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MEROMORPHIC SOLUTIONS TO DIFFERENCE PAINLEVÉ EQUATIONS I AND II

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Abstract. In this article, we study the solutions of difference Painlevé equations I and II. to do this, we first give some properties of solutions to difference Riccati equations. We show that there are no rational solutions for the difference Painlevé equation I. We also study rational solutions for the difference Painlevé equation II. Later, we present a new way to seek solutions of the difference Painlevé equation II by using the difference Riccati equation. At last, we construct new solutions of the difference Painlevé equation II through a known solution.

1. Introduction

An ordinary differential equation is said to possess the Painlevé property if all of its solutions are single-values about all movable singularities, see [7]. Over a century ago Painlevé [21, 22], Fuchs [4] and Gambier [5] completed a substantial classification work, which comprised of sieving through a large class of second-order differential equations by making use of a criterion proposed by Picard [23], now known as the Painlevé property. Painlevé and his colleagues discovered six new equations, later named as Painlevé equations, which were not solvable in terms of known functions.

Painlevé equations are the most complicated systems that one can solve (in a nontrivial way). Anything simpler becomes trivially integrable, anything more complicated becomes hopelessly non-integrable. Therefore, Painlevé equations are the borderline between trivial integrability and non-integrable, a fact that bestows upon them a lot of of interesting properties.

There are several candidates for the discrete analogue of the Painlevé property. The singularity confinement test by Grammaticos, Ramani and Papageorgiou [6] has been very successful in finding new discrete equations of Painlevé type [25]. Hietarinta and Viallet suggested to amend singularity confinement by demanding additionally that the algebraic complexity of the solutions remains relatively low as the iteration progresses [14]. The first approach using a low growth criterion of iterates as a sign of integrability is due to Veselov [28] who showed that generic exponential growth rate of the degree of iterates corresponds to the non-existence of certain types of first integrals. A method due to Roberts and Vivaldi relies on

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orbit dynamics over finite fields [27], while Halburd's Diophantine integrability [8] makes use of polynomial growth of heights of iterates over number fields.

The analogues of Painlevé property for difference equations in the complex plane have been discussed, for example, in [1], Ablowitz et.al. consider the solutions of difference Painlevé equations by Nevanlinna theory, which is a landmark in the application of Nevanlinna theory in the study of difference equations. They have observed that all of the relevant difference equations have obvious analytic versions, and hence can be studied by using the methods of complex analysis, and in particular Nevanlinna theory.

Halburd and Korhonen [12] showed that if the difference equation

$$\overline{w} + \underline{w} = R(z, w), \tag{1.1}$$

where $R(r, w)$ is rational in w with meromorphic coefficients, has an admissible meromorphic solution of finite order, then the class (1.1) can be reduced into a short list of equations, where the difference Painlevé equation I and the difference Painlevé equation II are included.

Halburd and Korhonen [13] showed that if the difference equation

$$\overline{w}\underline{w} = \frac{c_2(w - c_+)(w - c_-)}{(w - a_+)(w - a_-)} \tag{1.2}$$

has an admissible finite order meromorphic solution such that the order of its poles is bounded, then the equation (1.1) can be transformed by Möbius tranformation in w to a short list of equations, where the difference Painlevé III is included, unless w is the solution of difference Riccati equation. Ronkainen [26] removed the assumption on the boundedness of the order of the poles, difference painlevé equation III can be obtained from equation (1.2) when the coefficients are special solutions of equations. Similarly, difference Painlevé equation V is also considered by Ronkainen [26].

The aim of this article is to discuss the solutions of difference Painlevé equations I and II. The remainder of this paper is organized as follows. Difference version of Nevanlinna theory for studying the difference Painlevé equations is shown in Section 2. We give some properties of the solutions of the difference Riccati equation in Section 3. In section 4, we show the relations of the solutions between the difference Riccati equation and difference Painlevé equations I and II, respectively. We conclude that there are no rational solutions of the difference Painlevé equation I. We also discuss the rational solutions of the difference Painlevé equation II in four cases according to the coefficients of the equations. At the end of section 4, we give a way to seek a new solution of the difference Painlevé equation II.

2. Nevalinna theory and difference equations

Difference analogue lemma on the logarithmic derivative of meromorphic functions of finite order was found by Halburd and Korhonen [10, 11], and by Chiang and Feng [3], independently. This result was recently extended to meromorphic functions of hype order strictly less than one in [9], where it was also shown that this growth condition cannot be essentially relaxed further. Let us state latest version of difference analogue lemma on the logarithmic derivative of meromorphic functions of finite order as follows.

Lemma 2.1 ([9, Theorem 5.1]). *Let w be a non-constant meromorphic function and $c \in \mathbb{C}$. If w is of finite order, then*
$$m\Big(r, \frac{w(z+c)}{w(z)}\Big) = \Big(\frac{\log r}{r}T(r,w)\Big)$$
for all r outside of a set E satisfying
$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)}\, dt/t}{\log r} = 0,$$
i.e., outside of a set E of zero logarithmic density. If $\rho_2(w) = \rho_2 < 1$ and $\varepsilon > 0$, then
$$m\Big(r, \frac{w(z+c)}{w(z)}\Big) = o\Big(\frac{T(r,w)}{r^{1-\rho_2-\varepsilon}}\Big)$$
for all r outside of a set of finite logarithmic measure.

The following Lemma is improved version of [10, Lemma 2.2].

Lemma 2.2 ([9, Lemma 8.3]). *Let $T : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper order of T is strictly less than one, i.e.,*
$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1,$$
and $\delta \in (0, 1-\rho_2)$, then
$$T(r+s) = T(r) + o\Big(\frac{T(r)}{r^\delta}\Big),$$
where r runs to infinity outside of a set of finite logarithmic measure.

The difference version of a result due to A. Mohon'ko and V. Mohon'ko [20], is found by Halburd and Korhonen [11], and Laine and Yang [18, Theorem 2.4], independently. Let us state this result as follows.

Lemma 2.3 ([11, Corollary 3.4]). *Let $w(z)$ be a nonconstant finite order meromorphic solution of $P(z,w) = 0$, where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a small meromorphic function related to w, then*
$$m\Big(\frac{1}{w-a}\Big) = S(r,w)$$
for all r outside of a possible exceptional set with finite logarithmic measure.

The difference analogue of Clunie lemma, due to Laine and Yang, is the generalization of the result in [11, Theorem 3.1], which is stated as follows.

Lemma 2.4 ([18, Theorem 2.3]). *Let w be a transcendental meromorphic solution of finite order ρ of a difference equations of the form*
$$H(z,w)P(z,w) = Q(z,w)$$
where $H(z,f)$, $P(z,w)$, $Q(z,f)$ are difference polynomials such that total degree n in $w(z)$ and its shifts, and the total degree of $Q(z,w)$ is $\le n$. Then, for each $\varepsilon > 0$,
$$m(r, P(z,w)) = O(r^{\rho-1+\varepsilon}) + S(r,w),$$
possibly outside of an exceptional set of finite logarithmic measure.

3. Difference Riccati equations

About a century ago Malmquist [19] proved that if the differential equation

$$w' = R(z, w(z)), \tag{3.1}$$

where $R(z, w(z))$ is rational in its both arguments, has a non-rational meromorphic solution, then (3.1) reduces into a Riccati equation

$$w' = a(z)w^2 + b(z)w + c(z)$$

with rational coefficients. Similarly, if the difference equation

$$w(z+1) = R(z, w(z)), \tag{3.2}$$

where again $R(z, w(z))$ is rational in z and $w(z)$, admits at least one non-rational meromorphic solution of finite order (or of hyper-order strictly less than one), then (3.2) must in fact be a difference Riccati equation

$$w(z+1) = \frac{c(z)w(z) + d(z)}{a(z)w(z) + b(z)}$$

with rational coefficients. This is a known difference analogue of Malmquist's theorem, which follows by a more general result due to Yanagihara [29].

In what follows, the difference Riccati equations we mentioned have the coefficients of small functions related to w. If $a(z) \equiv 0$, then the difference Riccati equations are reduced to the linear difference equations. Thus, let us consider the case $a(z) \not\equiv 0$. In this section, we discuss some basic properties of the solutions of the difference Riccati equations, which will be used for investigating difference Painlevé equations I and II in the next section.

Lemma 3.1. *If w is the solution of*

$$w(z+1) = \frac{c(z)w(z) + d(z)}{a(z)w(z) + b(z)}, \tag{3.3}$$

then w also satisfies

$$w(z-1) = \frac{d(z-1) - b(z-1)w(z)}{a(z-1)w(z) - c(z-1)}. \tag{3.4}$$

Proof. If w is the solution of (3.3), then we have

$$w(z) = \frac{c(z-1)w(z-1) + d(z-1)}{a(z-1)w(z-1) + b(z-1)}.$$

Thus, by a simple calculation it yields that

$$w(z-1) = \frac{d(z-1) - b(z-1)w(z)}{a(z-1)w(z) - c(z-1)},$$

which is our assertion. □

Let us consider the rational solutions of the difference Riccati equations. In this case, all the coefficients of the difference Riccati equations are reduced to the constants.

Lemma 3.2. *Let w be a non-constant rational solution of the difference Riccati equations* (3.3)*, which is written as $w(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, then* $\deg P \le \deg Q$. *Moreover, if $d \neq 0$, then* $\deg P = \deg Q$.

Proof. Suppose that w is a rational function, let us write w as

$$w(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials. If w is a solution of (3.3), then it yields that by substituting it to (3.3)

$$\frac{P(z+1)}{Q(z+1)} = \frac{cP(z) + dQ(z)}{aP(z) + bQ(z)}.$$

From the equation above, we have the following equality

$$aP(z+1)P(z) + bP(z+1)Q(z) = cP(z)Q(z+1) + dQ(z)Q(z+1). \tag{3.5}$$

Let us denote $p = \deg P$ and $q = \deg Q$. If $p > q$, then the degree of polynomials of both side of the equation (3.5) are $2p$ and $p + q$ (or $2q$ if c=0), respectively. It implies that $p = q$, which is a contradiction. Thus, we have $\deg P \leq \deg Q$.

In particular, if $d \neq 0$, let us denote

$$P_*(z, w) = \overline{w}(aw + b) - (cw + d).$$

Since $w = 0$ is not the solution of $P_*(z, w) = 0$, it follows by Lemma 2.3 that

$$N\big(r, \frac{1}{w}\big) = q \log r + O(1) = T(r, w) + O(1). \tag{3.6}$$

Now let us rewrite difference Riccati equations (3.3) as follows

$$a\overline{w}w = cw - b\overline{w} + d.$$

Since a, b, c and d are small functions of w, it yields by $H(z, w) = aw$ and $Q(z, w) = cw - b\overline{w} + d$ that

$$N(r, w) = p \log r + O(1) = T(r, w) + O(1). \tag{3.7}$$

according to Lemma 2.4. Therefore, we conclude from (3.6) and (3.7) that $\deg P = \deg Q$, which is our assertion. □

Now we will find a linear transformation, by which every difference Riccati equation (3.3) can be reduced into a simple form

$$\overline{w} = \frac{w + D}{w + 1}, \tag{3.8}$$

where D related to a, b, c, d is a small function of w. Thus, we can only consider the solutions of difference Riccati equations (3.8).

Lemma 3.3. *Any difference Riccati equation* (3.3) *can be transformed by a linear mapping to the form* (3.8).

Proof. Since w satisfies (3.3), by the linear transformation $w = Af + B$ it yields

$$\overline{Af} + \overline{B} = \frac{cAf + cB + d}{aAf + aB + b},$$

where A, B are the small functions of f. By Nevanlinna First Main theory, it follows that A and B are also the small functions of w. Thus, f satisfies

$$\overline{Af} = \frac{(c - a\overline{B})\big(f + \frac{cB + d - aB\overline{B} - b\overline{B}}{cA - aA\overline{B}}\big)}{a\left(f + \frac{aB + b}{aA}\right)}$$

Let us set

$$a\overline{A} = c - a\overline{B}$$
$$aA = b + aB$$

then, we have our assertion if A and B satisfy

$$A = \frac{1}{2}\Big(\frac{c}{\underline{a}} + \frac{b}{a}\Big)$$
$$B = \frac{1}{2}\Big(\frac{c}{\underline{a}} - \frac{b}{a}\Big)$$

Moveover, we can express D by a, b, c and d. $\square$

Remark 3.4. It follows that there are two linearly independent meromorphic solutions of difference Riccati equations by Lemma 3.3 and discussion in [15] and [24]. We can construct other meromorphic solutions of difference Riccati equations by these two linearly independent solutions. Moveover, we could use it for investigating the solutions of difference Painleve equation II. Let us discuss it in the next section.

4. Difference Painlevé equations

Halburd and Korhonen proved that if the equation

$$\overline{w} + \underline{w} = R(z, w), \tag{4.1}$$

where $R(z, w)$ is rational in w and meromorphic in z, has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation

$$\overline{w} = \frac{\overline{p}w + q}{w + p},$$

where p, q are small functions related to w, or equations (4.1) can be transformed by a linear change in w to one of the following equations:

$$2\overline{w} + w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + k_1 \tag{4.2}$$
$$\overline{w} - w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + (-1)^z k_1 \tag{4.3}$$
$$\overline{w} + \underline{w} = \frac{\pi_1 z + \pi_3}{w} + \pi_2 \tag{4.4}$$
$$\overline{w} + \underline{w} = \frac{\pi_1 z + k_1}{w} + \frac{\pi_2}{w^2} \tag{4.5}$$
$$\overline{w} + \underline{w} = \frac{(\pi_1 z + k_1)w + \pi_2}{(-1)^{-z} - w^2} \tag{4.6}$$
$$\overline{w} + \underline{w} = \frac{(\pi_1 z + k_1)w + \pi_2}{1 - w^2} \tag{4.7}$$
$$\overline{w}w + w\underline{w} = p \tag{4.8}$$
$$\overline{w} + \underline{w} = pw + q \tag{4.9}$$

where π_k, k_k are finite order small functions related to w of period k.

Historically, Equation (4.2) is known as the difference Painlevé equation I, The choice of name stems from the fact that a continuous limit, such as $w = -1/2 + \varepsilon^2 u$, $k_1 = -3$, $\pi_1 z + \pi_2 = -(3 + 2\varepsilon^4 t)/4$, $\varepsilon \to 0$, may be used to map (4.2) to the Painlevé equation I $u'' = 6u^2 + t$. Equations (4.4) and (4.5) are still known as difference Painlevé equation I, while equation (4.7) is often referred to as the difference Painlevé II. In this section, we consider the solutions of difference Painleve equations I and II.

Theorem 4.1. *Let w be a non-constant solution of the difference Riccati equation* (3.3) *with rational coefficients. Then w is not the solution of the difference Painlevé equation I*
$$\overline{w}+\underline{w}=\frac{\pi_1 z+\pi_3}{w}+\pi_2,$$
where $\pi_i (i=1,2,3)$ are the small functions related to w with period of i respectively, if $\pi_2 \not\equiv 0$.

Proof. Suppose that w is the solution of
$$\overline{w}=\frac{cw+d}{aw+b},$$
where a, b, c and d are the small functions of w. By Lemma 3.1, it follows that w satisfies
$$\underline{w}=\frac{\underline{d}-\underline{b}w}{\underline{a}w-\underline{c}}.$$
According to discussion above, it yields that w satisfies that
$$\begin{aligned}\overline{w}+\underline{w}&=\frac{cw+d}{aw+b}+\frac{\underline{d}-\underline{b}w}{\underline{a}w-\underline{c}}\\&=\frac{(c\underline{a}-a\underline{b})w^2+(\underline{a}d-c\underline{c}+a\underline{d}-\underline{b}b)w+b\underline{d}-d\underline{c}}{a\underline{a}w^2+(b\underline{a}-a\underline{c})w-b\underline{c}}.\end{aligned}\tag{4.10}$$
Now, if w is the solution of the difference Painlevé equation I (4.4), then we have
$$\frac{(c\underline{a}-a\underline{b})w^2+(\underline{a}d-c\underline{c}+a\underline{d}-\underline{b}b)w+b\underline{d}-d\underline{c}}{a\underline{a}w^2+(b\underline{a}-a\underline{c})w-b\underline{c}}=\frac{\pi_1 z+\pi_3}{w}+\pi_2.$$
By a simple calculation, it yields
$$\begin{aligned}&(c\underline{a}-a\underline{b})w^3+(\underline{a}d-c\underline{c}+a\underline{d}-\underline{b}b)w^2+(b\underline{d}-d\underline{c})w\\&=\pi_2 a\underline{a}w^3+\pi_2(b\underline{a}-a\underline{c})w^2+\pi a\underline{a}w^2+\pi(b\underline{a}-a\underline{c})w-b\underline{c}k_1 w-b\underline{c}\pi,\end{aligned}$$
where $\pi=\pi_1 z+\pi_3$. Therefore, by comparing the coefficients related to w in both side of equations, we have
$$b\underline{c}(\pi_1 z+\pi_3)\equiv 0.$$
It is well-known that $\pi_1 z+\pi_3\not\equiv 0$, then $b\equiv 0$ or $c\equiv 0$. Let us suppose that $b\equiv 0$ at first, thus it follows that
$$\begin{gathered}\pi_2=\frac{c}{a}\\ \underline{a}d+a\underline{d}=\pi a\underline{a}\\ -d\underline{c}=-a\underline{c}\pi\end{gathered}$$
by comparing the coefficients of equations related to w. Since $\pi_2\not\equiv 0$, we have $c\not\equiv 0$. Then it shows that $\pi=d/a$. By submitting it to the Riccati difference equation, it yields that
$$\overline{w}=\frac{\pi_1 z+\pi_3}{w}+\pi_2$$
which implies that $\underline{w}\equiv 0$. It is impossible. Similarly, we have it is impossible if $c\equiv 0$. It is a contradiction with w is a solution of difference Painlevé equation I (4.4). $\square$

By the same way as the proof of Theorem 4.1, we have the following result.

Theorem 4.2. *Let w be a non-constant solution of the difference Riccati equation* (3.3) *with rational coefficients. Then w is not the solution of the difference Painlevé equation I*

$$\overline{w}+w+\underline{w}=\frac{\pi_1 z+\pi_2}{w}+k_1,$$

where π_i and $k_j(i=1,2;j=1)$ are the small functions related to w with period of i and j, respectively.

Let us discuss the case when the difference Painlevé equation I has a rational solution. It is well-known that the only small functions related to the rational functions are constants. If w is a rational solution of

$$\overline{w}+w+\underline{w}=\frac{\pi_1 z+\pi_2}{w}+k_1, \tag{4.11}$$

where π_i and $k_j(i=1,2;j=1)$ are the small functions related to w with period of i and j respectively, then π_1, π_2 and k_1 are reduced to constants. If $\pi_1\equiv 0$ and $\pi_2\equiv 0$, then (4.11) is reduced to a linear difference equation, which is trivial. Thus, we only need to consider the case that either $\pi_1\not\equiv 0$ or $\pi_2\not\equiv 0$.

Theorem 4.3. *Let w be a non-constant solution of the difference Painlevé equation I defined as in* (4.11)*, then w is transcendental.*

Proof. Suppose that w is a rational solution of (4.11). Let us write w as

$$w(z)=\frac{P(z)}{Q(z)}=\frac{a_m z^m+a_{m-1}z^{m-1}+\dots+a_1 z+a_0}{b_n z^n+b_{n-1}z^{n-1}+\dots+b_1 z+b_0}.$$

If we substitute w into the difference Painlevé equation I (4.11), then we have

$$\frac{P(z+1)}{Q(z+1)}+\frac{P(z)}{Q(z)}+\frac{P(z-1)}{Q(z-1)}=\frac{(\pi_1 z+\pi_2)Q(z)}{P(z)}+k_1.$$

According to the equation above, it follows that

$$\begin{aligned}
&P(z+1)P(z)Q(z)Q(z-1)+P(z)Q(z+1)Q(z-1)P(z)\\
&+P(z-1)P(z)Q(z)Q(z+1)\\
&=(\pi_1 z+\pi_2)Q(z)Q(z+1)Q(z)Q(z-1)\\
&\quad +k_1 Q(z+1)Q(z)Q(z-1)P(z).
\end{aligned}\tag{4.12}$$

There are three terms related to $P(z)$ and $Q(z)$ on the left-hand side of (4.12) with the same degree. Moreover, their coefficients of maximum degree are not different, so it is easy to see that the degree of the polynomial on the left-hand side of (4.12) is $2m+2n$.

Suppose that $\pi_1\neq 0$. If $n>m-1$, then the degree of the right-hand side of (4.12) is $4n+1$. It implies that $2m=2n+1$, which is impossible. If $n<m-1$, then the degree of the right-hand side of (4.12) is $3n+m$. It implies that $m=n$, which is impossible. If $m=n+1$, then the degree of the right-hand side of (4.12) is not larger than $3n+m$ or $4n+1$. It implies that $2m-1\le 2n$, which is impossible. Therefore, we have $\pi_1=0$.

If $\pi_1=0$ and $\pi_2\neq 0$, then the equation we considered is

$$\overline{w}+w+\underline{w}=\frac{\pi_2}{w}+k_1, \tag{4.13}$$

where k_1 and π_2 are reduced to constants if w is a rational solution of (4.13). Now let us denote

$$P_1(z,w) = w(\overline{w} + w + \underline{w}) - \pi_2 - k_1 w.$$

Since $w = 0$ is not the solution of $P_1(z,w)$, it follows from Lemma 2.3 that

$$N\big(r, \frac{1}{w}\big) = m \log r + O(1) = T(r,w) + O(1). \tag{4.14}$$

Since π_2 and k_1 are the small function of w, it yields by Lemma 2.4 that

$$N(r,w) = n \log r + O(1) = T(r,w) + O(1). \tag{4.15}$$

Therefore, we have $m = n$ from (4.14) and (4.15).

Let $A = \{z_i \in \mathbb{C}; i \in \mathbb{N}\}$ be the set of all zeros of $w(z) = 0$, and its cardinality of the set $A \cap \{z \in \mathbb{C} : |z| \le r\}$ is denoted by $n_A\big(r, \frac{1}{w}\big)$ according to its multiplicity, its counting functions is denoted by $N_A\big(r, \frac{1}{w}\big)$. Suppose that $z_0 \in A$, then from (4.13) z_0 is a pole of $w(z+1)$ or $w(z-1)$. If z_0 is the pole of $w(z+1)$ or $w(z-1)$, then $z_0 + 2$ or $z_0 - 2$ is also the pole of $w(z)$ by (4.13).

Let B be the subset of A of all points such that

$$w(z_0 - 1) = \infty \quad \text{and} \quad w(z_0 + 1) = \infty$$

hold. We denote by $N_B\big(r, \frac{1}{w}\big)$ the corresponding counting function, and

$$N_{A\setminus B}\big(r, \frac{1}{w}\big) = N_A\big(r, \frac{1}{w}\big) - N_B\big(r, \frac{1}{w}\big)$$

the counting function for those points $z_0 \in A \setminus B$. Therefore, for each points in $B \in \{z \in \mathbb{C} : |z| \le r\}$ there are at least two poles in the disc $\{z \in \mathbb{C} : |z| \le r+1\}$, which implies that

$$N_B\big(r, \frac{1}{w}\big) \le \frac{1}{2} N_B\big(r+1, w\big). \tag{4.16}$$

Similarly, one point in $A \setminus B \cap \{z \in \mathbb{C}; |z| \le r\}$ there exists exactly one pole in the disc $\{z \in \mathbb{C} : |z| \le r+1\}$. Therefore, we have

$$N_{A\setminus B}\big(r, \frac{1}{w}\big) \le N_{A\setminus B}(r+1, w). \tag{4.17}$$

We proceed to prove that the points in the set A are "almost all" in the set $A \setminus B$. Suppose that

$$N_{A\setminus B}(r+1, w) = \alpha T(r,w) + S(r,w),$$

where $0 \le \alpha \le 1$. From (4.14) to (4.17), we have

$$\begin{aligned} T(r,w) &= N\big(r, \frac{1}{w}\big) + S(r,w) = N_A\big(r, \frac{1}{w}\big) + S(r,w) \\ &= N_B\big(r, \frac{1}{w}\big) + N_{A\setminus B}\big(r, \frac{1}{w}\big) + S(r,w) \\ &\le \frac{1}{2} N_B(r+1, w) + N_{A\setminus B}(r+1, w) \\ &= \frac{1}{2} N_A(r+1, w) + \frac{1}{2} N_{A\setminus B}(r+1, w) \\ &\le (\frac{1}{2} + \frac{\alpha}{2}) T(r+1, w) + S(r,w). \end{aligned} \tag{4.18}$$

It yields that $\alpha = 1$ by Lemma 2.2, since w is a meromorphic function of finite order. Therefore, we have

$$N_B(r+1, w) = S(r,w).$$

It reveals that for "almost all" $z_0 \in A$, then $w(z_0+1)=\infty$ and $w(z_0-1)=\infty$ can not hold at the same time. If w is a rational function, then there is no point $z_0 \in A$ such that $w(z_0+1)=\infty$ and $w(z_0-1)=\infty$ hold at the same time.

Suppose that z_0 is the zero of w, then z_0 is the pole of $w(z+1)$ or $w(z-1)$. Let us suppose $w(z_0+1)=\infty$. Then $w(z_0+2)=\infty$ and $w(z_0+1)+w(z_0+2)=k_1$. Since w can be written as the quotient of $P(z)$ and $Q(z)$, then all the zeros of $P(z)$ are the zeros of $Q(z+1)$ and $Q(z+2)$. Keeping in mind that $\deg P=\deg Q$, it shows that

$$P(z)=\alpha Q(z+1)=\beta Q(z+2),$$

where α and β are constants. According to [30, Lemma 2.4], it follows that $P(z)$ and $Q(z)$ are constants, which is impossible. Thus, w is not a rational solution of (4.13). Now we have our assertion. $\square$

Chen and Shen investigate the rational solutions of the difference Painlevé equation I (4.4) in [2]. They show that if $\pi_1=0$ and $w=P(z)/Q(z)$ is the solution of (4.4), where $P(z)$ and $Q(z)$ are polynomials, then $\deg P \leq \deg Q$. By using the same way as Theorem 4.3, we conclude that there is no rational solution of (4.4), which improves the result of Chen and Shon.

In what follows, we consider the difference Painlevé equation II as follows,

$$\overline{w}+\underline{w}=\frac{(\pi_1 z+k_1)w+\pi_2}{1-w^2}, \tag{4.19}$$

where π_i and $k_j (i=1,2; j=1)$ are the small functions related to w with period of i and j respectively. If $\pi_1=0$, $\pi_2=0$ and $k_1=0$, then the difference Painlevé equation II (4.19) can be reduced into a linear difference equation. So we are interested in the case when not all of the coefficients of (4.19) are zeros.

We proceed to consider the rational solutions of the difference Painlevé equation II as follows. Now let us consider the case $\pi_2=0$ at first.

Theorem 4.4. *Let w be a non-constant meromorphic solution of finite order of the difference Painlevé equation II (4.19), if $\pi_1 \neq 0$ and $\pi_2=0$, then w is transcendental.*

Proof. If $\pi_2=0$, then the difference Painlevé equation we considered as

$$\overline{w}+\underline{w}=\frac{(\pi_1 z+k_1)w}{1-w^2}, \tag{4.20}$$

where π_1 and k_1 are the small functions related to w with period of 1. The equation (4.20) can be read as follows by $w(z)=P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, if w is a rational solution of (4.20)

$$(\overline{P}\underline{Q}+\underline{P}\overline{Q})(Q^2-P^2)=(\pi_1 z+k_1)PQ\overline{Q}\underline{Q}.$$

We conclude that $\deg Q=\deg P$. In fact, if $\deg P>\deg Q$, then the degree of the polynomial of the left-hand side is $3\deg P+\deg Q$, and the degree of the polynomial of the right-hand side is $\deg P+3\deg Q+1$, which implies that $2\deg Q+1=2\deg P$. It is impossible. If $\deg P<\deg Q$, then the degree of the polynomial of the left-hand side is $3\deg Q+\deg P$, and the degree of the polynomial of the right-hand side is also $\deg P+3\deg Q+1$, which is impossible. Therefore, we have $\deg Q=\deg P$.

Now, dividing by w on the both side of (4.20) implies that

$$\frac{\overline{w}}{w}+\frac{\underline{w}}{w}=\frac{(\pi_1 z+k_1)}{1-w^2}.$$

Since w is a rational function, Lemma 2.1 tells us that

$$m\Big(r, \frac{(\pi_1 z + k_1)w}{1-w^2}\Big) = m\Big(\frac{\overline{w}}{w} + \frac{\underline{w}}{w}\Big) = O(1). \tag{4.21}$$

From (4.21), it yields that by $w(z) = P(z)/Q(z)$

$$m\Big(r, \frac{(\pi_1 z + k_1)PQ}{Q^2 - P^2}\Big) = O(1),$$

which implies that $\deg P + \deg Q + 1 \le \max\{2\deg P, 2\deg Q\}$. It is a contradiction with $\deg Q = \deg P$. The proof is complte. $\square$

In the following, we consider another case of the difference Painlevé equation II that $\pi_1 = 0$ and $\pi_2 = 0$. We state our result as follows.

Theorem 4.5. *Let w be a non-constant rational solution of the difference Painlevé equation II (4.19), if $\pi_1 = 0$ and $\pi_2 = 0$, then $w = \frac{1}{z+\alpha}$, where $\alpha \in \mathbb{C}$, and $k_1 = 2$. Moreover, $-w$ is also a rational solution of the difference Painlevé equation II.*

Proof. If $\pi_1 = 0$ and $\pi_2 = 0$, then the difference Painlevé equation II we considered is

$$\overline{w} + \underline{w} = \frac{k_1 w}{1-w^2}, \tag{4.22}$$

where k_1 is a small function related to w with period of 1. If w is a rational solution of (4.22), then k_1 is reduced to a constant.

Let $A = \{z_i \in \mathbb{C}; i \in \mathbb{N}\}$ be the set of all zeros of

$$w(z) - 1 = 0 \quad \text{and} \quad w(z) + 1 = 0,$$

and its cardinality of the set $A \cap \{z \in \mathbb{C} : |z| \le r\}$ is denoted by $n_A(r, \frac{1}{w\pm1})$ according to its multiplicity, its counting functions is denoted by $N_A(r, \frac{1}{w\pm1})$. Suppose that $z_0 \in A$, then from (4.22) z_0 is the pole of $w(z+1)$ or $w(z-1)$. If z_0 is the pole of $w(z+1)$ or $w(z-1)$, then $z_0 + 2 \in A$ or $z_0 - 2 \in A$.

Let B be the subset of A of all points such that

$$w(z_0 - 1) = \infty \quad \text{and} \quad w(z_0 + 1) = \infty$$

hold. We denote by $N_B\Big(r, \frac{1}{w\pm1}\Big)$ the corresponding counting function, and

$$N_{A\setminus B}\Big(r, \frac{1}{w\pm1}\Big) = N_A\Big(r, \frac{1}{w\pm1}\Big) - N_B\Big(r, \frac{1}{w\pm1}\Big)$$

counting function for those points $z_0 \in A \setminus B$. Therefore, for each points in $B \in \{z \in \mathbb{C} : |z| \le r\}$ there is exactly one pole in the disc $\{z \in \mathbb{C} : |z| \le r+1\}$, which implies that

$$N_B(r-1, w) \le N_B\Big(r, \frac{1}{w\pm1}\Big) \le N_B(r+1, w). \tag{4.23}$$

Similarly, two points in $A \setminus B \cap \{z \in \mathbb{C}; |z| \le r\}$ there exists one pole in the disc $\{z \in \mathbb{C} : |z| \le r+1\}$. Therefore, we have

$$N_{A\setminus B}\Big(r, \frac{1}{w\pm1}\Big) \le 2N_{A\setminus B}(r+1, w). \tag{4.24}$$

Since w is the finite order meromorphic solution of (4.22), let

$$H(z, w) = (\overline{w} + \underline{w})(1 - w^2) - k_1 w.$$

It is easy to check that $w = \pm 1$ is not the solutions of $H(z, w) = 0$. It follows from Lemma 2.3 that

$$N_A\Big(r, \frac{1}{w \pm 1}\Big) = N\Big(r, \frac{1}{w \pm 1}\Big) = 2T(r, w) + S(r, w) \tag{4.25}$$

In the following, we proceed to prove that the points in the set A are "almost all" in the set $A \setminus B$. Suppose that

$$N_{A\setminus B}(r + 1, w) = \alpha T(r, w) + S(r, w),$$

where $0 \le \alpha \le 1$. From (4.23) to (4.25), we have

$$\begin{aligned} 2T(r, w) &= N\Big(r, \frac{1}{w \pm 1}\Big) + S(r, w) = N_A\Big(r, \frac{1}{w \pm 1}\Big) + S(r, w) \\ &= N_B\Big(r, \frac{1}{w \pm 1}\Big) + N_{A\setminus B}\Big(r, \frac{1}{w \pm 1}\Big) + S(r, w) \\ &\le N_B(r + 1, w) + 2N_{A\setminus B}(r + 1, w) \\ &= N_A(r + 1, w) + N_{A\setminus B}(r + 1, w) \\ &\le (1 + \alpha)T(r + 1, w) + S(r, w). \end{aligned} \tag{4.26}$$

It yields that $\alpha = 1$ by Lemma 2.2, since w is a meromorphic function of finite order. Therefore, we have

$$N_B(r + 1, w) = S(r, w).$$

It reveals that for "almost all" $z_0 \in A$, then $w(z_0 + 1) = \infty$ and $w(z_0 - 1) = \infty$ can not hold at the same time. If w is a rational function, then there is no point $z_0 \in A$ such that $w(z_0 + 1) = \infty$ and $w(z_0 - 1) = \infty$ hold at the same time. Now, set

$$w(z) = P(z)/Q(z)$$

where $P(z)$ and $Q(z)$ are polynomials, we can rewrite (4.22) as follows

$$\frac{P(z-1)Q(z+1) + P(z+1)Q(z-1)}{Q(z+1)Q(z-1)} = \frac{k_1 P(z)Q(z)}{Q^2(z) - P^2(z)}. \tag{4.27}$$

Then the numerator and the denominator on the left-hand side are relatively prime. In fact, if it is not true, then there is a point such that $Q(z_0+1) = 0$ and $Q(z_0-1) = 0$ hold, which is a contradiction with there is no point z_0 such that $w(z_0 + 1) = \infty$ and $w(z_0 - 1) = \infty$ hold, if w is a rational function. Moreover, the numerator and the denominator on the right-hand side are also relatively prime. Indeed, if this is not true, there is a point such that $P(z_1) = 0$ and $Q(z_1) = 0$ hold, which is impossible. Therefore, we have

$$Q(z + 1)Q(z - 1) = Q^2(z) - P^2(z) \tag{4.28}$$

holds, and at the same time the equality

$$P(z - 1)Q(z + 1) + P(z + 1)Q(z - 1) = k_1 P(z)Q(z) \tag{4.29}$$

holds. Comparing the degree of both side of (4.28), we conclude that $\deg Q > \deg P$. Let $n = \deg Q$ and $m = \deg P$, it is clear that $n > m$ and $n \ge 1$. Suppose that

$$\begin{gathered} P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0, \\ Q(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0, \end{gathered}$$

where $a_i (0 \leq i \leq n)$ and $b_j (0 \leq j \leq m)$ are constants, and $a_n \neq 0$ and $b_m \neq 0$. If we substitute $P(z)$ and $Q(z)$ into (4.28), and compare the coefficients of degree $2n-2$ of both side of (4.28), then we have

$$a_m^2 = nb_n^2, \tag{4.30}$$

which implies that $m = n-1$. Now, let us compare the coefficients of degree n of both side of (4.29), then it shows that $k_1 = 2$.

Suppose that $n > 2$, then we proceed to compare the coefficients of degree $2n-3$ of both side of (4.29), it yields that

$$(n-1)(n-2)a_{n-1}b_n = 0,$$

which is impossible. Thus, it shows that $n \leq 2$. Suppose that $n = 2$, then we can write

$$P(z) = a_1 z + a_0 \quad \text{and} \quad Q(z) = b_2 z^2 + b_1 z + b_0.$$

If we substitute $P(z)$ with degree 1 and $Q(z)$ with degree 2 into (4.28) and (4.29), and compare the coefficients of both side of two equations, then we have

$$a_1^2 = 2b_2^2 \tag{4.31}$$
$$a_0 a_1 = b_1 b_2 \tag{4.32}$$
$$a_1 b_1 = a_0 b_2 \tag{4.33}$$
$$b_2^2 + 2b_0 b_2 = b_1^2 - a_0^2. \tag{4.34}$$

From (4.32) and (4.33), we conclude that $a_1^2 = b_2^2$, which is a contradiction with (4.31). From the discussion above, it is known that $n = 1$. Therefore, it follows that by (4.30)

$$w = \frac{1}{z+\alpha},$$

where $\alpha \in \mathbb{C}$. Moreover, $-w$ is also a solution of difference Painlevé equations II (4.22). Therefore, we have our assertion. $\square$

From Theorem 4.5, we know that if the difference Painlevé equation II (4.19) with $\pi_1 = 0$ and $\pi_2 = 0$ solves a rational solution, then $k_1 = 2$, Thus, we have the following corollary.

Corollary 4.6. *There is no rational solutions of the difference Painlevé equation II*

$$\overline{w} + \underline{w} = \frac{k_1 w}{1 - w^2},$$

if $k_1 \neq 2$, where k_1 is a small function related to w with period 1.

The next example shows that there exists a rational solution of the difference Painlevé equation II (4.19), if $\pi_1 = 0$ and $\pi_2 = 0$.

Example 4.7. Let $w = 1/z$. Then w is a rational solution of the difference Painlevé equation II

$$\overline{w} + \underline{w} = \frac{2w}{1 - w^2}.$$

Also $-w$ is a rational solution of this equation.

In fact, we can obtain (4.28) the same way if $\pi_2 \neq 0$ and $\pi_1 = 0$. Thus, for the difference Painlevé equation II becomes

$$\overline{w} + \underline{w} = \frac{k_1 w + \pi_2}{1 - w^2}, \tag{4.35}$$

where π_2 and k_1 are non-zero small functions related to w, which are of period 2 and 1 respectively. If w is a rational solution of (4.35), then k_1 and π_2 are reduced to the non-zero constants. If we write $w(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, then (4.28) holds. It shows that $\deg P < \deg Q$. Let

$$H_1(z, w) = (\overline{w} + \underline{w})(1 - w^2) - (k_1 w + \pi_2).$$

It yields that $w = 0$ is not a solution of $H_1(z, w) = 0$ according to $\pi_2 \neq 0$. Thus, it follows from Lemma 2.3 that

$$m\big(r, \frac{1}{w}\big) = S(r, w),$$

which implies that $\deg p \geq \deg Q$. It is a contradiction. Therefore, w is not a rational solution of (4.35).

Theorem 4.8. *Let w be a non-constant meromorphic solution of finite order of the difference Painlevé equation II (4.19). If $\pi_1 = 0$ and $\pi_2 \neq 0$, then w is transcendental.*

The next theorem is a partial result on the rational solutions of the difference Painlevé equation II with $\pi_1 \neq 0$ and $\pi_2 \neq 0$.

Theorem 4.9. *Suppose that $w(z) = P(z)/Q(z)$ is a non-constant rational function. Let $P(z)$ and $Q(z)$ be the polynomials*

$$\begin{gathered} P(z) = a_m z^m + \ldots + a_1 z + a_0, \\ Q(z) = b_n z^n + \ldots + b_1 z + b_0, \end{gathered}$$

where $a_i (0 \leq i \leq m)$ and $b_j (0 \leq j \leq n)$ are constants. If w is a rational solution of the difference Painlevé equation II (4.19) such that $\pi_1 \neq 0$ and $\pi_2 \neq 0$, then $n = m + 1$ and $(\pi_2/\pi_1)^2 = n \in \mathbb{N}$.

Proof. Suppose that w is a rational solution of (4.19), then we have

$$\begin{aligned} &Q(z+1)Q(z-1)((\pi_1 z + k_1)P(z)Q(z) + \pi_2 Q^2(z)) \\ &= (Q^2(z) - P^2(z))(P(z+1)Q(z-1) + P(z-1)Q(z+1)). \end{aligned} \tag{4.36}$$

Let us compare the degree of both sides of (4.36), then we have $n = m + 1$ and

$$\pi_1 a_{n-1} + \pi_2 b_n = 0. \tag{4.37}$$

By the similar way as (4.30) in Theorem 4.5, it follows that $a_{n-1}^2 = n b_n^2$. This and (4.37), yield

$$\frac{\pi_2^2}{\pi_1^2} = n.$$

Therefore, we have our assertion. $\square$

From Theorem 4.9, we know that if the difference Painlevé equation II (4.19) with $\pi_1 \neq 0$ and $\pi_2 \neq 0$ solves a rational solution, then π_2^2/π_1^2 is an integer, Thus, we have the following corollary.

Corollary 4.10. *There is no rational solutions of the difference Painlevé equation II*

$$\overline{w}+\underline{w}=\frac{(\pi_1 z+k_1)w+\pi_2}{1-w^2},$$

if π_2^2/π_1^2 is not an integer, where π_i and $k_j (i=1,2; j=1)$ are the small functions related to w with period of i and j, respectively.

The next example shows that there exists a rational solution of the difference Painlevé equation II (4.19), if $\pi_1 \neq 0$ and $\pi_2 \neq 0$.

Example 4.11. Let $w=\frac{1}{z+\beta}$ and $\beta=\frac{k_1-2}{\pi_1}$, then w is a rational solution of the difference Painlevé equation II

$$\overline{w}+\underline{w}=\frac{(\pi_1 z+k_1)w+\pi_2}{1-w^2},$$

if $\pi_2+\pi_1=0$.

Next we proceed to seek new solutions from a known solution of the difference Painlevé equation II (4.19). To seek a new solution, let us reveal the relation between the solutions of the difference Riccati equation and the solutions of the difference Painlevé equation II in the following theorem.

Theorem 4.12. *If w is a solution of the difference Riccati equation*

$$\overline{w}=\frac{w+\left(1+\frac{\pi_2}{2}-\frac{\pi_1 z+k_1}{2}\right)}{1+w},$$

where π_i and k_j $(i=1,2; j=1)$ are the small functions related to w with period of i and j, respectively, then w is the solution of the difference Painlevé equation II (4.19).

Proof. Since w is a solution of difference Riccati equation, then by Lemma 3.1, w satisfies

$$\underline{w}=\frac{w-\left(1-\frac{\pi_2}{2}-\frac{\pi_1 z+k_1}{2}\right)}{1-w}.$$

Therefore,

$$\begin{aligned}\overline{w}+\underline{w}&=\frac{w+(1+\frac{\pi_2}{2}-\frac{\pi_1 z+k_1}{2})}{1+w}+\frac{w-(1-\frac{\pi_2}{2}-\frac{\pi_1 z+k_1}{2})}{1-w}\\&=\frac{(\pi_1 z+k_1)w+\pi_2}{1-w^2}.\end{aligned}$$

□

In what follows, we try to construct the meromorphic solutions of the difference Painlevé equation II . If we find a meromorphic solution w of the difference Riccati equations, then w solves the difference Painlevé equation II according to Theorem 4.12. The main idea is from [15].

Suppose that w is a meromorphic solution of the difference Ricciti equation

$$\underline{w}=\frac{w-\left(1-\frac{\pi_2}{2}-\frac{\pi_1 z+k_1}{2}\right)}{1-w}, \tag{4.38}$$

where π_i and $k_j (i = 1, 2; j = 1)$ are the small functions related to w with period of i and j, respectively. By denoting

$$w = \frac{f(z+1) - f(z)}{f(z)}, \tag{4.39}$$

we have the meromorphic function $f(z)$ satisfies the following equality

$$\frac{f(z+2) - f(z+1)}{f(z+1)} = \frac{\frac{f(z+1)-f(z)}{f(z)} + \pi}{1 + \frac{f(z+1)-f(z)}{f(z)}},$$

where $\pi = 1 - \frac{\pi_2}{2} - \frac{\pi_1 z + k_1}{2}$, which implies that $f(z)$ is a solution of the following second order linear homogeneous difference equation

$$f(z+2) - 2f(z+1) + (1-\pi)f(z) = 0. \tag{4.40}$$

In what follows, let us consider the simplest case that π is a constant. Thus,

$$f(z) = (1+\sqrt{\pi})^z Q_1(z) + (1-\sqrt{\pi})^z Q_2(z)$$

is a solution of (4.40), where Q_1 and Q_2 are functions with period 1. Therefore,

$$w = \frac{\sqrt{\pi}(1+\sqrt{\pi})^z Q_1(z) - \sqrt{\pi}(1-\sqrt{\pi})^z Q_2(z)}{(1+\sqrt{\pi})^z Q_1(z) + (1-\sqrt{\pi})^z Q_2(z)}$$

is a solutions of the difference Riccati equation (4.38) by (4.39). We conclude that w is a solution of the difference Painlevé equation II by Theorem 4.12.

Now we seek a new meromorphic solution by a known solution of the difference Painlevé equations II. For example, $w_1 = 1/z$ is also the solution of the difference Riccati equation

$$\overline{w} = \frac{w}{1+w}. \tag{4.41}$$

It follows from theorem 4.12 that w_1 is a solution of the difference Painlevé equation II

$$\overline{w} + \underline{w} = \frac{2}{1-w^2}. \tag{4.42}$$

If $f_1(z)$ is the solution of (4.39), then let us set

$$f_1(z+1) = \frac{z+1}{z} f_1(z), \tag{4.43}$$

then according to (4.40), we have $f_1(z)$ is a solution of the linear difference equation

$$f(z+2) - 2f(z+1) + f(z) = 0.$$

To seek a new solution from $w_1 = 1/z$, we need the following lemma. We state it as complex analysis version in [15]. It also works for the real line case, see [16, p.48] and [17, pp.115-116].

Lemma 4.13. *Let $R(z)$ be a rational function. We write $R(z)$ in the form*

$$R(z) = \rho \frac{\Pi_{k=1}^n (z-\alpha_k)}{\Pi_{j=1}^m (z-\beta_j)},$$

where $\rho \neq 0$, α_k, $k = 1, \dots, n$ and $j = 1, \dots, m$ are complex numbers. The first order linear homogeneous equations

$$y(z+1) = R(z)y(z)$$

can be solved as

$$y(z) = Q(z)\rho^z \frac{\Pi_{k=1}^{n}\Gamma(z-\alpha_k)}{\Pi_{j=1}^{m}\Gamma(z-\beta_j)}.$$

Since $f_1(z)$ is a solution of the first order difference equation (4.43), by Lemma 4.13 we have

$$f_1(z) = Q_1(z)\frac{\Gamma(z+1)}{\Gamma(z)} = zQ_1(z),$$

where $Q_1(z)$ is a function with period 1. It is known that $f_2(z) = Q_2(z)$ is another solution of (4.43), where $Q_2(z)$ is a function with period 1. By [24], we know that (4.43) has two linearly independent meromorphic solutions $Q_1(z)$ and $Q_2(z)$. Thus, every meromorphic solution of (4.43) can be formed by

$$f(z) = zQ_1(z) + Q_2(z).$$

Therefore, by using the transformation (4.39), we have another new meromorphic solution of (4.41), which can be written as

$$w(z) = \frac{Q_1(z)}{zQ_1(z) + Q_2(z)}. \tag{4.44}$$

Now we have found a new meormorphic solution (4.44) of difference Painlevé equation II (4.42) from a known solution $w_1 = 1/z$. In particular,

$$w = \frac{1}{z+\alpha}$$

solves the difference Painlevé equation II (4.42), where α is any complex constant.

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