum_{0<t_k<t} B^{-1}C(t-t_k)I_k x(t_k) \\ &\quad + \sum_{0<t_k<t} B^{-1}S(t-t_k)J_k x(t_k) \end{aligned} \tag{2.7}$$

is said to be a mild solution of problem (2.4)-(2.6) on $\mathcal{I}$.

If $x(\cdot)$ is a mild solution of (2.4)-(2.6), then by the properties of a second order differential equation and Lemma 2.3, we have

$$\begin{aligned} x'(t) &= B^{-1}C(t)[By_0 + f(0,x(0))] + B^{-1}AS(t)Bx_0 - B^{-1}f(t,x(t)) \\ &\quad - \int_0^t B^{-1}AS(t-s)f(s,x(s))ds + \int_0^t B^{-1}C(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &\quad + \int_0^t B^{-1}C(t-s)Gu(s)ds + \int_0^t B^{-1}C(t-s)\Big[g(s,x(s)) \\ &\quad + \int_0^s k(s,\tau,x(\tau))d\tau\Big]ds + \sum_{0<t_k<t} B^{-1}AS(t-t_k)I_k x(t_k) \\ &\quad + \sum_{0<t_k<t} B^{-1}C(t-t_k)J_k x(t_k), \quad t \in \mathcal{I}. \end{aligned}$$

To study the controllability problem, we assume the following hypotheses:

(H1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t),\ t \in \mathcal{I}\}$ of bounded linear operators in the Banach space X. There exist constants $\mathscr{M}_1 \geq 1$ and $\mathscr{M}_2, \mathscr{L}_D \geq 0$ such that $\|C(t)\| \leq \mathscr{M}_1$, $\|S(t)\| \leq \mathscr{M}_2$, and $\|\mathscr{D}(t-s)\| \leq \mathscr{L}_D$, for every $t \in \mathcal{I}$. Furthermore we take $\mathscr{M}_3 = \sup_{t\in\mathcal{I}} \|AS(t)\|$, $\mathscr{N}_1 = \|B^{-1}\|$, and $\mathscr{N}_2 = \|B\|$.

(H2) The linear operator $\mathscr{W}_1 : \mathscr{L}^2(\mathcal{I}, U) \to X$ defined by

$$\mathscr{W}_1 u = \int_0^b B^{-1} S(b-s) G u(s) ds$$

has an inverse operator $\mathscr{W}_1^{-1}$ which takes values in $\mathscr{L}^2(\mathcal{I}, U)/\ker \mathscr{W}_1$ and there exists a positive constant $\mathscr{K}_1$ such that $\|G\mathscr{W}_1^{-1}\| \leq \mathscr{K}_1$.

(H3) The linear operator $\mathscr{W}_2 : \mathscr{L}^2(\mathcal{I}, U) \to X$ defined by

$$\mathscr{W}_2 u = \int_0^b B^{-1} C(b-s) G u(s) ds$$

has an inverse operator $\mathscr{W}_2^{-1}$ which takes values in $\mathscr{L}^2(\mathcal{I}, U)/\ker \mathscr{W}_2$ and there exists a positive constant $\mathscr{K}_2$ such that $\|G\mathscr{W}_2^{-1}\| \leq \mathscr{K}_2$.

(H4) $\mathscr{W}_1 \mathscr{W}_2^{-1} x = \mathscr{W}_2 \mathscr{W}_1^{-1} x = 0$, for every $x \in X$.

(H5) The function $f : \mathcal{I} \times X \to X$ is continuous for a.e. $t \in \mathcal{I}$. and the function $f(.,x) : \mathcal{I} \times X \to X$ is strongly measurable, for each $x \in X$. Then there exist positive constants $\mathscr{L}_f > 0$, $\mathscr{F}_0 > 0$ such that

$$\|f(t, x_1(t)) - f(s, x_2(t))\| \leq \mathscr{L}_f[|t-s| + \|x_1 - x_2\|],$$

for $t, s \in \mathcal{I}$ and $x_i \in X$, $i = 1, 2$, and

$$\max_{t\in\mathcal{I}} \|f(t, 0)\| = \mathscr{F}_0.$$

(H6) The function $g : \mathcal{I} \times X \to X$ satisfies the following conditions:
   (i) For each $t \in \mathcal{I}$, the function $g(t, \cdot) : \mathcal{I} \times X \to X$ is continuous and for each $x \in X$, the function $g(\cdot, x) : \mathcal{I} \times X \to X$ is strongly measurable.
   (ii) There exist a constants $\mathscr{L}_g > 0, \mathscr{G}_0$ such that

$$\|g(t, x_1) - g(s, x_2)\| \leq \mathscr{L}_g[|t-s| + \|x_1 - x_2\|], \quad \text{for } t \in I \text{ and } x_i \in X,\ i = 1, 2$$

and

$$\max_{t\in\mathcal{I}} \|g(t, 0)\| \leq \mathscr{G}_0, \quad \text{for } t \in I.$$

(H7) The function $k : \mathcal{I}^2 \times X \to X$ satisfies the following condition:
   (i) For each $t, s \in \mathcal{I}$, the function $k(t, s, \cdot) : \mathcal{I}^2 \times X \to X$ is continuous and for each $x \in X$, the function $k(\cdot, \cdot, x) : \mathcal{I}^2 \times X \to X$ is strongly measurable.
   (ii) There exists a constant $\mathscr{L}_k > 0$, $\mathscr{K}_0$ such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq \mathscr{L}_k[\|x_1 - x_2\|], \quad \text{for } t, s \in \mathcal{I} \text{ and } x_i \in X,\ i = 1, 2$$

and

$$\max_{t\in\mathcal{I}} \|k(t, s, 0)\| \leq \mathscr{K}_0, \quad \text{for } t, s \in \mathcal{I}.$$

(H8) $I_k, J_k : X \to X$, $k = 1, 2, \dots, m$, are continuous and there exist constants $\mathscr{L}_I > 0$, $\mathscr{L}_J > 0$, $\mathcal{I}_0 > 0$ and $\mathcal{J}_0 > 0$ such that

$$\|I_k(x_1) - I_k(x_2)\| \leq \mathscr{L}_I \|x_1 - x_2\|,$$
$$\|J_k(x_1) - J_k(x_2)\| \leq \mathscr{L}_J \|x_1 - x_2\|,$$
$$\mathcal{I}_0 = \|I_k(0)\|, \quad \mathscr{J}_0 = \|J_k(0)\|, \quad k = 1, 2, \cdots .$$

for all $x_1, x_2 \in X$ and $k = 1, 2, \dots, m$.

(H9) There exist constants $\rho > 0$, $\widehat{\rho} > 0$ such that

$$\begin{aligned}
&\mathscr{N}_1\mathscr{M}_2[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{N}_2\mathscr{M}_1\|x_0\| + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_f + \mathscr{F}_0] \\
&+ b^2 r\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2\mathcal{S}_0 + b\mathscr{R}_1\mathscr{M}_2[r\mathscr{L}_g + \mathscr{G}_0 + b\{r\mathscr{L}_k + \mathscr{K}_0\}] \\
&+ \mathscr{R}_1\mathscr{M}_1\sum_{k=0}^{m}[r\mathscr{L}_I + \mathcal{I}_0] + \mathscr{R}_1\mathscr{M}_2\sum_{k=0}^{m}[r\mathscr{L}_J + \mathscr{J}_0] \le \rho
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{N}_1\mathscr{M}_1[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_3\mathscr{N}_2\|x_0\| + \mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] + b\mathscr{N}_1\mathscr{M}_3[r\mathscr{L}_f + \mathscr{F}_0] \\
&+ b^2 r\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1\mathcal{S}_0 + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_g + \mathscr{G}_0 + b\{r\mathscr{L}_k + \mathscr{K}_0\}] \\
&+ \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}[r\mathscr{L}_I + \mathcal{I}_0] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}[r\mathscr{L}_J + \mathscr{J}_0] \le \widehat{\rho},
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{S}_0 \\
&= \mathscr{K}_1\Big[\|x_b\| + \mathscr{N}_1\mathscr{M}_2[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_1\mathscr{N}_2\|x_0\| + b\mathscr{M}_1\mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] \\
&\quad + b^2 r\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[r\mathscr{L}_g + \mathscr{G}_0 + b\{r\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}[r\mathscr{L}_I + \mathcal{I}_0] \\
&\quad + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^{m}[r\mathscr{L}_J + \mathscr{J}_0]\Big] + \mathscr{K}_2\Big[\|y_b\| + \mathscr{N}_1\mathscr{M}_1[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_3\mathscr{N}_2\|x_0\| \\
&\quad + \mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] + b\mathscr{M}_3\mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] + b^2 r\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}[r\mathscr{L}_I + \mathcal{I}_0] \\
&\quad + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_g + \mathscr{G}_0 + b\{r\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}[r\mathscr{L}_J + \mathscr{J}_0]\Big].
\end{aligned}$$

Definition 2.5 ([13])**.** System (2.4)-(2.6) is said to be controllable on the interval $\mathcal{I}$, if for every initial functions $x_0,\ x_b \in X$ and $y_0,\ y_b \in X$, there exists a control $u \in \mathscr{L}^2(\mathcal{I}, U)$ such that the solution $x(\cdot)$ of (2.4)-(2.6) satisfies $x(0) = x_0$, $x(b) = x_b$ and $x'(0) = y_0,\ x'(b) = y_b$.

3. Controllability result

Theorem 3.1. *If assumptions* (H1)–(H9) *hold and if* $0 \le \Lambda_1,\ \Lambda_2 < 1$, *then system* (2.4)–(2.6) *is controllable on* $\mathcal{I}$, *provided that there exist constants*

$$\begin{aligned}
\Lambda_1 &= (1 + b\mathscr{N}_1\mathscr{M}_2\mathscr{K}_1)\Big[b\mathscr{M}_1\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[\mathscr{L}_g + b\mathscr{L}_k] \\
&\quad + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_I + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^{m}\mathscr{L}_J\Big] + b\mathscr{N}_1\mathscr{M}_2\mathscr{K}_2\Big[\mathscr{N}_1\mathscr{L}_f + b\mathscr{M}_3\mathscr{N}_1\mathscr{L}_f \\
&\quad + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1\mathscr{L}_g + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}\mathscr{L}_I + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_J\Big]
\end{aligned}$$

and

$$\Lambda_2 = b\mathscr{N}_1\mathscr{M}_1\mathscr{W}_1\Big[b\mathscr{M}_1\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[\mathscr{L}_g + b\mathscr{L}_k] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_I \\ + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^{m}\mathscr{L}_J\Big] + (1 + b\mathscr{N}_1\mathscr{M}_1)\mathscr{W}_2\Big[\mathscr{N}_1\mathscr{L}_f + b\mathscr{M}_3\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D \\ + b\mathscr{N}_1\mathscr{M}_1\mathscr{L}_g + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}\mathscr{L}_I + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_J\Big].$$

Proof. Using (H2), (H3) for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u(t) = \mathscr{W}_1^{-1}\Big[&x_b - B^{-1}S(b)[By_0 + f(0, x(0))] - B^{-1}C(b)Bx_0 \\ &+ \int_0^b B^{-1}C(b-s)f(s, x(s))ds - \int_0^b B^{-1}S(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &- \int_0^b B^{-1}S(b-s)\big[g(s, x(s)) + \int_0^s k(s, \tau, x(\tau))d\tau\big]ds \\ &- \sum_{0<t_k<b} B^{-1}C(b-t_k)I_k x(t_k) + \sum_{0<t_k<b} B^{-1}S(b-t_k)J_k x(t_k)\Big](t) \\ &+ \mathscr{W}_2^{-1}\Big[y_b - B^{-1}C(b)[By_0 + f(0, x(0))] - B^{-1}AS(b)Bx_0 + f(b, x(b)) \\ &+ \int_0^b B^{-1}AS(b-s)f(s, x(s))ds - \int_0^b B^{-1}C(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &- \int_0^b B^{-1}C(b-s)\big[g(s, x(s)) + \int_0^s k(s, \tau, x(\tau))d\tau\big]ds \\ &- \sum_{0<t_k<b} B^{-1}AS(b-t_k)I_k x(t_k) + \sum_{0<t_k<b} B^{-1}C(b-t_k)J_k x(t_k)\Big](t). \end{aligned}$$

Now we have to show that, when using this control $u(t)$, the nonlinear operator

$$\mathscr{P} : \mathcal{PC} \to \mathcal{PC}$$

defined by

$$\begin{aligned} &(\mathscr{P}x)(t) \\ &= B^{-1}S(t)[By_0 + f(0, x(0))] + B^{-1}C(t)Bx_0 \\ &\quad - \int_0^t B^{-1}C(t-s)f(s, x(s))ds + \int_0^t B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &\quad + \int_0^t B^{-1}S(t-s)\Big\{G\mathscr{W}_1^{-1}\Big[x_b - B^{-1}S(b)[By_0 + f(0, x(0))] - B^{-1}C(b)Bx_0 \\ &\quad + \int_0^b B^{-1}C(b-s)f(s, x(s))ds - \int_0^b B^{-1}S(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &\quad - \int_0^b B^{-1}S(b-s)\big[g(s, x(s)) + \int_0^s k(s, \tau, x(\tau))d\tau\big]ds \\ &\quad - \sum_{0<t_k<b} B^{-1}C(b-t_k)I_k x(t_k) + \sum_{0<t_k<b} B^{-1}S(b-t_k)J_k x(t_k)\Big](s) \end{aligned}$$

$$
\begin{aligned}
&+ G\mathscr{W}_2^{-1}\Big[y_b - B^{-1}C(b)[By_0 + f(0,x(0))] - B^{-1}AS(b)Bx_0 + B^{-1}f(b,x(b)) \\
&+ \int_0^b B^{-1}AS(b-s)f(s,x(s))ds - \int_0^b B^{-1}C(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\
&- \int_0^b B^{-1}C(b-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]ds \\
&- \sum_{0<t_k<b} B^{-1}AS(b-t_k)I_kx(t_k) + \sum_{0<t_k<b} B^{-1}C(b-t_k)J_kx(t_k)\Big](s)\Big\}ds \\
&+ \int_0^t B^{-1}S(t-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]ds \\
&+ \sum_{0<t_k<t} B^{-1}C(t-t_k)I_kx(t_k) + \sum_{0<t_k<t} B^{-1}S(t-t_k)J_kx(t_k)
\end{aligned}
$$

has a fixed point $x(\cdot)$, which is the solution of the system (2.4)–(2.6). Clearly $x(b) = x_b,\ x'(b) = y_b$, which imply that the system is controllable. Since all the functions involved in the operator are continuous, $\mathscr{P}$ is continuous. For convenience, let

$$
\begin{aligned}
&\mathscr{S}(s,x) \\
&= G\mathscr{W}_1^{-1}\Big[x_b - B^{-1}S(b)[By_0 + f(0,x(0))] - B^{-1}C(b)Bx_0 \\
&\quad + \int_0^b B^{-1}C(b-s)f(s,x(s))ds - \int_0^b B^{-1}S(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\
&\quad - \int_0^b B^{-1}S(b-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]ds \\
&\quad - \sum_{0<t_k<b} B^{-1}C(b-t_k)I_kx(t_k) + \sum_{0<t_k<b} B^{-1}S(b-t_k)J_kx(t_k)\Big](s) \\
&\quad + G\mathscr{W}_2^{-1}\Big[y_b - B^{-1}C(b)[By_0 + f(0,x(0))] - B^{-1}AS(b)Bx_0 + B^{-1}f(b,x(b)) \\
&\quad + \int_0^b B^{-1}AS(b-s)f(s,x(s))ds - \int_0^b B^{-1}C(b-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\
&\quad - \int_0^b B^{-1}C(b-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]ds \\
&\quad - \sum_{0<t_k<b} B^{-1}AS(b-t_k)I_kx(t_k) + \sum_{0<t_k<b} B^{-1}C(b-t_k)J_kx(t_k)\Big](s).
\end{aligned}
$$

From assumptions (H1)–(H9), we have

$$
\begin{aligned}
&\|\mathscr{S}(s,x)\| \\
&\le \mathscr{K}_1\Big[\|x_b\| + \mathscr{N}_1\mathscr{M}_2[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_1\mathscr{N}_2\|x_0\| + b\mathscr{M}_1\mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] \\
&\quad + b^2r\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[r\mathscr{L}_g + \mathscr{G}_0 + b\{r\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^m[r\mathscr{L}_I + \mathscr{I}_0] \\
&\quad + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^m[r\mathscr{L}_J + \mathscr{J}_0]\Big] + \mathscr{K}_2\Big[\|y_b\| + \mathscr{N}_1\mathscr{M}_1[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_3\mathscr{N}_2\|x_0\| \\
&\quad + \mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] + b\mathscr{M}_3\mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0] + b^2r\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_g + \mathscr{G}_0
\end{aligned}
$$

$$
\begin{aligned}
&+ b\{r\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_3 \sum_{k=0}^{m} [r\mathscr{L}_I + \mathcal{I}_0] + \mathscr{N}_1\mathscr{M}_1 \sum_{k=0}^{m} [r\mathscr{L}_J + \mathscr{J}_0]\Big] \\
&= \mathcal{S}_0
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\mathscr{S}(s,x_1) - \mathscr{S}(s,x_2)\| \\
&\le \Big\{\mathscr{K}_1\Big[b\mathscr{M}_1\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2\mathscr{L}_g + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_1 \sum_{k=0}^{m} \mathscr{L}_I \\
&\quad + \mathscr{N}_1\mathscr{M}_2 \sum_{k=0}^{m} \mathscr{L}_J\Big] + \mathscr{K}_2\Big[\mathscr{N}_1\mathscr{L}_f + b\mathscr{M}_3\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1\mathscr{L}_g \\
&\quad + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_3 \sum_{k=0}^{m} \mathscr{L}_I + \mathscr{N}_1\mathscr{M}_1 \sum_{k=0}^{m} \mathscr{L}_J\Big]\Big\}\|x_1 - x_2\|.
\end{aligned}
$$

First we show that $\mathscr{P}$ maps $\mathcal{PC}$ into itself. Now

$$
\begin{aligned}
\|(\mathscr{P}x)(t)\| &\le \|B^{-1}S(t)[By_0 + f(0,x(0))]\| + \|B^{-1}C(t)Bx_0\| \\
&\quad + \int_0^t \|B^{-1}C(t-s)f(s,x(s))\|ds + \int_0^t \|B^{-1}S(t-s)\mathscr{S}(s,x)\|ds \\
&\quad + \int_0^t \|B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau\|ds \\
&\quad + \int_0^t \|B^{-1}S(t-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]\|ds \\
&\quad + \sum_{0<t_k<t} \|B^{-1}C(t-t_k)I_k x(t_k)\| + \sum_{0<t_k<t} \|B^{-1}S(t-t_k)J_k x(t_k)\| \\
&\le \mathscr{N}_1\mathscr{M}_2[\mathscr{N}_2\|y_0\| + \mathscr{L}_f] + \mathscr{N}_1\mathscr{M}_1\mathscr{N}_2\|x_0\| + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_f + \mathscr{F}_0] \\
&\quad + b^2 r\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2\mathcal{S}_0 + b\mathscr{N}_1\mathscr{M}_2[r\mathscr{L}_g + \mathscr{G}_0 \\
&\quad + b\{r\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_1 \sum_{k=0}^{m} [r\mathscr{L}_I + \mathcal{I}_0] + \mathscr{N}_1\mathscr{M}_2 \sum_{k=0}^{m} [r\mathscr{L}_J + \mathscr{J}_0] \\
&< \rho
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\mathscr{P}x)'(t)\| \\
&\le \|B^{-1}C(t)[By_0 + f(0,x(0))]\| + \|B^{-1}AS(t)Bx_0\| + \|B^{-1}f(t,x(t))\| \\
&\quad + \int_0^t \|B^{-1}AS(t-s)f(s,x(s))\|ds + \int_0^t \|B^{-1}C(t-s)\mathscr{S}(s,x)\|ds \\
&\quad + \int_0^t \|B^{-1}C(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau\|ds \\
&\quad + \int_0^t \|B^{-1}C(t-s)\big[g(s,x(s)) + \int_0^s k(s,\tau,x(\tau))d\tau\big]\|ds \\
&\quad + \sum_{0<t_k<t} \|B^{-1}AS(t-t_k)I_k x(t_k)\| + \sum_{0<t_k<t} \|B^{-1}C(t-t_k)J_k x(t_k)\| \\
&\le \mathscr{N}_1\mathscr{M}_1[\mathscr{N}_2\|y_0\| + \mathscr{F}_0] + \mathscr{N}_1\mathscr{M}_3\mathscr{N}_2\|x_0\| + \mathscr{N}_1[r\mathscr{L}_f + \mathscr{F}_0]
\end{aligned}
$$

$$
\begin{aligned}
&+ b\mathscr{N}_1\mathscr{M}_3[r\mathscr{L}_f + \mathscr{F}_0] + b^2 r\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1\mathcal{S}_0 + b\mathscr{N}_1\mathscr{M}_1[r\mathscr{L}_g + \mathscr{G}_0 \\
&+ br\{\mathscr{L}_k + \mathscr{K}_0\}] + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}[r\mathscr{L}_I + \mathcal{I}_0] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}[r\mathscr{L}_J + \mathscr{J}_0] < \widehat{\rho}.
\end{aligned}
$$

Therefore $\mathscr{P}$ maps from $\mathcal{PC}$ into itself. Moreover, if $x_1, x_2 \in \mathcal{PC}$, then

$$
\begin{aligned}
&\|(\mathscr{P}x_1)(t) - (\mathscr{P}x_2)(t)\| \\
&\le \Big\| \int_0^t B^{-1}C(t-s)[f(s,x_1(s)) - f(s,x_2(s))]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)[x_1(\tau) - x_2(\tau)]d\tau ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}S(t-s)[\mathscr{S}(s,x_1) - \mathscr{S}(s,x_2)]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}S(t-s)[g(s,x_1(s)) - g(s,x_2(s))]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}S(t-s)\int_0^s [(k(s,\tau,x_1(\tau)) - k(s,\tau,x_2(\tau)))d\tau]ds \Big\| \\
&\quad + \Big\| \sum_{0<t_k<t} B^{-1}C(t-t_k)[I_k x_1(t_k) - I_k x_2(t_k)] \Big\| \\
&\quad + \Big\| \sum_{0<t_k<t} B^{-1}S(t-t_k)[J_k x_1(t_k) - J_k x_2(t_k)] \Big\| \\
&\le \Big\{ b\mathscr{N}_1\mathscr{M}_1\mathscr{L}_F + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[\mathscr{L}_g + b\mathscr{L}_k] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_I \\
&\quad + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^{m}\mathscr{L}_J + b\mathscr{N}_1\mathscr{M}_2\mathscr{K}_1\Big[b\mathscr{M}_1\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2\mathscr{L}_g \\
&\quad + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_I + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^{m}\mathscr{L}_J \Big] + b\mathscr{N}_1\mathscr{M}_2\mathscr{K}_2\Big[\mathscr{N}_1\mathscr{L}_f \\
&\quad + b\mathscr{M}_3\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1\mathscr{L}_g + b\mathscr{L}_k + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^{m}\mathscr{L}_I \\
&\quad + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^{m}\mathscr{L}_J \Big]\Big\} \|x_1 - x_2\| = \Lambda_1\|x_1 - x_2\|.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\|(\mathscr{P}x_1)'(t) - (\mathscr{P}x_2)'(t)\| \\
&\le \Big\| B^{-1}[f(s,x_1(s)) - f(s,x_2(s))] \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}AS(t-s)[f(s,x_1(s)) - f(s,x_2(s))]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}C(t-s)\int_0^s \mathscr{D}(s-\tau)[x_1(\tau) - x_2(\tau)]d\tau ds \Big\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \Big\| \int_0^t B^{-1}C(t-s)[\mathscr{S}(s,x_1) - \mathscr{S}(s,x_2)]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}C(t-s)[g(s,x_1(s)) - g(s,x_2(s))]ds \Big\| \\
&\quad + \Big\| \int_0^t B^{-1}C(t-s) \int_0^s [(k(s,\tau,x_1(\tau)) - k(s,\tau,x_2(\tau))d\tau)]ds \Big\| \\
&\quad + \Big\| \sum_{0<t_k<t} B^{-1}AS(t-t_k)[I_k x_1(t_k) - I_k x_2(t_k)] \Big\| \\
&\quad + \Big\| \sum_{0<t_k<t} B^{-1}C(t-t_k)[J_k x_1(t_k) - J_k x_2(t_k)] \Big\| \\
&\le \Big\{ \mathscr{N}_1\mathscr{L}_f + b\mathscr{N}_1\mathscr{M}_3\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_1[\mathscr{L}_g + b\mathscr{L}_k] \\
&\quad + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^m \mathscr{L}_I + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^m \mathscr{L}_J + b\mathscr{N}_1\mathscr{M}_1\mathscr{K}_1\Big[b\mathscr{M}_1\mathscr{N}_1\mathscr{L}_f \\
&\quad + b^2\mathscr{N}_1\mathscr{M}_2\mathscr{L}_D + b\mathscr{N}_1\mathscr{M}_2[\mathscr{L}_g + b\mathscr{L}_k] + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^m \mathscr{L}_I \\
&\quad + \mathscr{N}_1\mathscr{M}_2\sum_{k=0}^m \mathscr{L}_J\Big] + b\mathscr{N}_1\mathscr{M}_1\mathscr{K}_2\Big[\mathscr{N}_1\mathscr{L}_f + b\mathscr{M}_3\mathscr{N}_1\mathscr{L}_f + b^2\mathscr{N}_1\mathscr{M}_1\mathscr{L}_D \\
&\quad + b\mathscr{N}_1\mathscr{M}_1[\mathscr{L}_g + b\mathscr{L}_k] + \mathscr{N}_1\mathscr{M}_3\sum_{k=0}^m \mathscr{L}_I + \mathscr{N}_1\mathscr{M}_1\sum_{k=0}^m \mathscr{L}_J\Big]\Big\}\|x_1 - x_2\| \\
&= \Lambda_2\|x_1 - x_2\|.
\end{aligned}
$$

Since $\Lambda_1 < 1$ and $\Lambda_2 < 1$, the operator $\mathscr{P}$ is a contraction. Consequently by the Banach contraction fixed point theorem, there exists a unique fixed point $x \in \mathcal{PC}$ such that $(\mathscr{P}x)(t) = x(t)$. This fixed point is then the solution of the problem (2.4)-(2.6). Then clearly, $(\mathscr{P}x)(b) = x(b) = x_b$, $(\mathscr{P}x)'(b) = x'(b) = y_b$ which implies that the system (2.4)-(2.6) is controllable on $\mathcal{I}$. Thus the proof is complete. □

Now to study the controllability of (2.1)-(2.3), we impose the following additional hypotheses:

(H10) The function $f : \mathcal{I} \times X \times X \to X$ is continuous for a.e. $t \in \mathcal{I}$. and the function $f(.,x,y) : \mathcal{I} \times X \times X \to X$ is strongly measurable, for each $x \in X$. Then there exist positive constants $\mathscr{L}_F > 0,\ F_0 > 0$ such that

$$\|f(t,x_1(t),y_1(t)) - f(s,x_2(t),y_2(t))\| \le \mathscr{L}_F[|t-s| + \|x_1 - x_2\| + \|y_1 - y_2\|],$$

for $t,s \in \mathcal{I},\ x_i,y_i \in X,\ i = 1,2$ and

$$\max_{t\in\mathcal{I}} \|f(t,0,0)\| = F_0.$$

(H11) The function $g : \mathcal{I} \times X \times X \to X$ satisfies the following conditions:
  - (i) For each $t \in \mathcal{I}$, the function $g(t,\cdot,\cdot) : \mathcal{I} \times X \times X \to X$ is continuous and for each $x \in X$, the function $g(\cdot,x,\ y) : \mathcal{I}\times X\times X \to X$ is strongly measurable.
  - (ii) There exist a constants $\mathscr{L}_G > 0,\ G_0 > 0$ such that

$$\|g(t,x_1,y_1) - g(s,x_2,y_2)\| \le \mathscr{L}_G[|t-s| + \|x_1 - x_2\| + \|y_1 - y_2\|],$$

for $t, s \in \mathcal{I}$, and $x_i, y_i \in X,\ i = 1, 2$, and

$$\max_{t\in\mathcal{I}} \|g(t, 0, 0)\| \le G_0, \quad \text{for } t \in I.$$

(H12) The function $k : \mathcal{I}^2 \times X \times X \to X$ satisfies the following conditions:
 (i) For each $t, s \in \mathcal{I}$, the function $k(t, s, \cdot,\cdot) : \mathcal{I}^2 \times X \times X \to X$ is continuous and for each $x \in X$, the function $k(\cdot, \cdot, x, y) : \mathcal{I}^2 \times X \times X \to X$ is strongly measurable.
 (ii) There exists a constant $\mathscr{L}_K > 0,\ K_0 > 0$ such that

$$\|k(t, s, x_1, y_1) - k(t, s, x_2, y_2)\| \le \mathscr{L}_K[\|x_1 - x_2\| + \|y_1 - y_2\|],$$

for $t, s \in \mathcal{I}$, and $x_i, y_i \in X$, $i = 1, 2$, and

$$\max_{t\in\mathcal{I}} \|k(t, s, 0, 0)\| \le K_0, \quad \text{for } t, s \in \mathcal{I}$$

(H13) $I_k, J_k : X \times X \to X, k = 1, 2, \dots, m$, are continuous and there exist constants $\mathcal{L}_I > 0$, $\mathcal{L}_J > 0$, $I_0 > 0$ and $J_0 > 0$ such that

$$\begin{aligned}
\|I_k(x_1, y_1) - I_k(x_2, y_2)\| &\le \mathcal{L}_I[\|x_1 - x_2\| + \|y_1 - y_2\|],\\
\|J_k(x_1, y_1) - J_k(x_2, y_2)\| &\le \mathcal{L}_J[\|x_1 - x_2\| + \|y_1 - y_2\|]
\end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in X$ and $k = 1, 2, \dots, m$, and

$$I_0 = \|I_k(0)\|, \quad J_0 = \|J_k(0)\|, \quad k = 1, 2, \dots, m.$$

Definition 3.2. A continuous solution $x(\cdot)$ of the integral equation

$$\begin{aligned}
x(t) &= B^{-1}S(t)[By_0 + f(0, x(0), x'(0))] + B^{-1}C(t)Bx_0\\
&\quad - \int_0^t B^{-1}C(t-s)f(s, x(s), x'(s))ds\\
&\quad + \int_0^t B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds + \int_0^t B^{-1}S(t-s)Gu(s)ds\\
&\quad + \int_0^t B^{-1}S(t-s)\big[g(s, x(s), x'(s)) + \int_0^s k(s, \tau, x(\tau), x'(\tau))d\tau\big]ds\\
&\quad + \sum_{0<t_k<t} B^{-1}C(t-t_k)I_k(x(t_k), x'(t_k))\\
&\quad + \sum_{0<t_k<t} B^{-1}S(t-t_k)J_k(x(t_k), x'(t_k))
\end{aligned} \tag{3.1}$$

is said to be a mild solution of (2.1)-(2.3) on $\mathcal{I}$.

If $x(\cdot)$ is a mild solution of (2.1)-(2.3), then by the properties of a second order differential equation and Lemma 2.3, we have

$$\begin{aligned} x'(t) &= B^{-1}C(t)[By_0 + f(0,x(0),x'(0))] + B^{-1}AS(t)Bx_0 \\ &\quad - B^{-1}f(t,x(t),x'(t)) - \int_0^t B^{-1}AS(t-s)f(s,x(s),x'(s))ds \\ &\quad + \int_0^t B^{-1}C(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\ &\quad + \int_0^t B^{-1}C(t-s)\big[g(s,x(s),x'(s)) + \int_0^s k(s,\tau,x(\tau),x'(\tau))d\tau\big]ds \\ &\quad + \int_0^t B^{-1}C(t-s)Gu(s)ds + \sum_{0<t_k<t} B^{-1}AS(t-t_k)I_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{0<t_k<t} B^{-1}C(t-t_k)J_k(x(t_k),x'(t_k)), \quad t \in \mathcal{I}. \end{aligned} \tag{3.2}$$

Theorem 3.3. *If assumptions* (H1)–(H4), (H10)–(H13) *hold, then system* (2.1)-(2.3) *is controllable on* $\mathcal{I}$.

The proof of the above is similar to Theorem 3.1 and hence, is omitted.

4. Nonlocal Initial Conditions

The study of abstract nonlocal initial value problems was initiated by Byszewski [8]. Because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Several authors have discussed the nonlocal problem in abstract spaces [5, 6]. The importance of nonlocal is studied in [3, 8]. In this section we consider a second order Sobolev type neutral integrodifferential equations with nonlocal initial condition

$$x(0) + \sum_{i=1}^n p(x_i) = x_0 \quad x'(0) + \sum_{i=1}^n w(x_i) = y_0 \tag{4.1}$$

In addition the assumptions in Section 2 and 3, we also assume the following hypotheses.

(H14) The function $p, w : \mathcal{PC}(\mathcal{I}, X) \to X$ is continuous function, and then there exist positive constants $\mathscr{P}_\alpha > 0, \mathscr{Q}_\alpha > 0$ such that

$$\begin{gathered} \|\sum_{i=1}^n p(x_i)\| \le \mathscr{P}_\alpha, \quad \|\sum_{i=1}^n w(x_i)\| \le \mathscr{Q}_\alpha \\ \|\sum_{i=1}^n p(x_i) - \sum_{i=1}^n p(y_i)\| \le \mathscr{P}_\alpha[\|x-y\|], \\ \|\sum_{i=1}^n w(x_i) - \sum_{i=1}^n w(y_i)\| \le \mathscr{Q}_\alpha[\|x-y\|], \end{gathered}$$

for $x_i, y_i \in X$, $i = 1, 2, \dots, n$.

Definition 4.1. A continuous solution $x(\cdot)$ of the integral equation

$$
\begin{aligned}
x(t) &= B^{-1}S(t)\Big[B\{y_0 - \sum_{i=1}^{n} w(x_i)\} + f(0, x(0), x'(0))\Big] \\
&\quad + B^{-1}C(t)B[x_0 - \sum_{i=1}^{n} p(x_i)] - \int_0^t B^{-1}C(t-s)f(s, x(s), x'(s))ds \\
&\quad + \int_0^t B^{-1}S(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds \\
&\quad + \int_0^t B^{-1}S(t-s)\big[g(s, x(s), x'(s)) + \int_0^s k(s,\tau,x(\tau),x'(\tau))d\tau\big]ds \\
&\quad + \int_0^t B^{-1}S(t-s)Gu(s)ds + \sum_{0<t_k<t} B^{-1}C(t-t_k)I_k(x(t_k), x'(t_k)) \\
&\quad + \sum_{0<t_k<t} B^{-1}S(t-t_k)J_k(x(t_k), x'(t_k))
\end{aligned} \tag{4.2}
$$

is said to be a mild solution of (2.1)-(2.3) and (4.1) on $\mathcal{I}$.

If $x(\cdot)$ is a mild solution of (2.1)-(2.3) and (4.1), then by the properties of a second order differential equation and Lemma 2.3, we have

$$
\begin{aligned}
x'(t) &= B^{-1}C(t)\Big[B\big\{y_0 - \sum_{i=1}^{n} w(x_i)\big\} + f(0, x(0), x'(0))\Big] \\
&\quad + B^{-1}AS(t)B[x_0 - \sum_{i=1}^{n} p(x_i)] - B^{-1}f(t, x(t), x'(t)) \\
&\quad - \int_0^t B^{-1}AS(t-s)f(s, x(s), x'(s))ds \\
&\quad + \int_0^t B^{-1}C(t-s)\int_0^s \mathscr{D}(s-\tau)x(\tau)d\tau ds + \int_0^t B^{-1}C(t-s)Gu(s)ds \\
&\quad + \int_0^t B^{-1}C(t-s)\big[g(s, x(s), x'(s)) + \int_0^s k(s,\tau,x(\tau),x'(\tau))d\tau\big]ds \\
&\quad + \sum_{0<t_k<t} B^{-1}AS(t-t_k)I_k(x(t_k), x'(t_k)) \\
&\quad + \sum_{0<t_k<t} B^{-1}C(t-t_k)J_k(x(t_k), x'(t_k)), \quad t \in \mathcal{I}.
\end{aligned} \tag{4.3}
$$

Theorem 4.2. *If assumptions* (H1)–(H4),(H10)–(H14) *hold, then system* (2.1)-(2.3) *and* (4.1) *is controllable on* $\mathcal{I}$.

The of the above theorem is similar to Theorem 3.1 and hence, is omitted.

5. Example

Consider the partial integrodifferential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big[z_t(t,y) - \frac{1}{2}\cos z(t,y)\Big] \\
&= \frac{\partial^2}{\partial y^2} z(t,y) + \mu(t,y) + b_1(s,y)\Big(t, \frac{1}{2}e^{-t}\sin z_t(t,y)), \\
&\quad \int_0^t \sin z_s(s,y) e^{-s} z_s(\sin s, y)ds\Big) \int_0^a l(t,\tau) z_\tau(t,y)d\tau \\
&\quad + \int_{-\infty}^t b_2(s,y)\sin z_t(s,y)ds, \quad y \in [0,\pi],\ t \in \mathcal{I},
\end{aligned} \tag{5.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in \mathcal{I}, \tag{5.2}$$

$$z(0,y) + \sum_{i=1}^m \gamma_i \Phi_{t_i}(s,y) = z_0(y) \quad 0 < y < 1, \quad t \in \mathcal{I}; \tag{5.3}$$

$$\Delta z|_{t=t_k} = I_k(z(y)) = \int_0^\pi \gamma_k(y,s)\cos^2 z(s,y)ds, \quad z \in X,\ 1 \le k \le p, \tag{5.4}$$

where $\mu(t,y) : \mathcal{I} \times [0,\pi] \to [0,\pi]$ is continuous on $0 \le y \le \pi, t \in \mathcal{I}$ and the constant γ_i are small. Let $X = \mathcal{L}^2[0,\pi]$ be endowed with the usual norm $\|\cdot\|_{\mathcal{L}_2}$, and let $x(t) = z(t,y)$ be continuous,

$$\begin{gathered}
f(t,x(t),x'(t)) = \frac{1}{2}\cos z(t,y), \\
g(t,x(t),x'(t)) = b_1(s,y)\Big(t, \frac{1}{2}e^{-t}\sin z_t(t,y)), \int_0^t \sin z_s(s,y)e^{-s} z_s(\sin s,y)ds\Big), \\
\int_0^t k(t,s,x_s)ds = \int_0^a l(t,\tau)z_\tau(t,y)d\tau + \int_{-\infty}^t b_2(s,y)\sin z_t(s,y)ds, \\
\sum_{i=1}^n p(x_i) = \sum_{i=1}^m \gamma_i \Phi_{t_i}(s,y), \\
I_i(z(x)) = \int_0^\pi \gamma_k(y,s)\cos^2 z(s,y)ds.
\end{gathered}$$

Define the operator $A : \mathcal{D}(A) \subset X \to X$ and $E : \mathcal{D}(E) \subset X \to X$ by

$$Az = -z_{xx}, \quad Ez = z - z_{xx},$$

where each domain $\mathcal{D}(A)$ and $\mathcal{D}(E)$ is given by

$$\{z \in X : z, z_x \text{ are absolutely continuous, } z_{xx} \in X,\ z(0) = z(\pi) = 0\}.$$

Then A and E can be written, respectively, as

$$Az = \sum_{n=1}^\infty n^2 \langle z, z_n\rangle z_n,\ z \in \mathcal{D}(A), \quad Ez = \sum_{n=1}^\infty (1+n^2)\langle z, z_n\rangle z_n,\ z \in \mathcal{D}(E),$$

where $z_n(x) = \sqrt{2/\pi}\sin(nx)$, $n = 1,2,\dots$, is the orthogonal set of vectors of A. Furthermore for $z \in X$, we have

$$E^{-1}z = \sum_{n=1}^\infty \frac{1}{1+n^2}\langle z, z_n\rangle z_n, \quad -AE^{-1}z = \sum_{n=1}^\infty \frac{-n^2}{1+n^2}\langle z, z_n\rangle z_n,$$

$$S(t)z = \sum_{n=1}^{\infty} \exp\Big(\frac{-n^2 t}{1+n^2}\Big)\langle z, z_n\rangle z_n.$$

Further, the linear operators $\mathscr{W}_1,\ \mathscr{W}_2 : \mathscr{L}^2(\mathcal{I}, U) \to X$ defined by

$$\mathscr{W}_1 u = \int_0^b B^{-1} S(b-s) G u(s) ds, \quad \mathscr{W}_2 u = \int_0^b B^{-1} C(b-s) G u(s) ds$$

has a bounded inverse operators and satisfies the condition (H2) and (H3).

We see that (5.1)–(5.4) can be formulated abstractly as (2.1)–(2.3). Hence all the conditions stated in the Theorem 3.1 are satisfied and it is possible choose b_1, b_2, γ_i. Hence by the Theorem 3.1, equation (5.1)–(5.4) is controllable on $\mathcal{I}$.

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