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NONEXISTENCE OF POSITIVE GLOBAL SOLUTIONS TO THE DIFFERENTIAL EQUATION $u''(t) - t^{-p-1}u^p = 0$

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Abstract. A blow-up result for positive solutions to the differential equation $u''(t) - t^{-p-1}u^p = 0$ is derived. Our result is different from the one obtained in the [1], and our conditions are less restrictive.

1. Introduction

Blow up of solutions for differential equations in finite time is a well known phenomena. For details on the blow-up and the existence of global solution, we refer the reader to the standard books [2, 3].

In a recent work Li et al. [1], discussed the nonexistence of positive global solutions to a second-order initial value problem

$$\begin{gathered} u'' - t^{-p-1}u^p = 0, \quad t > 1,\ p \in (1,\infty), \\ u(1) = u_0, \quad u'(1) = u_1, \quad u_0, u_1 \in \mathbb{R}. \end{gathered} \tag{1.1}$$

We remark that the existence and uniqueness of classical local solutions for (1.1) follows by standard arguments when the function $t^{-p-1}u^p$ with $p > 1$, $u \geq 0$ and $t \geq 1$ is locally Lipschitz.

As shown in [1], via the substitutions $u(t) = tv(t)$, $v(t) = w(t)$, and $s = \ln t$, problem (1.1) is transformed into

$$w_{ss} + w_s = w^p, \tag{1.2}$$

$$u(0) = w_0 = u_0, \quad u_1 - u_0 = w_1 = u_1 - u_0. \tag{1.3}$$

The objective of this note is to study the blow-up of solutions for problem (1.1) via a test function approach. The proof of our result is simpler, and different from the one presented in [1]. Furthermore, we impose a condition only on u_1, that it is less restrictive than the conditions imposes on both u_0 and u_1 in [1].

2. Blow-up solution

Theorem 2.1. *Assume that* $u_1 \geq 0$. *Then any solution of problem* (1.2)-(1.3) *blows-up in a finite time.*

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Proof. Assume that a solution of problem (1.2)-(1.3) is global. Multiplying (1.2) by a function $\phi(s)$ of class C^2 such that $\phi(0)=1$, $\phi'(0)=0$, $\phi(T)=0$, $\phi'(T)=0$, $T>0$ and integrating by parts, we obtain

$$\int_0^T w^p\phi\,ds+u_1=-\int_0^T w\phi'\,ds+\int_0^T w\phi''\,ds. \tag{2.1}$$

Writing

$$w|\phi'|=w\phi^{1/p}|\phi'|\phi^{-1/p},\quad w|\phi''|=w\phi^{1/p}|\phi''|\phi^{-1/p}$$

in (2.1) and using the Hölder's inequality with ε, we obtain

$$\begin{aligned}\int_0^T w^p\phi\,ds+u_1\le\varepsilon\int_0^T w^p\phi\,ds+C_\varepsilon\int_0^T|\phi'|^{p'}\phi^{-p'/p}\,ds\\ +\varepsilon\int_0^T w^p\phi ds+C_\varepsilon\int_0^T|\phi''|^{p'}\phi^{-p'/p}\,ds.\end{aligned} \tag{2.2}$$

Taking $\varepsilon=1/4$ (for example), we obtain

$$\int_0^T w^p\phi\,ds+u_1\le C\Big(\int_0^T|\phi'|^{p'}\phi^{-p'/p}\,ds+\int_0^T|\phi''|^{p'}\phi^{-p'/p}\,ds\Big). \tag{2.3}$$

At this stage, we choose

$$\phi(s)=\begin{cases}1, & 0\le s\le T/2,\\ \searrow, & T/2\le s\le T,\\ 0, & s\ge T,\end{cases} \tag{2.4}$$

and introduce the change of variable $s=\tau T$ in the integrals on the right hand side of (2.3) to obtain

$$\int_0^T w^p\phi ds+u_1\le C\left(T^{-p'+1}+T^{-2p'+1}\right). \tag{2.5}$$

As $p'>1$, letting $T\to+\infty$ in (2.5), we obtain

$$\int_0^T w^p\phi\,ds+u_1\le 0, \tag{2.6}$$

which is a contradiction as $w>0$ and $u_1\ge 0$. This completes the proof. □

3. Estimation of the blow-up time

The solution cannot exist for $T>T_*$, where

$$T_*=\min\Big(\Big(\frac{2C}{u_1}\Big)^{\frac{1}{p'-1}},\Big(\frac{2C}{u_1}\Big)^{\frac{1}{2p'-1}}\Big). \tag{3.1}$$

Indeed, from (2.5), the solution cannot exist for

$$u_1\le\big(T^{-p'+1}+T^{-2p'+1}\big). \tag{3.2}$$

Then, the estimate (3.1) is obtained by considering the two cases $T\le 1$ and $T\ge 1$.

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