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SOLUTIONS TO SEMILINEAR ELLIPTIC PDE'S WITH BIHARMONIC OPERATOR AND SINGULAR POTENTIAL

MOUSOMI BHAKTA

Abstract. We study the existence and nonexistence of positive solution to the problem

$$\begin{gathered}\Delta^2 u - \mu a(x)u = f(u) + \lambda b(x) \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega,\\ u = 0 = \Delta u \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. We show the existence of a value $\lambda^* > 0$ such that when $0 < \lambda < \lambda^*$, there is a solution and when $\lambda > \lambda^*$ there is no solution in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Moreover as $\lambda \uparrow \lambda^*$, the minimal positive solution converges to a solution. We also prove that there exists $\tilde{\lambda}^* < \infty$ with $\lambda^* \le \tilde{\lambda}^*$, and for $\lambda > \tilde{\lambda}^*$, such that the above problem does not have solution even in the distributional sense/very weak sense, and there is a complete *blow-up*. Under an additional integrability condition on b, we establish the uniqueness of positive solution.

1. Introduction

In this article we study the semilinear fourth-order elliptic problem with singular potential,

$$\begin{gathered}\Delta^2 u - \mu a(x)u = f(u) + \lambda b(x) \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega,\\ u = 0 = \Delta u \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \ge 5$. a, b, f are nonnegative functions. $a \in L^1_{\rm loc}(\Omega)$, $b \in L^2(\Omega)$, $b \not\equiv 0$. μ, λ are (small) positive constants. We assume that

$$f : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a convex } C^1 \text{ function with } f(0) = 0 = f'(0) \tag{1.2}$$

and satisfying the growth conditions:

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty, \tag{1.3}$$

$$\int_1^\infty g(s)ds < \infty \quad \text{and } sg(s) < 1 \quad \text{for } s > 1, \tag{1.4}$$

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where, for $s \geq 1$, we define

$$g(s) = \sup_{t>0} \frac{f(t)}{f(ts)}. \tag{1.5}$$

It is easy to see that g is nonincreasing, nonnegative function. Since by convexity $t \to \frac{f(t)}{t}$ is increasing and $f(0) = 0$, it follows that $s \to sg(s)$ is nonincreasing.

As in the literature, $W^{k,p}(\Omega)$ has the usual norm $\left(\int_\Omega \sum_{0\leq|\alpha|\leq k} |D^\alpha u|^p dx\right)^{1/p}$. Thanks to interpolation theory, one can neglect intermediate derivatives and see that

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\int_\Omega |u|^p dx + \int_\Omega |D^k u|^p dx\Big)^{1/p}, \tag{1.6}$$

defines a norm which is equivalent to the usual norm in $W^{k,p}(\Omega)$ (see [1]). As Ω is a smooth bounded domain and $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$, invoking [11, Theorem 2.2] we find that

$$\|u\|_{W_0^{k,p}(\Omega)} = \Big(\int_\Omega |D^k u|^p dx\Big)^{1/p}, \tag{1.7}$$

defines an equivalent norm to (1.6). Now onwards we will consider $W_0^{k,p}(\Omega)$ endowed with the norm defined in (1.7). The inner product in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is defined by

$$(u, v)_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} = \int_\Omega \Delta u \Delta v dx,$$

which induces the norm

$$\|u\|_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} = |\Delta u|_{L^2(\Omega)}, \tag{1.8}$$

is equivalent to (1.7) with $k = p = 2$ (for details see [11, 12]).

We assume $a \in L^1_{\rm loc}(\Omega)$ and there exists a positive constant $\gamma > 0$ such that

$$\int_\Omega (|\Delta u|^2 - a(x)^2 u^2) dx \geq \gamma \int_\Omega u^2 \quad \forall u \in C_0^\infty(\Omega). \tag{1.9}$$

Using Fatou's lemma and the standard density argument, it is easy to check that (1.9) holds for every $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$. Therefore we write

$$\int_\Omega (|\Delta u|^2 - a(x)^2 u^2) dx \geq \gamma \int_\Omega u^2 \quad \forall u \in W^{2,2} \cap W_0^{1,2}(\Omega). \tag{1.10}$$

We note that if $a(x) = \alpha/|x|^2$ where $\alpha < \bar{\alpha} := \frac{N(N-4)}{4}$, applying the following Rellich inequality [13, 14]:

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \bar{\alpha}^2 \int_{\mathbb{R}^N} |x|^{-4}|u|^2 dx \quad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.11}$$

and the Poincare inequality along with the norm equivalence established above, it is not difficult to check that (1.10) holds. When $a(x) = \frac{\bar{\alpha}}{|x|^2}$, (1.10) is the improved Hardy-Rellich inequality (see [10], [15]).

We also assume

$$0 < \mu < \sqrt{\gamma}. \tag{1.12}$$

Using (1.9) and (1.12) it follows that

$$\mu \int_\Omega a(x)u^2 dx \leq \mu\Big(\int_\Omega a(x)^2 u^2 dx\Big)^{1/2} \Big(\int_\Omega u^2 dx\Big)^{1/2} \leq \frac{\mu}{\sqrt{\gamma}} |\Delta u|^2_{L^2(\Omega)} \tag{1.13}$$

for all $u \in C_0^\infty(\Omega)$. Therefore,

$$\|u\|_H^2 := \int_\Omega [|\Delta u|^2 - \mu a(x) u^2] dx,$$

is a norm in $C_0^\infty(\Omega)$ and completion of $C_0^\infty(\Omega)$ with respect to this norm yields the Hilbert space H. By (1.13), (1.12) and (1.7), it follows that $\|u\|_H$ is equivalent to $\|u\|_{W_0^{2,2}(\Omega)}$. Thanks to (1.13), the norm equivalence established above and the Poincare inequality, there exists $\tilde{\gamma} > 0$ such that

$$\int_\Omega (|\Delta u|^2 - \mu a(x) u^2) dx \geq \tilde{\gamma} \int_\Omega u^2 dx \quad \forall \ \ u \in \mathcal{C}_0^\infty(\Omega).$$

Using Fatou's lemma and the standard density argument it is easy to check that

$$\int_\Omega (|\Delta u|^2 - \mu a(x) u^2) dx \geq \tilde{\gamma} \int_\Omega u^2 dx \quad \forall u \in W^{2,2} \cap W_0^{1,2}(\Omega). \tag{1.14}$$

This inequality implies that the first eigenvalue of $\Delta^2 - \mu a(x)$ is strictly positive.

Definition 1.1. We say that $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of (1.1) if $u > 0$ a.e., $f(u) \in L^2(\Omega)$ and u satisfies

$$\int_\Omega (\Delta u \Delta \phi - \mu a(x) u \phi) dx = \int_\Omega (f(u) + \lambda b(x)) \phi \, dx \quad \forall \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Similarly $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is called a supersolution (subsolution) if $f(u) \in L^2(\Omega)$ and for all positive $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,

$$\int_\Omega (\Delta u \Delta \phi - \mu a(x) u \phi) dx \geq (\leq) \int_\Omega (f(u) + \lambda b(x)) \phi \ dx.$$

Definition 1.2. We say that $u \in L^1(\Omega)$ is a distributional solution or very weak solution of (1.1) if $u > 0$ a.e., $\mu a(x) u + f(u) \in L^1_{\rm loc}(\Omega)$ and u satisfies (1.1) in the distributional sense, i.e.,

$$\int_\Omega u(\Delta^2 \phi - \mu a(x) \phi) dx = \int_\Omega (f(u) + \lambda b(x)) \phi \ dx \quad \forall \phi \in C_0^\infty(\Omega).$$

Similar type of problem with the Laplace operator in much more generalized sense was extensively studied by Dupaigne and Nedev in [8]. In [8], the authors proved a necessary and sufficient condition for the existence of L^1 solution and they have also established an estimate from above and below for the solution. We also refer [4, 5, 7] (and the references therein) for the related problems in the second order case.

Higher order problems are quite different compared to the second order case. In this case a possible failure of the maximum principle causes several technical difficulties. Possibly because of this reason the knowledge on higher order nonlinear problems is far from being reasonably complete, as it is in the second-order case. In the case of fourth-order problem Navier boundary conditions play an important role to prove existence results as under this boundary condition, equation with biLaplacian operator can be rewritten as a second order system with Dirichlet boundary value problems. Then using classical elliptic theory, one can easily prove a Maximum Principle. As a consequence, one can deduce a Comparison Principle which plays as one of the key factor in proving existence results. In a recent work [12], an equation similar to ((1.1)) with $a(x) = 1/|x|^4$ and $f(u) = u^p$ has been studied. More precisely, in [12] the authors have studied the optimal power p for existence/nonexistence of distributional solutions. In recent years there are many papers dealing with $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ solution of semilinear elliptic and parabolic problem with biLaplacian operator and some specific nonlinearities. We quote a few among them [2, 3, 6, 9] (also see the references therein). Semilinear elliptic equations with biharmonic operator arise in continuum mechanics, bio- physics, differential geometry. In particular in the modeling of thin elastic plates, clamped plates and in the study of the Paneitz-Branson equation and the Willmore equation (see [11] and the references therein for more details).

This article is organized as follows: In Section 2 we recall some useful lemmas from [12] and prove some important lemmas regarding existence. In Section 3 we prove our main existence result. More precisely, under some hypothesis on f, we prove there exists $\lambda^*>0$ such that if $0<\lambda<\lambda^*$, problem (1.1) has a minimal solution u_λ in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Moreover, if $\lambda>\lambda^*$, then (1.1) does not have any solution which belongs to $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Under an additional mild growth condition on f at infinity, we also prove when $\lambda\uparrow\lambda^*$, there exists $u^*\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ such that minimal solution u_λ of (1.1) converges to u^* in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ and u^* happens to be a solution of (1.1) with $\lambda=\lambda^*$. Section 4 deals with the case for which (1.1) does not have any solution even in the very weak sense. In this case we establish *complete blow-up* phenomenon (see Definition 4.2). Section 5 is devoted to the stability result where the minimal positive solution in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ already exists. In this section, under some better integrability condition on b, we also prove (1.1) with $\lambda=\lambda^*$ has a unique solution in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$.

2. Preliminary lemmas

Definition 2.1. We say that $u\in L^1(\Omega)$ is a weak supersolution (subsolution) to
$$\Delta^2 u = g(x,u) \quad \text{in } \Omega,$$
in the sense of distribution if $g(x,u)\in L^1(\Omega)$ and for all positive $\phi\in C_0^\infty(\Omega)$, we have
$$\int_\Omega u\Delta^2\phi dx \geq (\leq) \int_\Omega g(x,u)\phi dx.$$
If u is a weak supersolution and as well a weak subsolution in the sense of distribution, then we say that u is a distributional solution.

Next we recall three important lemmas from [12] which we will use frequently in this paper.

Lemma 2.2 (Strong Maximum Principle)**.** *Let u be a nontrivial supersolution of*
$$\begin{gathered} \Delta^2 u = 0 \quad \textit{in } \Omega,\\ u = 0 = \Delta u \quad \textit{on } \partial\Omega. \end{gathered} \tag{2.1}$$
Then $-\Delta u>0$ and $u>0$ in Ω.

For a proof of the above lemma see [12, Lemma 3.2].

Lemma 2.3 (Comparison Principle)**.** *Let u and v satisfy the following:*
$$\begin{gathered} \Delta^2 u \geq \Delta^2 v \quad \textit{in } \Omega,\\ u \geq v \quad \textit{on } \partial\Omega,\\ -\Delta u \geq -\Delta v \quad \textit{on } \partial\Omega. \end{gathered} \tag{2.2}$$

Then, $-\Delta u \geq -\Delta v$ and $u \geq v$ in Ω.

For a proof of the above lemma see [12, Lemma 3.3].

Lemma 2.4 (Weak Harnack Principle [12, Lemma 3.4])**.** *Let u be a positive distributional supersolution to (2.1). Then for any $B_R(x_0) \Subset \Omega$, there exists a positive constant $C = C(\theta, \rho, q, R)$, $0 < q < \frac{N}{N-2}$, $0 < \theta < \rho < 1$, such that*

$$\|u\|_{L^q(B_{\rho R}(x_0))} \leq C \operatorname{ess\,inf}_{B_{\theta R}(x_0)} u.$$

Lemma 2.5. *Let $a \in L^1_{\rm loc}(\Omega)$, $b \in L^2(\Omega)$, $a, b \geq 0$ a.e., $b \not\equiv 0$, μ be a positive constant satisfying (1.12) and a satisfy (1.10). Then the equation*

$$\begin{gathered} \Delta^2 u - \mu a(x) u = b \quad \text{in } \Omega, \\ u = 0 = \Delta u \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

has a positive solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Proof. Given $b \in L^2(\Omega)$, we know there exists unique $u_1 \in W^{2,2} \cap W_0^{1,2}(\Omega)$ satisfying the following:

$$\begin{gathered} \Delta^2 u_1 = b \quad \text{in } \Omega, \\ u_1 = 0 = \Delta u_1 \quad \text{on } \partial\Omega. \end{gathered}$$

Applying strong maximum principle (Lemma 2.2) we obtain $u_1 > 0$. Now define u_n ($n \geq 2$) as follows:

$$\begin{gathered} \Delta^2 u_n = \mu a(x) u_{n-1} + b \quad \text{in } \Omega, \\ u_n = 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{gathered} \tag{2.4}$$

By (1.10), we have $\mu a(x) u_{n-1} \in L^2(\Omega)$. This in turn implies the existence of unique $u_n \in W^{2,2} \cap W_0^{1,2}(\Omega)$ which satisfies (2.4). Also by comparison principle we have $0 < u_1 \leq \dots \leq u_{n-1} \leq u_n \leq \dots$.

Claim: $\{u_n\}$ is a Cauchy sequence in $W^{2,2} \cap W_0^{1,2}(\Omega)$.

To see this, we note that $\Delta^2(u_{n+1} - u_n) = \mu a(x)(u_n - u_{n-1})$. By taking $(u_{n+1} - u_n)$ as a test function and using (1.10), we obtain

$$\begin{aligned} |\Delta(u_{n+1} - u_n)|^2_{L^2(\Omega)} &= \mu \int_\Omega a(x)(u_n - u_{n-1})(u_{n+1} - u_n) dx \\ &\leq \mu (\int_\Omega a(x)^2 (u_n - u_{n-1})^2 dx)^{1/2} (\int_\Omega (u_{n+1} - u_n)^2 dx)^{1/2} \\ &\leq \frac{\mu}{\sqrt{\gamma}} |\Delta(u_n - u_{n-1})|_{L^2(\Omega)} |\Delta(u_{n+1} - u_n)|_{L^2(\Omega)}. \end{aligned}$$

Therefore

$$|\Delta(u_{n+1} - u_n)|_{L^2(\Omega)} \leq \frac{\mu}{\sqrt{\gamma}} |\Delta(u_n - u_{n-1})|_{L^2(\Omega)} \leq \dots \leq (\frac{\mu}{\sqrt{\gamma}})^{n-1} |\Delta(u_2 - u_1)|_{L^2(\Omega)}.$$

As $\mu < \sqrt{\gamma}$, from the above estimate we can conclude that $\{u_n\}$ is a Cauchy sequence in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Hence, there exists $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that $u_n \to u$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Moreover, $u > 0$ since $u_n > 0$ for all $n \geq 1$. As $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ solves (2.4), we have

$$\int_\Omega \Delta u_n \Delta \phi dx = \mu \int_\Omega a(x) u_{n-1} \phi dx + \int_\Omega b \phi dx \quad \forall \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Taking the limit as $n \to \infty$, we obtain u is a solution to (2.3). $\square$

Lemma 2.6. *Let $a \in L^1_{\rm loc}(\Omega)$, $b \in L^2(\Omega)$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ (f convex) be nonnegative functions. Let $\mu, \lambda > 0$, $\mu < \sqrt{\gamma}$. Suppose there exists a nonnegative supersolution $\tilde{u} \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ of (1.1) (respectively for (2.3)). Then there exists a unique solution $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ to (1.1) which satisfies $0 \le u \le \tilde{w}$ for any supersolution $\tilde{w} \ge 0$ of (1.1) (respectively for (2.3)). u is called the minimal nonnegative solution of (1.1) (respectively for (2.3)). By strong maximum principle it also follows that $u > 0$ in Ω.*

Remark 2.7. We denote the minimal positive solution of (2.3) by ζ_1 and denote $G(b) = \zeta_1$. The function $0 < u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ solving (1.1) (respectively (2.3)) also solves (1.1) (2.3) in the distributional sense (see definition (1.2)).

Proof. The proof is the same for both the equations (1.1) and (2.3), therefore we present only the proof for (1.1). First we will show that if minimal solution exists then it is unique. To see this, let u_1 and u_2 are two solutions which satisfy $0 \le u_i \le \tilde{w}, (i = 1, 2)$ for every nonnegative supersolution $\tilde{w}$. Thus $u_1 \le u_2$ and $u_2 \le u_1$. Hence $u_1 = u_2$.

Next, let $\tilde{u} \ge 0$ be a supersolution to (1.1) and $u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ be a positive solution of

$$\begin{gathered} \Delta^2 u_0 = \lambda b \quad \text{in } \Omega, \\ u_0 = 0 = \Delta u_0 \quad \text{on } \partial\Omega. \end{gathered}$$

By comparison principle we obtain $0 < u_0 \le \tilde{u}$ in Ω. Next, using iteration we will show that there exists $u_n \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ for $n = 1, 2, \dots$ such that u_n solves the problem

$$\begin{gathered} \Delta^2 u_n = \mu a(x) u_{n-1} + f(u_{n-1}) + \lambda b(x) \quad \text{in } \Omega, \\ u_n = 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

Since $\tilde{u}$ is a weak supersolution to (1.1), we have $f(\tilde{u}) \in L^2(\Omega)$. Thanks to the fact that $0 < u_0 \le \tilde{u}$ and f is convex (thus f is nondecreasing), we obtain $f(u_0) \le f(\tilde{u})$. Thus $f(u_0) + \lambda b(x) \in L^2(\Omega)$. Also, by (1.10) it follows that $\mu a(x) u_0 \in L^2(\Omega)$. Therefore u_1 is well defined and by comparison principle $0 < u_0 \le u_1 \le \tilde{u}$. Using the induction method, similarly we can show that u_n is well defined and $0 < u_0 \le u_1 \le \dots \le u_n \le \dots \le \tilde{u}$.

Claim: $\{u_n\}$ is uniformly bounded in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

To see this, let us note that from (2.5) we can write

$$\begin{aligned} |\Delta u_n|^2_{L^2(\Omega)} &= \int_\Omega (\mu a(x) u_{n-1} + f(u_{n-1}) + \lambda b(x)) u_n dx \\ &\le \int_\Omega (\mu a(x) \tilde{u}^2 + f(\tilde{u})\tilde{u} + \lambda b \tilde{u}) dx \\ &\le \big[\mu |a(x)\tilde{u}|_{L^2(\Omega)} + |f(\tilde{u})|_{L^2(\Omega)} + \lambda |b|_{L^2(\Omega)}\big] |\tilde{u}|_{L^2(\Omega)} \le C. \end{aligned}$$

As a consequence there exists $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$. From (2.5) we have,

$$\int_\Omega \Delta u_n \Delta \phi dx = \int_\Omega [\mu a(x) u_{n-1} + f(u_{n-1}) + \lambda b] \phi dx \quad \forall \phi \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega).$$

Using Vitaly's convergence theorem we can pass to the limit $n \to \infty$ on the right-hand side and obtain u is a solution to (1.1). Also $u > 0$ since $u_n > 0$ for all $n \ge 1$.

Let $\tilde{w}$ be another supersolution, then by comparison principle it follows that $u_0 \le \tilde{w}$ and $u_n \le \tilde{w}$ for every $n \ge 1$. Taking the limit $n \to \infty$, it gives us that $u \le \tilde{w}$. Hence the lemma follows. $\square$

3. Existence and nonexistence results

Theorem 3.1. *Assume $a \in L^1_{\rm loc}(\Omega)$, $0 \not\equiv b \in L^2(\Omega)$, a, b, f are nonnegative functions, (1.10), (1.12), (1.2), (1.3), (1.4) and (1.5) are satisfied. Let $G = (\Delta^2 - \mu a(x))^{-1}$ and $\zeta_1 = G(b)$, as proved in Lemma 2.5 (also see Remark 2.7). Suppose there exists constants $\epsilon > 0$ and $C > 0$ such that*

$$f(\epsilon\zeta_1) \in L^2(\Omega) \quad \text{and} \quad G(f(\epsilon\zeta_1)) \le C\zeta_1 \quad a.e. \tag{3.1}$$

Then there exists $0 < \lambda^* = \lambda^*(N, a(x), b(x), f, \mu)$ such that if $\lambda < \lambda^*$, then (1.1) has a minimal positive solution $u_\lambda \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and $u_\lambda \ge \lambda\zeta_1$.

If $\lambda > \lambda^*$ then (1.1) has no positive solution in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

Moreover, if $\lambda > 0$ is small then

$$\lambda\zeta_1 \le u_\lambda \le 2\lambda\zeta_1.$$

The assumption (3.1) is motivated from the work of Dupaigne and Nedev (see [8, Theorem 1]). To prove this theorem, first we need to prove a lemma and a proposition.

Lemma 3.2. *Let the functions a, b and the constant μ satisfy the assumptions in Theorem 3.1. $\zeta_1 = G(b)$ as in theorem 3.1 and assume that (1.2) is satisfied. If*

$$f(2\zeta_1) \in L^2(\Omega) \quad \text{and} \quad G(f(2\zeta_1)) \le \zeta_1,$$

then (1.1) with $\lambda = 1$ admits a solution $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

Proof. Let $f(2\zeta_1) \in L^2(\Omega)$ and $G(f(2\zeta_1)) \le \zeta_1$. We define, $v := G(f(2\zeta_1)) + \zeta_1$. Clearly $v > 0$ and $v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ since ζ_1 and $G(f(2\zeta_1))$ are in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ by Lemma 2.5. Also,

$$v - \zeta_1 = G(f(2\zeta_1)), \quad v \le 2\zeta_1, \quad f(v) \in L^2(\Omega).$$

Thus we have

$$\Delta^2(v - \zeta_1) - \mu a(x)(v - \zeta_1) = f(2\zeta_1) \quad \text{in } \Omega,$$

i.e.,

$$\Delta^2 v - \mu a(x)v = f(2\zeta_1) + b \ge f(v) + b \quad \text{in } \Omega$$

and $v = 0 = \Delta v$ on $\partial\Omega$. As a result, v is a positive supersolution of (1.1) with $\lambda = 1$. Finally, by applying Lemma 2.6 we obtain the existence of minimal positive solution $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ of (1.1) with $\lambda = 1$. $\square$

Proposition 3.3. *Suppose there exists $\tilde{\lambda} > 0$ such that $(P_{\tilde{\lambda}})$ has a positive solution $u_{\tilde{\lambda}} \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$. Then for every $0 < \lambda < \tilde{\lambda}$, (1.1) has a solution in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.*

Proof. Let $u_{\tilde{\lambda}} \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ denote a positive solution corresponding to (1.1) with $\tilde{\lambda}$ instead of λ. Therefore by definition (see Definition 1.1) $f(u_{\tilde{\lambda}}) \in L^2(\Omega)$. Define, $v = \tilde{\lambda}\zeta_1$. Note that,

$$\Delta^2\Big(\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}\Big) - \mu a(x)\Big(\frac{u_{\tilde{\lambda}}}{\tilde{\lambda}}\Big) = \frac{1}{\tilde{\lambda}}(f(u_{\tilde{\lambda}}) + \tilde{\lambda}b) = \frac{f(u_{\tilde{\lambda}})}{\tilde{\lambda}} + b \ge b \quad \text{in } \Omega.$$

This implies, $\frac{u_{\tilde\lambda}}{\tilde\lambda}$ is a positive supersolution to (2.3). Therefore by minimality of ζ_1 it follows, $\zeta_1 \le \frac{u_{\tilde\lambda}}{\tilde\lambda}$, which in turn implies $v \le u_{\tilde\lambda}$. Let $0<\lambda<\tilde\lambda$ and define, $w = u_{\tilde\lambda} - v + \lambda\zeta_1$. Clearly $w>0$. Using the definition of v and λ we also get $w \le u_{\tilde\lambda}$. By convexity of f, it follows $\frac{f(t)}{t}$ is increasing and thus f is nondecreasing. As a consequence, $f(w) \le f(u_{\tilde\lambda})$ and hence $f(w)\in L^2(\Omega)$. Also,

$$\Delta^2 w - \mu a(x)w = f(u_{\tilde\lambda}) + \tilde\lambda b - (\tilde\lambda-\lambda)b = f(u_{\tilde\lambda}) + \lambda b \ge f(w)+\lambda b.$$

As a result, $w\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ is a positive supersolution to (1.1). Hence by Lemma 2.6, there exists minimal positive solution of (1.1). □

Proof of Theorem 3.1. We assume (3.1) holds.

Step 1: We show that if $\lambda>0$ is small then (1.1) has a positive a solution $u_\lambda \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. We will prove this step in the spirit of [8]. By Lemma 3.2, it follows that (1.1) has a solution as long as it holds

$$f(2\lambda\zeta_1)\in L^2(\Omega) \quad \text{and} \quad G(f(2\lambda\zeta_1)) \le \lambda\zeta_1. \tag{3.2}$$

From the definition of g (see definition (1.5)), it follows that $g(\frac{\epsilon}{2\lambda}) \ge \frac{f(t)}{f(t\frac{\epsilon}{2\lambda})}$ for all $t>0$. Choosing $t=2\lambda\zeta_1$, we obtain $f(2\lambda\zeta_1) \le f(\epsilon\zeta_1)g(\frac{\epsilon}{2\lambda})$. Applying (3.1), we have $f(2\lambda\zeta_1)\in L^2(\Omega)$ and $G(f(2\lambda\zeta_1))$ is well defined. Also by minimality of $G(f(2\lambda\zeta_1))$ and by assumption (3.1), we obtain

$$G(f(2\lambda\zeta_1)) \le g(\frac{\epsilon}{2\lambda})G(f(\epsilon\zeta_1)) \le Cg(\frac{\epsilon}{2\lambda})\zeta_1.$$

To show (3.2) holds for $\lambda>0$ small, it is enough to prove that

$$\lim_{\lambda\to0}\frac{1}{\lambda}g(\frac{\epsilon}{2\lambda}) = 0 \quad \text{or equivalently} \quad \lim_{K\to\infty} Kg(K)=0.$$

Since $s\to sg(s)$ is nonincreasing, the above limit is well defined, i.e. there exists $C'\ge 0$ such that $\lim_{K\to\infty}Kg(K)=C'$. If $C'>0$, then $g(K)\sim\frac{C}{K}$ near ∞ and this contradicts (1.4). Hence $C'=0$ and (3.2) holds for $\lambda>0$ small.

Step 2: Define,

$$\Lambda = \{\lambda>0 : (P_\lambda) \text{ has a minimal positive solution } u_\lambda\},$$

By Step 1 and Proposition 3.3, it follows that Λ is a non-empty interval. We define,

$$\lambda^* = \sup\Lambda.$$

Then it is easy to see that, if $\lambda<\lambda^*$, (1.1) has a minimal positive solution and for $\lambda>\lambda^*$, (1.1) does not have any positive solution in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$.

Step 3: From $G(b)=\zeta_1$, it is easy to see that $G(\lambda b)=\lambda\zeta_1$. If $\lambda<\lambda^*$ and u_λ denotes the corresponding minimal positive solution of (1.1), then it is not difficult to check that u_λ is a supersolution to the equation satisfied by $\lambda\zeta_1$. Therefore by minimality of $\lambda\zeta_1$, we obtain

$$u_\lambda \ge \lambda\zeta_1. \tag{3.3}$$

Step 4: We show that if $\lambda>0$ is small, then

$$\lambda\zeta_1 \le u_\lambda \le 2\lambda\zeta_1.$$

By Step 1, (3.2) holds since $\lambda>0$ is small. Define, $w=G(f(2\lambda\zeta_1))+\lambda\zeta_1$. Therefore

$$w\le 2\lambda\zeta_1 \quad \text{and} \quad w-\lambda\zeta_1 = G(f(2\lambda\zeta_1)).$$

As in the proof of Lemma 3.2, we can establish that $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a positive supersolution of (1.1). Thus $u_\lambda \le w \le 2\lambda\zeta_1$. Combining this with (3.3), we have $\lambda\zeta_1 \le u_\lambda \le 2\lambda\zeta_1$. □

Define

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x), \quad x \in \Omega. \tag{3.4}$$

Theorem 3.4. *Assume the assumptions in Theorem 3.1 are satisfied, u_λ denotes the minimal positive solution of* (1.1) *for $0 < \lambda < \lambda^*$ and u^* is as defined in* (3.4). *In addition suppose f satisfies the condition*

$$\lim_{s \to \infty} \frac{sf'(s)}{f(s)} > 1. \tag{3.5}$$

Then $u^* \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and u^* is a solution to (1.1) *with λ^* instead of λ. Moreover, $u_\lambda \to u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Remark 3.5. Since f is convex and C^1, (3.5) is a mild assumption. It is easy to see that if $f \in C^2$ and strictly convex, then (3.5) is obvious.

Proof of Theorem 3.4. u_λ begin a solution of (1.1) implies

$$\int_\Omega \Delta u_\lambda \Delta v = \mu \int_\Omega a(x) u_\lambda v + \int_\Omega f(u_\lambda) v + \lambda \int_\Omega b(x) v \quad \forall v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{3.6}$$

By Theorem 5.2, it follows that u_λ is a stable solution of (1.1) (see Definition 5.1). Therefore $\int_\Omega (|\Delta u_\lambda|^2 - \mu a(x) u_\lambda^2 - f'(u_\lambda) u_\lambda^2) dx \ge 0$. Hence by taking $v = u_\lambda$ in (3.6) we have

$$\int_\Omega f'(u_\lambda) u_\lambda^2 dx \le \int_\Omega (|\Delta u_\lambda|^2 - \mu a(x) u_\lambda^2) dx = \int_\Omega (f(u_\lambda) u_\lambda + \lambda b(x) u_\lambda) dx. \tag{3.7}$$

Moreover, using (3.5) we can write, for every $\epsilon > 0$ there exists $C > 0$ such that

$$(1+\epsilon) f(s) s \le f'(s) s^2 + C \quad \forall s \ge 0. \tag{3.8}$$

Hence combining (3.7) and (3.8) we obtain

$$(1+\epsilon) \int_\Omega (f'(u_\lambda) u_\lambda^2 - \lambda b(x) u_\lambda) dx \le (1+\epsilon) \int_\Omega f(u_\lambda) u_\lambda dx \le \int_\Omega (f'(u_\lambda) u_\lambda^2 + C) dx.$$

As a result,

$$\epsilon \int_\Omega f'(u_\lambda) u_\lambda^2 dx \le C|\Omega| + (1+\epsilon)\lambda \int_\Omega b u_\lambda dx.$$

Consequently,

$$\int_\Omega f(u_\lambda) u_\lambda dx \le C_1 + C_2 \lambda \int_\Omega b u_\lambda dx, \tag{3.9}$$

for some constants $C_1, C_2 > 0$. Since $\lambda < \lambda^*$, by taking $v = u_\lambda$ in (3.6) and applying Holder inequality and (3.9) we have

$$\begin{aligned}
\int_\Omega |\Delta u_\lambda|^2 dx &= \mu \int_\Omega a(x) u_\lambda^2 + \int_\Omega f(u_\lambda) u_\lambda + \lambda \int_\Omega b u_\lambda \\
&\le \mu |a(x) u_\lambda|_{L^2(\Omega)} |u_\lambda|_{L^2(\Omega)} + \lambda^*(1 + C_2) \int_\Omega b u_\lambda dx + C_1.
\end{aligned}$$

Applying (1.10) and Cauchy-Schwartz inequality with $\delta > 0$ on the above estimate, we obtain

$$\begin{aligned}\int_\Omega |\Delta u_\lambda|^2 dx &\le \frac{\mu}{\sqrt{\gamma}}|\Delta u_\lambda|^2_{L^2(\Omega)} + C_3|b|_{L^2(\Omega)}|u_\lambda|_{L^2(\Omega)} + C_1 \\ &\le \frac{\mu}{\sqrt{\gamma}}|\Delta u_\lambda|^2_{L^2(\Omega)} + \frac{C_3}{\sqrt{\gamma}}|b|_{L^2(\Omega)}|\Delta u_\lambda|_{L^2(\Omega)} + C_1 \\ &\le \frac{\mu}{\sqrt{\gamma}}|\Delta u_\lambda|^2_{L^2(\Omega)} + \delta|\Delta u_\lambda|^2_{L^2(\Omega)} + c(\delta)|b|^2_{L^2(\Omega)} + C_1.\end{aligned}$$

Since $\mu < \sqrt{\gamma}$ (by (1.12)), we can choose $\delta > 0$ such that $\frac{\mu}{\sqrt{\gamma}} + \delta < 1$. Hence from the above estimate we have

$$\int_\Omega |\Delta u_\lambda|^2 dx \le C_4|b|^2_{L^2(\Omega)} + C_1 \le C',$$

for some constant $C' > 0$. This implies $\{u_\lambda\}$ is uniformly bounded in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for $\lambda < \lambda^*$. Consequently, by (3.4) we conclude that $u_\lambda \rightharpoonup u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Passing to the limit $\lambda \to \lambda^*$ in (3.6), via Lebesgue monotone convergence theorem, it is easy to check that u^* is a solution to (1.1) with λ^* instead of λ. When $\lambda \to \lambda^*$, using monotone convergence theorem we also have

$$\begin{aligned}\|u_\lambda\|^2_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} &= \int_\Omega |\Delta u_\lambda|^2 dx \\ &= \mu\int_\Omega a(x)u_\lambda^2 + \int_\Omega f(u_\lambda)u_\lambda + \lambda\int_\Omega b u_\lambda \\ &\to \mu\int_\Omega a(x){u^*}^2 + \int_\Omega f(u^*)u^* + \lambda^*\int_\Omega b u^* \\ &= \int_\Omega |\Delta u^*|^2 dx = \|u^*\|^2_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)}\end{aligned}$$

Hence $\|u_\lambda\|_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} \to \|u^*\|_{W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)}$. Combining this along with the weak convergence, we conclude $u_\lambda \to u^*$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. □

We denote by u_{λ^*}, the minimal positive solution of (1.1) with λ^* instead of λ.

4. Nonexistence of very weak solution and complete blow-up

Define

$$\tilde{\lambda}^* = \sup\{\lambda > 0 :\ (1.1) \text{ has a very weak solution/distributional solution }\}.$$

It is not difficult to check that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution to (1.1) in the sense of Definition 1.1, then u is a very weak solution of (1.1) as well. Therefore $\tilde{\lambda}^* \ge \lambda^*$.

Lemma 4.1. $\tilde{\lambda}^* < \infty$.

Proof. Assume (1.1) has a very weak solution $u \in L^1(\Omega)$. Therefore

$$\int_\Omega u(\Delta^2\phi - \mu a(x)\phi)dx = \int_\Omega (f(u) + \lambda b(x))\phi\, dx \quad \forall \phi \in C_0^\infty(\Omega). \tag{4.1}$$

Let $\tilde{\Omega} \Subset \Omega$ and $\psi \in C_0^\infty(\Omega)$ be a nonnegative function such that $\operatorname{supp}(\psi) \subset \tilde{\Omega}$. We choose ϕ as follows:

$$\Delta^2\phi = \psi \quad \text{in } \Omega,$$

$$\phi = 0 = \Delta\phi \quad \text{on } \partial\Omega.$$

Clearly $\phi \in C^\infty(\Omega)$ and by strong maximum principle $\phi > 0$ in Ω. Thus there exists $c > 0$ such that $\phi \geq c > 0$ in $\tilde{\Omega}$. Substituting this ϕ in (4.1), we have

$$\mu \int_\Omega a(x)u\phi\, dx + \int_\Omega f(u)\phi\, dx + \lambda \int_\Omega b(x)\phi\, dx = \int_\Omega u\psi\, dx = \int_{\tilde{\Omega}} u\psi\, dx. \tag{4.2}$$

Since f satisfies (1.3), it is easy to check that, for $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$u \leq C_\epsilon + \epsilon f(u).$$

Therefore from the right-hand side of (4.2) we obtain

$$\int_{\tilde{\Omega}} u\psi\, dx \leq C_\epsilon \int_\Omega \psi dx + \epsilon \int_{\tilde{\Omega}} f(u)\psi dx \leq C_\epsilon \int_\Omega \psi dx + \epsilon |\frac{\psi}{\phi}|_{L^\infty(\tilde{\Omega})} \int_\Omega f(u)\phi dx$$

Now choose $\epsilon > 0$ such that $\epsilon|\frac{\psi}{\phi}|_{L^\infty(\tilde{\Omega})} < 1/2$. Thus from (4.2) we have

$$\mu \int_\Omega a(x)u\phi\, dx + \frac{1}{2}\int_\Omega f(u)\phi\, dx + \lambda \int_\Omega b(x)\phi\, dx \leq C \int_\Omega \psi dx \leq C'.$$

This implies $\tilde{\lambda}^* < \infty$. In particular there are no solutions of (1.1) for $\lambda > \tilde{\lambda}^*$, even in the very weak sense. □

Definition 4.2. Let $\{a_n(x)\}$, $\{b_n(x)\}$ and $\{f_n\}$ be increasing sequence of bounded functions converging pointwise respectively to $a(x)$, $b(x)$ and f. (Since $f \in C^1(\mathbb{R}^+)$, without loss of generality we can also assume $f_n \in C(\mathbb{R}^+)$). Let $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be the minimal nonnegative solution of

$$\begin{gathered} \Delta^2 u_n - \mu a_n(x) u_n = f_n(u_n) + \lambda b_n(x) \quad \text{in } \Omega, \\ u_n = 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{gathered} \tag{4.3}$$

We say that there is a complete blow-up in (1.1), if given any such $\{a_n(x)\}$, $\{b_n(x)\}$, $\{f_n\}$ and u_n,

$$u_n(x) \to \infty \quad \forall\, x \in \Omega.$$

We remark that the existence of u_n follows from Theorem 6.3. The next theorem is proved in the spirit of [12].

Theorem 4.3. *Fix* $\lambda > 0$. *Suppose* (1.1) *does not have any solution, even in the very weak sense. Then there is complete blow up.*

Proof. Let $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be the minimal nonnegative solution of (4.3). Using the monotonicity property of a_n, b_n and f_n, we obtain u_{n+1} is a supersolution of the equation satisfied by u_n. Thus $u_n \leq u_{n+1}$. Therefore to establish the blow-up result, it is sufficient to show the complete blow-up for the family of minimal solutions u_n.

We prove this by the method of contradiction. Assume there exists $x_0 \in \Omega$ and a positive constant C such that $u_n(x_0) \leq C$. Thus applying weak Harnack inequality (Lemma 2.4) we have

$$|u_n|_{L^1(B_{\rho R}(x_0))} \leq C \operatorname{ess\,inf}_{B_{\theta R}(x_0)} u_n \leq C u_n(x_0) \leq C',$$

where $0 < \theta < \rho < 1$. Then following the same argument as in [12], we can show that there exists $r > 0$ and a positive constant $C = C(r)$ such that

$$\int_{B_r(0)} u_n dx \leq C, \quad \text{uniformly for } n \in \mathbb{N}.$$

Therefore, applying the monotone convergence theorem we see that, there exists $u \geq 0$ such that $u_n \to u$ in $L^1(B_r(0))$.

Let ϕ be the solution to the problem

$$\begin{gathered} \Delta^2\phi = \chi_{B_r(0)} \quad \text{in } \Omega, \\ \phi = 0 = \Delta\phi \quad \text{on } \partial\Omega. \end{gathered}$$

Clearly $\phi \in W^{4,p}(\Omega)$ since $\chi_{B_r(0)} \in L^p(\Omega)$ for all $p \geq 1$. Taking ϕ as a test function in (4.3), we have

$$\int_\Omega (a_n(x)u_n\phi + f_n(u_n)\phi + \lambda b_n\phi)dx = \int_{B_r(0)} u_n dx \leq C.$$

By monotone convergence theorem and Fatou's lemma, it follows that

$$\begin{gathered} a_n(x)u_n \uparrow a(x)u \quad \text{in } L^1_{\rm loc}(B_r(0)), \\ f_n(u_n) \to f(u) \quad \text{in } L^1_{\rm loc}(B_r(0)) \quad \text{and} \quad b_n(x) \uparrow b(x) \quad \text{in } L^1_{\rm loc}(B_r(0)). \end{gathered}$$

Hence as in [12, Theorem 5.1], we can conclude that u is a very weak solution to (1.1) in $B_{r_1}(0) \Subset B_r(0)$ and this contradicts the assumption of this theorem. $\square$

Combining Lemma 4.1 and Theorem 4.3, we obtain the following corollary.

Corollary 4.4. *If $\lambda > \tilde{\lambda}^*$, then there is complete blow-up.*

5. Stability results

Definition 5.1. We say that $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ is a stable solution, if the first eigenvalue of the linearized operator of the equation (1.1) is nonnegative, i.e., if

$$\inf_{\phi \in C^\infty_0(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta\phi|^2 - \mu a(x)\phi^2 - f'(u)\phi^2)dx}{\int_\Omega \phi^2 dx} \geq 0.$$

Theorem 5.2. *Suppose all the assumptions in Theorem 3.1 are satisfied and for $0 < \lambda < \lambda^*$, let u_λ denote the minimal positive solution of* (1.1)*. Then u_λ is stable.*

Proof. Following the idea of Dupaigne and Nedev [8], we prove this theorem. Let $a_n(x) = \min(a(x), n)$, $b_n = \min(b(x), n)$ and $u_n \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ denote the minimal positive solution of the problem

$$\begin{gathered} \Delta^2 u_n - \mu a_n(x)u_n = f(u_n) + \lambda b_n(x) \quad \text{in } \Omega, \\ u_n = 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{gathered} \tag{5.1}$$

By Lemma 2.6, u_n is well defined since u_λ is a supersolution of (5.1). Let $\lambda_1^n(\Delta^2 - \mu a_n(x) - f'(u_n))$ denote the 1st eigenvalue of the linearized operator $\Delta^2 - \mu a_n(x) - f'(u_n)$.

Claim: $\lambda_1^n(\Delta^2 - \mu a_n(x) - f'(u_n)) \geq 0$.

To prove this claim, we choose $p > N$. Define, $I : \mathbb{R} \times W^{4,p}(\Omega) \to L^p(\Omega)$ as follows

$$I(\lambda, u) = \Delta^2 u - \mu a_n(x)u - f(u) - \lambda b_n.$$

An easy computation using (1.14) and implicit function theorem, (see [8]) it follows that there exists a unique maximal curve $\lambda \in [0, \lambda^\#) \to u(\lambda)$ such that

$$I(\lambda, u(\lambda)) = 0 \quad \text{and} \quad I_u(\lambda, u(\lambda)) \in Iso(W^{4,p}.L^p).$$

If $0 < \lambda < \lambda^{\#}$, then $u_n \le u(\lambda)$, since u_n is the minimal positive solution of (5.1). Thus $f(u_n) \le f(u(\lambda))$. Moreover, $I(\lambda, u(\lambda)) = 0$ implies $f(u(\lambda)) = \Delta^2 u(\lambda) - \mu a_n(x)u(\lambda) - \lambda b_n(x) \in L^p(\Omega)$, which in turn implies $f(u_n) \in L^p(\Omega)$. Therefore by elliptic regularity theory, u_n is in the domain of I and hence $u_n = u(\lambda)$.

Following the same method as in [8], we can show that if $0 < \lambda < \lambda^*$, u_n is in the domain of I. Thus $\lambda^{\#} = \lambda^*$ (otherwise we could extend the curve $u(\lambda)$ beyond $\lambda^{\#}$ contradicting its maximality). We also claim that the first eigenvalue of $I_u(\lambda, u_n)$ does not vanish for any $\lambda < \lambda^*$. To see this, assume ϕ is an the eigenfunction corresponding to this first eigenvalue. If the first eigenvalue vanishes for some $\lambda_0 < \lambda^*$, then we have $\Delta^2\phi - \mu a(x)\phi - f'(u_n)\phi = 0$, i.e., $I_u(\lambda_0, u_n) = 0$ but we know that $I_u(\lambda, u)$ can not vanish for any $\lambda < \lambda^{\#}$ (otherwise $u(\lambda)$ will not be the maximal curve). Consequently, since $\lambda^{\#} = \lambda^*$, we can say that the first eigenvalue of $I_u(\lambda, u_n)$ does not vanish for any $\lambda < \lambda^*$. Moreover, by (1.14) we know first eigenvalue of $I_u(0, 0)$ is strictly positive. Therefore we conclude that $\lambda_1^n(\Delta^2 - \mu a_n(x) - f'(u_n)) \ge 0$ for every $\lambda \in [0, \lambda^*)$.

Also, $\{u_n\}$ is a nondecreasing sequence and converges to a solution of (1.1) in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Since $u_n \le u_\lambda$, $\lim_{n\to\infty} u_n$ has to be the minimal solution u_λ. Therefore by monotone convergence theorem we conclude the first eigenvalue $\lambda_1(\Delta^2 - \mu a(x) - f'(u_\lambda)) \ge 0$ which completes the proof. □

Theorem 5.3. *Suppose the assumptions in Theorem 3.1 hold and u_λ is the minimal positive solution of* (1.1). *Also assume* (3.5) *is satisfied. If $\lambda = \lambda^*$ and $b \in L^p(\Omega)$ for some $p > \frac{N}{3}$, then u_{λ^*} is the only positive solution of* (1.1), *with λ^* instead of λ, which belongs to $\in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

Proof. Suppose the theorem does not hold and u and v are two distinct positive solutions of (1.1), with λ^* instead of λ, where $u, v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Let u be the minimal positive solution. Therefore $u \le v$. Applying strong maximal principle we can easily check that $u < v$ in Ω. Since u and v are solution, by Definition (1.1) we have $f(u), f(v) \in L^2(\Omega)$. Thus applying (1.10), we obtain $\mu a(x)u + f(u) + \lambda^* b \in L^2(\Omega)$. This together with the elliptic regularity theory gives $u \in W^{4,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Similarly same result holds for v as well. Define $w = \frac{u+v}{2}$. Then $w \in W^{4,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and by convexity of f, we have

$$f(w) = f\big(\frac{u+v}{2}\big) \le \frac{f(u) + f(v)}{2} \in L^2(\Omega).$$

Thus,

$$\Delta^2 w - \mu a(x)w = \frac{f(u) + f(v)}{2} + \lambda^* b \ge f(w) + \lambda^* b.$$

Thus w is a supersolution of (1.1) with λ^* instead of λ. By Lemma 6.1, it follows that w is a solution to (P_{λ^*}). As a consequence, inequality on the above expression becomes equality and by convexity of f we conclude that f is linear on $[u(x), v(x)]$ for almost every $x \in \Omega$. For $\epsilon \in (0, 1)$, define $\theta = \epsilon u + (1 - \epsilon)v$. Therefore $f''(\theta(x))$ exists for a.e $x \in \Omega$ and $f''(\theta(x)) = 0$ a.e. $x \in \Omega$. This implies $\nabla(f'(\theta)) = 0$ a.e. in Ω, which in turn implies $f'(\theta) = C$ a.e. in Ω and $f(\theta) = C\theta + D$ a.e. in Ω for some constant C and D. Moreover, using convexity of f, this implies $f(t) = Ct + D$ for $t \in [\text{ess inf } \theta, \operatorname{ess\,sup} \theta]$. Applying Lemma 6.2, we have $\operatorname{ess\,inf} \theta = 0$. Since $f(0) = 0 = f'(0)$, we obtain $f \equiv 0$ on $[0, \operatorname{ess\,sup} \theta]$. As $\epsilon > 0$ arbitrary, we can conclude $f \equiv 0$ on $[0, \operatorname{ess\,sup} v]$. Therefore u and v both satisfy

$$\begin{gathered}\Delta^2 u - \mu a(x)u = \lambda^* b(x) \quad \text{in } \Omega, \\ u = 0 = \Delta u \quad \text{on } \partial\Omega.\end{gathered}$$

This in turn implies, $v - u$ satisfies

$$\begin{gathered}\Delta^2(v-u) - \mu a(x)(v-u) = 0 \quad \text{in } \Omega, \\ v - u = 0 = -\Delta(v-u) \quad \text{on } \partial\Omega.\end{gathered}$$

This contradicts (1.14) since $v - u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Hence $u = v$. $\square$

6. Appendix

Lemma 6.1. *If $b \in L^p(\Omega)$ for some $p > \max\{2, \frac{N}{3}\}$ and $w \in W^{4,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a supersolution of (1.1) with λ^* instead of λ, then w is a solution of (1.1) with λ^* instead of λ.*

Proof. Let w be a supersolution of (1.1) with λ^* instead of λ and not a solution. Define, $\nu \in \mathcal{D}'(\Omega)$ by

$$\nu(\phi) = \int_\Omega w(\Delta^2\phi) - (\mu a(x)w + f(w) + \lambda^* b)\phi \quad \forall \phi \in C_0^\infty(\Omega).$$

Since w is a supersolution, by Definition 1.1 we have $f(w) \in L^2(\Omega)$. Therefore thanks to (1.10), we obtain $\nu \in L^2(\Omega)$. Moreover, w is a supersolution implies $\nu \geq 0$. w is not a solution implies $\nu \not\equiv 0$. Consider the problem

$$\begin{gathered}\Delta^2\psi = \nu \quad \text{in } \Omega, \\ \psi = 0 = \Delta\psi \quad \text{on } \partial\Omega.\end{gathered}$$

We can break this problem into system of second-order Dirichlet problem by defining

$$\begin{gathered}-\Delta\psi = \tilde{\psi} \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega, \\ -\Delta\tilde{\psi} = \nu \quad \text{in } \Omega, \quad \tilde{\psi} = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Then by the weak maximum principle it is easy to check that $\psi > \epsilon\delta(x)$ for some $\epsilon > 0$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Next we consider the problem

$$\begin{gathered}\Delta^2\eta = b \quad \text{in } \Omega, \\ \eta = 0 = \Delta\eta \quad \text{on } \partial\Omega.\end{gathered}$$

As before we break this problem into system of equations as follows:

$$\begin{gathered}-\Delta\eta = \tilde{\eta} \quad \text{in } \Omega, \quad \eta = 0 \quad \text{on } \partial\Omega, \\ -\Delta\tilde{\eta} = b \quad \text{in } \Omega, \quad \tilde{\eta} = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Since $b \in L^p(\Omega)$ for some $p > \frac{N}{3}$, using theory of elliptic regularity and Soblev embedding theorem, we obtain $\tilde{\eta} \in L^{p^*}(\Omega)$ where $p^* = \frac{Np}{N-2p} > N$. Therefore $\eta \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Hence $\eta < C\delta(x)$ in Ω for some $C \in (0, \infty)$. Define, $v = w + \epsilon C^{-1}\eta - \psi$. Clearly $v < w$ in Ω and $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Also,

$$\begin{aligned}\Delta^2 v &= \Delta^2 w + \epsilon C^{-1} b - \nu \\ &= \mu a(x)w + f(w) + \lambda^* b + \nu + \epsilon C^{-1} b - \nu \\ &\geq \mu a(x)v + f(v) + (\lambda^* + \epsilon C^{-1})b.\end{aligned}$$

As a result, v is a supersolution to (1.1) with $\lambda^* + \epsilon C^{-1}$ instead of λ. Hence (1.1), with $\lambda^* + \epsilon C^{-1}$ instead of λ, has a solution contradicting the extremality of λ^*. $\square$

The next lemma is in the spirit of [8, Lemma 3.2].

Lemma 6.2. *If $u \in L^1(\Omega)$ is an nonnegative distributional solution of $\Delta^2 u = h$ in Ω, where $h \in L^1(\Omega)$, then $\operatorname{ess\,inf} u = 0$.*

Proof. Assume the lemma does not hold, that is, there exists $\epsilon > 0$ such that $u \geq \epsilon > 0$ a.e. in Ω. We extend u and h by 0 in $\mathbb{R}^N \setminus \Omega$. Let ρ_n denote the standard molifier. Define $u_n = u \star \rho_n$ and $h_n = h \star \rho_n$. Following the same argument as in [8, Lemma 3.2], we can show that, there exists $\alpha > 0$ such that for n large enough $u_n \geq \alpha\epsilon$ everywhere in Ω and given $\omega \Subset \Omega$ and n large enough, $\Delta^2 u_n = h_n$ everywhere in ω. Let ϕ solve the following problem

$$\begin{gathered} \Delta^2 \phi = 1 \quad \text{in } \omega, \\ \phi = 0 = \Delta\phi \quad \text{on } \partial\omega. \end{gathered} \tag{6.1}$$

Integrating by parts we obtain

$$\begin{aligned} \int_\omega u_n dx &= \int_\omega u_n \Delta^2 \phi dx \\ &= \int_\omega \Delta u_n \Delta\phi dx + \int_{\partial\omega} \frac{\partial}{\partial n}(\Delta\phi) u_n ds \\ &= \int_\omega h_n \phi dx + \int_{\partial\omega} \frac{\partial}{\partial n}(\Delta\phi) u_n ds. \end{aligned}$$

Thus,

$$\int_\omega h_n \phi dx - \int_\omega u_n dx = -\int_{\partial\omega} \frac{\partial}{\partial n}(\Delta\phi) u_n ds \leq -\alpha\epsilon|\omega|,$$

since $\int_{\partial\omega} \frac{\partial}{\partial n}(\Delta\phi) ds = |\omega|$ (follows from (6.1) after integrating by parts). Since $u_n \to u$ in $L^1(\Omega)$, $h_n \to h$ in $L^1(\Omega)$ we obtain

$$\int_\omega h\phi dx - \int_\omega u dx \leq -\alpha\epsilon|\omega|.$$

Next we choose $\omega = \omega_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\}$, $n \to \infty$. Let ϕ_n denote the corresponding solution to (6.1) in ω_n. Then $\phi_n \uparrow \phi$ where ϕ solves

$$\begin{gathered} \Delta^2 \phi = 1 \quad \text{in } \Omega, \\ \phi = 0 = \Delta\phi \quad \text{on } \partial\Omega. \end{gathered}$$

Taking the limit $n \to \infty$ in $\int_{\omega_n} h\phi_n dx - \int_{\omega_n} u dx \leq -\alpha\epsilon|\omega_n|$ and using $\Delta^2 u = h$ in Ω, we have $0 \leq -\alpha\epsilon|\Omega|$. This gives a contradiction. $\square$

Theorem 6.3. *Assume* (1.12) *is satisfied. Then problem* (4.3) *has a nonnegative minimal solution for every $\lambda > 0$.*

Proof. **Step 1:** Assume $a \in L^1_{\rm loc}(\Omega)$ which satisfies (1.10). Let $b \in L^\infty(\Omega)$ and $f \in L^\infty(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ be nonnegative functions, $b \not\equiv 0$ and $\lambda > 0$. Then there exists $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ such that u solves (1.1) for all $\lambda > 0$.

To prove step 1, let $u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ be a positive solution to

$$\begin{gathered} \Delta^2 u_0 = \lambda b \quad \text{in } \Omega, \\ u_0 = 0 = \Delta u_0 \quad \text{on } \partial\Omega. \end{gathered}$$

Since $\lambda b \in L^\infty(\Omega) \subset L^2(\Omega)$ we obtain $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Next, using iteration we will show that there exists $u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for $n = 1, 2, \dots$ such that u_n solves the problem

$$\begin{gathered} \Delta^2 u_n = \mu a(x) u_{n-1} + f(u_{n-1}) + \lambda b(x) \quad \text{in } \Omega, \\ u_n = 0 = \Delta u_n \quad \text{on } \partial\Omega. \end{gathered} \tag{6.2}$$

Thanks to (1.10) and the assumptions that $f, b \in L^\infty(\Omega)$, it follows that $\mu a(x) u_0 + f(u_0) + \lambda b(x) \in L^2(\Omega)$. Therefore u_1 is well defined. Moreover, by comparison principle $0 < u_0 \le u_1$. Using the induction method, similarly we can show u_n is well defined and $0 < u_0 \le u_1 \le \dots \le u_n \le \dots$.

Claim: $\{u_n\}$ is uniformly bounded in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

To see this, note that from (6.2) we can write

$$|\Delta u_n|^2_{L^2(\Omega)} = \int_\Omega (\mu a(x) u_{n-1} + f(u_{n-1}) + \lambda b(x)) u_n dx. \tag{6.3}$$

Using Holder inequality, (1.10) and Young's inequality, the terms on the right-hand side can be simplified as follows

$$\begin{aligned} \lambda \int_\Omega b u_n dx &\le \lambda |b|_{L^\infty(\Omega)} |\Omega|^{1/2} |u_n|_{L^2(\Omega)} \\ &\le \frac{C}{\sqrt{\gamma}} |b|_{L^\infty(\Omega)} |\Delta u_n|_{L^2(\Omega)} \\ &\le \epsilon |\Delta u_n|^2_{L^2(\Omega)} + c(\epsilon) |b|^2_{L^\infty(\Omega)}, \\ \int_\Omega f(u_{n-1}) u_n dx &\le |f|_{L^\infty(\Omega)} |\Omega|^{1/2} |u_n|_{L^2(\Omega)} \\ &\le \frac{C}{\sqrt{\gamma}} |f|_{L^\infty(\Omega)} |\Delta u_n|_{L^2(\Omega)} \\ &\le \epsilon |\Delta u_n|^2_{L^2(\Omega)} + c(\epsilon) |f|^2_{L^\infty(\Omega)}, \\ \mu \int_\Omega a(x) u_{n-1} u_n dx &\le \mu \int_\Omega a(x) u_n^2 dx \\ &\le \mu |a(x) u_n|_{L^2(\Omega)} |u_n|_{L^2(\Omega)} \\ &\le \frac{\mu}{\sqrt{\gamma}} |\Delta u_n|^2_{L^2(\Omega)}. \end{aligned}$$

Since $\mu/\sqrt{\gamma} < 1$, we can choose $\epsilon > 0$ such that $2\epsilon + \frac{\mu}{\sqrt{\gamma}} < 1$. Substituting this ϵ in above three inequalities and combining them with (6.3), we have

$$|\Delta u_n|^2_{L^2(\Omega)} \le C(|b|_{L^\infty(\Omega)} + |f|_{L^\infty(\Omega)}).$$

This proves the claim. As a consequence there exists $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$. Therefore we can conclude the theorem as we did in Lemma 2.6.

Step 2: Let $\{b_n(x)\}$ and $\{f_n\}$ be increasing sequence of bounded functions converging pointwise respectively to $b(x)$ and f (f_n is continuous for $n = 1, 2, \dots$). Then by Step 1, there exists a nonnegative minimal solution $v_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of the problem

$$\begin{gathered} \Delta^2 v_n - \mu a(x) v_n = f_n(v_n) + \lambda b_n(x) \quad \text{in } \Omega, \\ v_n = 0 = \Delta v_n \quad \text{on } \partial\Omega. \end{gathered} \tag{6.4}$$

Clearly v_n is a nonnegative supersolution to (4.3). Therefore the theorem follows from Lemma 2.6. $\square$

Final Remark. The results of this article can be easily extended to the equations of the form

$$\Delta^2 u - \mu a(x) u = c(x) f(u) + \lambda b(x) \quad \text{in } \Omega,$$

where $c \in L^1_{\rm loc}(\Omega)$ is a nonnegative function. In particular, (3.1) will be changed to

$$c f(\epsilon\zeta_1) \in L^2(\Omega) \quad \text{and} \quad G(c(x) f(\epsilon\zeta_1)) \leq C\zeta_1.$$

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Mousomi Bhakta
Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhaba road, Pune-411008, India
E-mail address: mousomi@iiserpune.ac.in