

CONVERGENCE OF EXTERIOR SOLUTIONS TO RADIAL CAUCHY SOLUTIONS FOR $\partial_t^2 U - c^2 \Delta U = 0$

HELGE KRISTIAN JENSSEN, CHARIS TSIKKOU

ABSTRACT. Consider the Cauchy problem for the 3-D linear wave equation $\partial_t^2 U - c^2 \Delta U = 0$ with radial initial data $U(0, x) = \Phi(x) = \varphi(|x|)$, $U_t(0, x) = \Psi(x) = \psi(|x|)$. A standard result states that U belongs to $C([0, T]; H^s(\mathbb{R}^3))$ whenever $(\Phi, \Psi) \in H^s \times H^{s-1}(\mathbb{R}^3)$. In this article we are interested in the question of how U can be realized as a limit of solutions to initial-boundary value problems on the exterior of vanishing balls B_ε about the origin. We note that, as the solutions we compare are defined on different domains, the answer is not an immediate consequence of H^s well-posedness for the wave equation.

We show how explicit solution formulae yield convergence and optimal regularity for the Cauchy solution via exterior solutions, when the latter are extended continuously as constants on B_ε at each time. We establish that for $s = 2$ the solution U can be realized as an H^2 -limit (uniformly in time) of exterior solutions on $\mathbb{R}^3 \setminus B_\varepsilon$ satisfying vanishing Neumann conditions along $|x| = \varepsilon$, as $\varepsilon \downarrow 0$. Similarly for $s = 1$: U is then an H^1 -limit of exterior solutions satisfying vanishing Dirichlet conditions along $|x| = \varepsilon$.

1. INTRODUCTION

Notation. We use the $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_0^+ = [0, \infty)$. For function of time and spatial position, the time variable t is always listed first, and the spatial variable (x or r) is listed last. We indicate by subscript “rad” that the functions under consideration are spherically symmetric, e.g. $H_{\text{rad}}^2(\mathbb{R}^3)$ denotes the set of $H^2(\mathbb{R}^3)$ -functions Φ with the property that $\Phi(x) = \varphi(|x|)$ for some function $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$. For a radial function we use the same symbol whether it is considered as a function of x or of $r = |x|$.

Throughout we fix $T > 0$ and $c > 0$ and set

$$\square_{1+1} := \partial_t^2 - c^2 \partial_r^2, \quad \square_{1+3} := \partial_t^2 - c^2 \Delta,$$

where Δ is the 3-D Laplacian. The open ball of radius r about the origin in \mathbb{R}^3 is denoted B_r . We write ∂_r for the directional derivative in the (outward) radial direction while ∂_i denotes ∂_{x_i} . Finally, for two functions $A(t)$ and $B(t)$ we write

$$A(t) \lesssim B(t)$$

to mean that there is a number C , possibly depending on the time T , c , the fixed cutoff functions β and χ (see (4.2)-(4.3) and (6.2)-(6.3)), as well as the initial data

2010 *Mathematics Subject Classification.* 35L05, 35L15, 35L20.

Key words and phrases. Linear wave equation; Cauchy problem; radial solutions; exterior solutions; Neumann and Dirichlet conditions.

©2016 Texas State University.

Submitted July 22, 2016. Published September 30, 2016.

Φ, Ψ , but independent of the vanishing radii ε , such that

$$A(t) \leq C \cdot B(t) \quad \text{holds for all } t \in [0, T].$$

2. RADIAL CAUCHY SOLUTIONS AS LIMITS OF EXTERIOR SOLUTIONS

Consider the Cauchy problem for the 3-D linear wave equation with radial initial data:

$$\begin{aligned} \square_{1+3}U &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3 \\ U(0, x) &= \Phi(x) \quad \text{on } \mathbb{R}^3 \\ U_t(0, x) &= \Psi(x) \quad \text{on } \mathbb{R}^3, \end{aligned} \tag{2.1}$$

where

$$\Phi \in H_{\text{rad}}^s(\mathbb{R}^3), \quad \Psi \in H_{\text{rad}}^{s-1}(\mathbb{R}^3),$$

with

$$\Phi(x) = \varphi(|x|) \quad \Psi(x) = \psi(|x|). \tag{2.2}$$

Throughout we refer to the unique solution U of (2.1) as the *Cauchy solution*.

In this work we consider how the radial Cauchy solution U can be realized as a limit of solutions to initial-boundary value problems posed on the exterior of vanishing balls B_ε ($\varepsilon \downarrow 0$) about the origin. The precise issue will be formulated below. We shall consider exterior solutions satisfying either a vanishing Neumann condition or a vanishing Dirichlet condition along $|x| = \varepsilon$.

It is well known that the Sobolev spaces H^s provide a natural setting for the Cauchy problem for the wave equation; see [3] and (3.3)-(3.4) below. The choice of space dimension 3 is for convenience: it is particularly easy to generate radial solutions in this case. Next, both the choice of spaces for the initial data for (2.1), as well as the boundary condition imposed on the exterior solutions, will influence the convergence of exterior solutions toward the Cauchy solution. For the wave equation in \mathbb{R}^3 the different convergence behavior of exterior Neumann and exterior Dirichlet solutions is brought out by considering H^2 vs. H^1 initial data; see Remark 2.1 below.

The scheme of generating radial solutions to Cauchy problems as limits of exterior solutions has been applied to a variety of evolutionary PDE problems; see [2] for references and discussion. In our earlier work [2] we used the 3-D wave equation to gauge the effectiveness of this general scheme in a case where “everything is known.” In order that the results be relevant to other (possibly nonlinear) problems, the analysis in [2] deliberately avoided any use of explicit solution formulae. Based on energy arguments and strong convergence alone, it was found that the exterior solutions do converge to the Cauchy solution as the balls vanish. However, the arguments did not yield optimal information about the regularity of the limiting Cauchy solution. Specifically, for $s = 2$ we obtained the Cauchy solution as a limit only in H^1 (via exterior Neumann solutions) or in L^2 (via exterior Dirichlet solutions). This is strictly less regularity than what is known to be the case, see (3.4).

Thus, in general, while limits of exterior solutions to evolutionary PDEs may be used to establish existence for radial Cauchy problems, one should not expect optimal regularity information about the Cauchy solution via this approach.

On the other hand, for the particular case of the 3-D wave equation with radial data, it is natural to ask what type of convergence we can establish if we exploit solution formulae (for the Cauchy solution as well as for the exterior solutions). The

present work addresses this question, and our findings are summarized in Theorem 3.1 below.

We stress that while [2] dealt with the issue of using exterior solutions as a stand-alone method for obtaining existence of radial Cauchy problems, the setting for the present work is different. We are now exploiting what is known about the solution of the Cauchy solutions as well as exterior solutions for the radial 3-D linear wave equation, and the only issue is how the former solutions are approximated by the latter.

Remark 2.1. Before starting the detailed analysis we comment on a slightly subtle point. As recorded in our main result (Theorem 3.1), we establish H^2 -regularity of the limiting Cauchy solution U when the initial data (Φ, Ψ) belong to $H^2 \times H^1$, and H^1 -regularity when the data belong to $H^1 \times L^2$. This is as it should be according to (3.3). Now, in the former case U is obtained as a limit of exterior Neumann solutions, while in the latter case it is obtained as a limit of exterior Dirichlet solutions. This raises a natural question: what regularity is obtained for U in the case of $H^2 \times H^1$ -data, if we insist on approximating by exterior Dirichlet solutions?

To answer this we need to specify how we compare the everywhere defined Cauchy solution U to exterior solutions U^ε , which are defined only on the exterior domains $\mathbb{R}_\varepsilon^3 := \mathbb{R}^3 \setminus B_\varepsilon$. There are at least two ways to do this:

- (a) by calculating $\|U(t) - U^\varepsilon(t)\|_{H^s(\mathbb{R}_\varepsilon^3)}$;
- (b) by first defining a suitable extension \tilde{U}^ε of U^ε to all of \mathbb{R}^3 , and then calculating $\|U(t) - \tilde{U}^\varepsilon(t)\|_{H^s(\mathbb{R}^3)}$.

(When using exterior solutions to establish existence for (2.1) (as in [2]), there is no such choice: one must produce approximations to U that are everywhere defined.) With (b), which is what we do in this paper, the natural choice is to extend $U^\varepsilon(t)$ continuously as a constant on B_ε at each time. I.e., for exterior Dirichlet solutions, we let $\tilde{U}^\varepsilon(t, x)$ vanish identically on B_ε , while for exterior Neumann solutions its value there is that of $U^\varepsilon(t, x)$ along the $|x| = \varepsilon$.

It turns out that regardless of whether we use (a) or (b) to compare the Cauchy solution to the exterior solutions, the answer to the question above is that we obtain only H^1 -convergence when exterior Dirichlet solutions are used. In fact, for (b) this is immediate: the exterior Dirichlet solution \tilde{U}^ε will typically have a nonzero radial derivative at $r = \varepsilon +$ so that its extension \tilde{U}^ε contains a “kink” along $|x| = \varepsilon$. Thus, second derivatives of \tilde{U}^ε will typically contain a δ -function along $|x| = \varepsilon$, and \tilde{U}^ε does not even belong to $H^2(\mathbb{R}^3)$ in this case. For (a) it suffices to consider the situation at time zero. With Φ as above we consider smooth cutoffs Φ^ε (see (6.2) below). A careful calculation, carried out in [2], shows that $\|\Phi - \Phi^\varepsilon\|_{H^2(\mathbb{R}_\varepsilon^3)}$ blows up as $\varepsilon \downarrow 0$.

These remarks highlight the unsurprising but relevant fact that exterior Dirichlet solutions are more singular than exterior Neumann solutions; see [2] for a discussion.

The goal is to show that the Cauchy solution U of (2.1) can be approximated, uniformly on compact time intervals, in H^2 -norm by suitably chosen exterior Neumann solutions and in H^1 -norm by suitably chosen exterior Dirichlet solutions.

As indicated we shall use explicit solution formulae for both the Cauchy problem (2.1) as well as for the exterior Neumann and Dirichlet problems. These formulae for radial solutions are readily available in 3 dimensions and exploits the fact that

radial solutions of $\square_{1+3}U = 0$ admit the representation

$$U(t, x) = \frac{u(t, |x|)}{|x|}$$

where $u(t, r)$ solves $\square_{1+1}u = 0$ on the half-line \mathbb{R}^+ . (Exterior Neumann solutions require a little work to write down explicitly; see (4.5).)

Of course, with the explicit formulae in place, it is a matter of computation to analyze the required norm differences. However, it is a rather involved computation since the formulae involve different expressions in several different regions. Also, the answers do not follow by appealing to well-posedness for the wave equation (see (3.4) below): the Cauchy solution and the exterior solutions are defined on different domains. As noted above we opt to extend the exterior solutions to the interior of the balls B_ε , before comparing them to the Cauchy solution.

Instead of a direct comparison we prefer to estimate the H^2 - and H^1 -differences in question by employing the natural energies for the wave equation. These energies will majorize the L^2 -distances of the first and second derivatives, and will also provide an estimate on the L^2 -distance of the functions themselves.

There are two advantages of this approach: first, it is straightforward to calculate the exact rates of change of the energies in question, and second, these rates depend only on what takes place at or within radius $r = \varepsilon$. The upshot is that it suffices to analyze fewer terms than required by a direct approach. Finally, to estimate the rates of change of the relevant energies we make use of the explicit solution formulae.

3. SETUP AND STATEMENT OF MAIN RESULT

3.1. Cauchy solution. A standard result (see e.g. [1, 3]) shows that the radial Cauchy solution U of (2.1) may be calculated explicitly by using the representation

$$U(t, x) = \frac{u(t, |x|)}{|x|},$$

where $u(t, r)$ solves the half-line problem (Half-line)

$$\begin{aligned} \square_{1+1}u &= 0 \quad \text{on } (0, T) \times \mathbb{R}^+ \\ u(0, r) &= r\varphi(r) \quad \text{for } r \in \mathbb{R}^+ \\ u_t(0, r) &= r\psi(r) \quad \text{for } r \in \mathbb{R}^+ \\ u(t, 0) &\equiv 0 \quad \text{for } t > 0, \end{aligned} \tag{3.1}$$

where φ and ψ are as in (2.2). By using the d'Alembert formula for the half-line problem (see [1]) we obtain that

$$U(t, r) = \begin{cases} \frac{1}{2r} [(ct+r)\varphi(ct+r) - (ct-r)\varphi(ct-r)] + \frac{1}{2cr} \int_{ct-r}^{ct+r} s\psi(s) ds & \text{if } 0 \leq r \leq ct \\ \frac{1}{2r} [(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)] + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi(s) ds. & \text{if } r \geq ct \end{cases} \tag{3.2}$$

In addition to the solution formula (3.2) we shall also exploit the following well-known stability property [3, 4]: with data $\Phi \in H^s(\mathbb{R}^3)$ and $\Psi \in H^{s-1}(\mathbb{R}^3)$ (radial or not), the Cauchy problem (2.1) admits a unique solution U which satisfies

$$U \in C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3)), \tag{3.3}$$

$$\|U(t)\|_{H^s(\mathbb{R}^3)} + \|U_t(t)\|_{H^{s-1}(\mathbb{R}^3)} \leq C_T (\|\Phi\|_{H^s(\mathbb{R}^3)} + \|\Psi\|_{H^{s-1}(\mathbb{R}^3)}), \quad (3.4)$$

for each $T > 0$, where C_T is a number of the form $C_T = \bar{C} \cdot (1 + T)$, and \bar{C} a universal constant.

3.2. Exterior solutions and their extensions. With $\Phi \in H_{\text{rad}}^s(\mathbb{R}^3)$ and $\Psi \in H_{\text{rad}}^{s-1}(\mathbb{R}^3)$, $s = 1$ or 2 , the goal is to show that the solution U of (2.1) can be “realized as a limit of exterior solutions” defined outside of B^ε as $\varepsilon \downarrow 0$. To make this precise we need to specify:

- (1) precisely which exterior solutions we consider: which boundary conditions do they satisfy along ∂B_ε , and how are their initial data related to the given Cauchy data Φ, Ψ ;
- (2) how we compare the everywhere defined Cauchy solution U with exterior solutions U^ε , that are defined only outside of B_ε ; and
- (3) which norm we use for comparing U and U^ε .

Concerning (1) we shall consider exterior solutions that satisfy either vanishing Neumann or vanishing Dirichlet conditions along ∂B^ε . In either case, the initial data for the exterior problem are generated by a two-step procedure: we first approximate the original Cauchy data by $C_{c,\text{rad}}^\infty(\mathbb{R}^3)$ -functions, and then apply an appropriate modification of these smooth approximations near the origin. These modifications use smooth cut-off functions and are made so that the result satisfies vanishing Neumann or Dirichlet conditions along $|x| = \varepsilon$. (See (4.2)-(4.3) and (6.2)-(6.3) for details.) In either case we denote the exterior, radial solutions corresponding to the approximate, smooth data by $U^\varepsilon(t, x) \equiv U^\varepsilon(t, r)$; they are given explicitly in (4.5) and (6.4) below.

As mentioned in Remark 2.1, for (2) we opt to compare the Cauchy solution U to the natural extensions \tilde{U}^ε of the smooth exterior solution U^ε : at each time t , $\tilde{U}^\varepsilon(t, x)$ takes the constant value $U^\varepsilon(t, \varepsilon)$ on B_ε , and coincides with $U^\varepsilon(t, x)$ for $|x| \geq \varepsilon$. Thus, in the case of Dirichlet data, $\tilde{U}^\varepsilon(t, x)$ vanishes identically on B_ε , while for Neumann data its value there is that of $U^\varepsilon(t, x)$ along the boundary $|x| = \varepsilon$.

Finally, concerning (3), Remark 2.1 above also explains the choice of H^2 -norm for comparing the Cauchy solution U to exterior Neumann solutions, and H^1 -norm for comparison to exterior Dirichlet solutions. Our main result is as follows.

Theorem 3.1. *Let $T > 0$ be given and let U denote the solution of the radial Cauchy problem (2.1) for the linear wave equation in three space dimensions with initial data (Φ, Ψ) .*

- (i) *For initial data in $H_{\text{rad}}^2(\mathbb{R}^3) \times H_{\text{rad}}^1(\mathbb{R}^3)$ the Cauchy solution U can be realized as a $C([0, T]; H^2(\mathbb{R}^3))$ -limit of suitable extended exterior Neumann solutions as $\varepsilon \downarrow 0$.*
- (ii) *For initial data in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ the Cauchy solution U can be realized as a $C([0, T]; H^1(\mathbb{R}^3))$ -limit of suitable extended exterior Dirichlet solutions as $\varepsilon \downarrow 0$.*

We point out that, e.g. in part (i), we do not claim that the extended Neumann solutions \tilde{U}^ε converge to U in H^2 -norm. In fact, we establish this latter property only for the case with $C_c^\infty(\mathbb{R}^3)$ initial data. However, thanks to the stability property (3.4), this is sufficient to obtain (i); see Proposition 3.2 below.

The rest of the paper is organized as follows. After reducing to the case with smooth and compactly supported data in Section (3.3), we treat H^2 -convergence of exterior Neumann solutions in Sections (4)-(5), while H^1 -convergence of exterior Dirichlet solutions is established in Sections (6)-(7).

3.3. Reduction to smooth case. The first step of the proof is to use well-posedness (3.4) for the Cauchy problem to reduce to the case of smooth initial data.

Proposition 3.2. *With the setup in Theorem 3.1, let $\tilde{U}^{N,\varepsilon}$ and $\tilde{U}^{D,\varepsilon}$ denote the extensions of the exterior Neumann and Dirichlet solutions, respectively, as described in Section (3.2). Then, Theorem 3.1 follows once it is established that*

$$\sup_{0 \leq t \leq T} \|U(t) - \tilde{U}^{N,\varepsilon}(t)\|_{H^2(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0, \quad (3.5)$$

$$\sup_{0 \leq t \leq T} \|U(t) - \tilde{U}^{D,\varepsilon}(t)\|_{H^1(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0, \quad (3.6)$$

for any initial data $\Phi, \Psi \in C_{c,\text{rad}}^\infty(\mathbb{R}^3)$.

Proof. For concreteness consider the case of exterior Neumann solutions, and let arbitrary data $\Phi \in H_{\text{rad}}^2(\mathbb{R}^3)$, $\Psi \in H_{\text{rad}}^1(\mathbb{R}^3)$ be given. Fix any $\delta > 0$. We first choose Φ_0, Ψ_0 in $C_{c,\text{rad}}^\infty(\mathbb{R}^3)$ with

$$\|\Phi - \Phi_0\|_{H^2} + \|\Psi - \Psi_0\|_{H^1} < \frac{\delta}{2C_T},$$

where C_T is as in (3.4). The existence of such Φ_0, Ψ_0 may be established in a standard manner via convolution (using a radial mollifier) and smooth cutoff at large radii. Let U_0 denote the solution of (2.1) with data Φ_0, Ψ_0 . Also, for any $\varepsilon > 0$ let $\tilde{U}_0^{N,\varepsilon}(t, x)$ denote the extension of the exterior Neumann solution with data $\Phi_0^\varepsilon, \Psi_0^\varepsilon$, as described in Section (3.2). Then, assuming that (3.5) has been established, we can choose $\varepsilon > 0$ sufficiently small to guarantee that

$$\sup_{0 \leq t \leq T} \|U_0(t) - \tilde{U}_0^{N,\varepsilon}(t)\|_{H^2} < \frac{\delta}{2}.$$

Hence, for any $t \in [0, T]$ we have

$$\begin{aligned} \|U(t) - \tilde{U}_0^{N,\varepsilon}(t)\|_{H^2} &\leq \|U(t) - U_0(t)\|_{H^2} + \|U_0(t) - \tilde{U}_0^{N,\varepsilon}(t)\|_{H^2} \\ &\stackrel{(3.4)}{\leq} C_T (\|\Phi - \Phi_0\|_{H^2} + \|\Psi - \Psi_0\|_{H^1}) + \frac{\delta}{2} < \delta, \end{aligned}$$

by the choice of Φ_0, Ψ_0 . □

From now on we therefore consider an arbitrary but fixed pair of functions $\Phi, \Psi \in C_{c,\text{rad}}^\infty(\mathbb{R}^3)$. Note that we then have that the functions φ and ψ in (2.2) are smooth on \mathbb{R}_0^+ and satisfy $\varphi'(0+) = \psi'(0+) = 0$.

4. EXTERIOR NEUMANN SOLUTIONS

In this section and the next we consider the case of exterior Neumann solutions. For the fixed initial data $\Phi, \Psi \in C_{c,\text{rad}}^\infty(\mathbb{R}^3)$ and any $\varepsilon > 0$ we derive a formula for $U^\varepsilon(t, x) \equiv U^{N,\varepsilon}(t, x)$, defined for $|x| \geq \varepsilon$ and satisfying $\partial_r U^\varepsilon|_{r=\varepsilon} = 0$. We refer to U^ε as the *exterior Neumann solution* corresponding to the solution U of (2.1) with

data Φ, Ψ . In Section (5) we will then estimate how it (really, its extension $\tilde{U}^\varepsilon(t, x)$ to all of \mathbb{R}^3) approximates the solution $U(t)$ in $H^2(\mathbb{R}^3)$ at fixed times.

To generate the exterior Neumann solution U^ε we fix a smooth, nondecreasing function $\beta : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with

$$\beta \equiv 1 \quad \text{on } [0, 1], \quad \beta(s) = s \quad \text{for } s \geq 2. \quad (4.1)$$

Then, with φ and ψ as in (2.2), we define

$$\Phi^\varepsilon(x) \equiv \varphi^\varepsilon(|x|) := \varphi\left(\varepsilon\beta\left(\frac{|x|}{\varepsilon}\right)\right), \quad (4.2)$$

$$\Psi^\varepsilon(x) \equiv \psi^\varepsilon(|x|) := \psi\left(\varepsilon\beta\left(\frac{|x|}{\varepsilon}\right)\right). \quad (4.3)$$

We refer to $(\Phi^\varepsilon, \Psi^\varepsilon)$ as the *Neumann data* corresponding to the original Cauchy data (Φ, Ψ) for (2.1). Note that the Neumann data are actually defined on all of \mathbb{R}^3 , that they are constant (equal to $\varphi(\varepsilon)$ and $\psi(\varepsilon)$, respectively) on B_ε , and that their restrictions to the exterior domain $\{x \in \mathbb{R}^3 : |x| \geq \varepsilon\}$ satisfy homogeneous Neumann conditions along $|x| = \varepsilon$.

The exterior Neumann solution U^ε is now defined as the unique radial solution of the initial-boundary value problem

$$\begin{aligned} \square_{1+3}V &= 0 \quad \text{on } (0, T) \times \{|x| > \varepsilon\} \\ V(0, x) &= \Phi^\varepsilon(x) \quad \text{for } |x| > \varepsilon \\ V_t(0, x) &= \Psi^\varepsilon(x) \quad \text{for } |x| > \varepsilon \\ \partial_r V(t, x) &= 0 \quad \text{along } |x| = \varepsilon \text{ for } t > 0. \end{aligned}$$

To obtain a formula for U^ε we exploit the fact that V is a radial solution of the 3-D wave equation if and only if $v = rV$ solves the 1-D wave equation. Setting

$$u^\varepsilon(t, r) := rU^\varepsilon(t, r),$$

we obtain that u^ε solves the corresponding 1-D problem on $\{r > \varepsilon\}$: (ε -Half-line)

$$\begin{aligned} \square_{1+1}u &= 0 \quad \text{on } (0, T) \times \{r > \varepsilon\} \\ u(0, r) &= r\varphi^\varepsilon(r) \quad \text{for } r > \varepsilon \\ u_t(0, r) &= r\psi^\varepsilon(r) \quad \text{for } r > \varepsilon \\ u_r(t, \varepsilon) &= \frac{1}{\varepsilon}u(t, \varepsilon) \quad \text{for } t > 0. \end{aligned}$$

Note that the Neumann condition for the 3-D solution corresponds to a Robin condition for the 1-D solution. (A direct calculation shows that the initial data for u^ε and u_t^ε both satisfy this Robin condition.)

The solution u^ε to the ε -Half-line problem is explicitly given via d'Alembert's formula:

$$u^\varepsilon(t, r) = \begin{cases} \frac{1}{2}[(ct+r)\varphi^\varepsilon(ct+r) + (ct-r+2\varepsilon)\varphi^\varepsilon(ct-r+2\varepsilon)] \\ \quad + \frac{1}{2c} \int_{ct-r+2\varepsilon}^{ct+r} s\psi^\varepsilon(s) ds \\ \quad + e^{\frac{r-ct-2\varepsilon}{\varepsilon}} \int_{\varepsilon}^{ct-r+2\varepsilon} \left[\frac{s\psi^\varepsilon(s)}{c} - \frac{s\varphi^\varepsilon(s)}{\varepsilon} \right] e^{s/\varepsilon} ds \\ \quad \text{if } \varepsilon \leq r \leq ct + \varepsilon, \\ \frac{1}{2}[(r+ct)\varphi^\varepsilon(r+ct) + (r-ct)\varphi^\varepsilon(r-ct)] \\ \quad + \frac{1}{2c} \int_{r-ct}^{r+ct} s\psi^\varepsilon(s) ds \\ \quad \text{if } r \geq ct + \varepsilon. \end{cases} \quad (4.4)$$

(One way to solve the 1-Dimensional Robin IBVP is to first solve the IBVP with general Dirichlet data $u^\varepsilon(t, \varepsilon) = h(t)$ along $r = \varepsilon$, for which a d'Alembert formula is readily available (see John [1], p. 8); one may then identify the h which gives $u_r = \frac{1}{\varepsilon}u$ along $r = \varepsilon$.) A direct calculation shows that u^ε is a classical solution on $\mathbb{R}_t \times \{r > \varepsilon\}$. From this we obtain the radial exterior Neumann solution $U^\varepsilon(t, r) := \frac{u^\varepsilon(t, r)}{r}$.

$$U^\varepsilon(t, r) = \begin{cases} \frac{1}{2r}[(ct+r)\varphi^\varepsilon(ct+r) + (ct-r+2\varepsilon)\varphi^\varepsilon(ct-r+2\varepsilon)] \\ + \frac{1}{2cr} \int_{ct-r+2\varepsilon}^{ct+r} s\psi^\varepsilon(s) ds \\ + \frac{1}{r} e^{\frac{r-ct-2\varepsilon}{\varepsilon}} \int_{\varepsilon}^{ct-r+2\varepsilon} \left[\frac{s\psi^\varepsilon(s)}{c} - \frac{s\varphi^\varepsilon(s)}{\varepsilon} \right] e^{s/\varepsilon} ds \\ \text{if } \varepsilon \leq r \leq ct + \varepsilon \\ \frac{1}{2r}[(r+ct)\varphi^\varepsilon(r+ct) + (r-ct)\varphi^\varepsilon(r-ct)] \\ + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi^\varepsilon(s) ds \\ \text{if } r \geq ct + \varepsilon. \end{cases} \quad (4.5)$$

We finally extend U^ε at each time to obtain an everywhere defined approximation of the Cauchy solution $U(t, x)$. As discussed earlier we use the natural choice of extending U^ε continuously as a constant on B_ε at each time:

$$\tilde{U}^\varepsilon(t, x) = \begin{cases} U^\varepsilon(t, \varepsilon) & \text{for } 0 \leq |x| \leq \varepsilon \\ U^\varepsilon(t, x) & \text{for } |x| \geq \varepsilon. \end{cases} \quad (4.6)$$

For later use we record that the value along the boundary is explicitly given as

$$U^\varepsilon(t, \varepsilon) = \frac{1}{\varepsilon}(ct + \varepsilon)\varphi^\varepsilon(ct + \varepsilon) + \frac{1}{\varepsilon} e^{-\frac{ct+\varepsilon}{\varepsilon}} \int_{\varepsilon}^{ct+\varepsilon} \left[\frac{s\psi^\varepsilon(s)}{c} - \frac{s\varphi^\varepsilon(s)}{\varepsilon} \right] e^{s/\varepsilon} ds, \quad (4.7)$$

and we also note that

$$\tilde{U}^\varepsilon(0, x) = \Phi^\varepsilon(x), \quad \tilde{U}_t^\varepsilon(0, x) = \Psi^\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^3. \quad (4.8)$$

5. COMPARING CAUCHY AND EXTERIOR NEUMANN SOLUTIONS

The issue now is to estimate the H^2 -distance

$$\|U(t) - \tilde{U}^\varepsilon(t)\|_{H^2(\mathbb{R}^3)}$$

as $\varepsilon \downarrow 0$. As explained in Section (2) we prefer to estimate this H^2 -difference by employing the natural energies for the wave equation. These energies will majorize the L^2 -distances of the first and second derivatives of $U(t)$ and $\tilde{U}^\varepsilon(t)$, and also provide control of the L^2 -distance of the functions themselves.

5.1. Energies. For any function $W(t, x)$ which is twice weakly differentiable on $\mathbb{R} \times \mathbb{R}^3$ we define the following 1st and 2nd order energies (note their domains of integration):

$$\begin{aligned} \mathcal{E}_W(t) &:= \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t W(t, x)|^2 + c^2 |\nabla W(t, x)|^2 dx, \\ \mathcal{E}_W^\varepsilon(t) &:= \frac{1}{2} \int_{|x| \geq \varepsilon} |\partial_t W(t, x)|^2 + c^2 |\nabla W(t, x)|^2 dx, \end{aligned}$$

and

$$\mathbb{E}_W(t) := \sum_{i=1}^3 \mathcal{E}_{\partial_i W}(t) = \sum_{i=1}^3 \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \partial_i W(t, x)|^2 + c^2 |\nabla \partial_i W(t, x)|^2 dx,$$

$$\mathbb{E}_W^\varepsilon(t) := \sum_{i=1}^3 \mathcal{E}_{\partial_i W}^\varepsilon(t) = \sum_{i=1}^3 \frac{1}{2} \int_{|x|>\varepsilon} |\partial_t \partial_i W(t, x)|^2 + c^2 |\nabla \partial_i W(t, x)|^2 dx.$$

The first goal is to estimate the energies

$$\begin{aligned} \mathcal{E}^\varepsilon(t) &:= \mathcal{E}_{U-\tilde{U}^\varepsilon}(t), \\ \mathbb{E}^\varepsilon(t) &:= \mathbb{E}_{U-\tilde{U}^\varepsilon}(t), \end{aligned}$$

which majorizes the L^2 -distances between the 1st and 2nd derivatives of U and \tilde{U}^ε , respectively. As a first step we observe the following facts.

Lemma 5.1. *With U and U^ε as defined above we have: each of the energies*

$$\mathcal{E}_U(t), \quad \mathcal{E}_{U^\varepsilon}^\varepsilon(t), \quad \mathcal{E}_{\partial_i U}(t), \quad \text{and} \quad \mathcal{E}_{\partial_i U^\varepsilon}^\varepsilon(t)$$

are constant in time.

Proof. The constancy of the first three energies is standard, while the constancy of $\mathcal{E}_{\partial_i U^\varepsilon}^\varepsilon(t)$ is a consequence of the fact that we consider radial solutions. Indeed, as U^ε is radial and satisfies vanishing Neumann conditions along $|x| = \varepsilon$, we have that $\nabla U^\varepsilon(t, x) \equiv 0$ along $|x| = \varepsilon$. Thus, $U_{x_i t}^\varepsilon \equiv 0$ for each $i = 1, 2, 3$ along $|x| = \varepsilon$. Differentiating in time, using that U^ε is a solution of the wave equation, and integrating by parts, we therefore have

$$\begin{aligned} \dot{\mathcal{E}}_{\partial_i U^\varepsilon}^\varepsilon(t) &= \int_{|x|>\varepsilon} U_{x_i t}^\varepsilon U_{x_i t t}^\varepsilon + c^2 \nabla U_{x_i}^\varepsilon \cdot \nabla U_{x_i t}^\varepsilon dx \\ &= c^2 \int_{|x|>\varepsilon} U_{x_i t}^\varepsilon \Delta U_{x_i}^\varepsilon + \nabla U_{x_i}^\varepsilon \cdot \nabla U_{x_i t}^\varepsilon dx \\ &= c^2 \int_{\partial\{|x|>\varepsilon\}} U_{x_i t}^\varepsilon \frac{\partial U_{x_i}^\varepsilon}{\partial \nu} dS = 0. \end{aligned}$$

□

Next, to estimate $\mathcal{E}^\varepsilon(t)$, we expand the integrand and use that $\nabla \tilde{U}^\varepsilon$ vanishes on B_ε (by our choice of extension), to get

$$\begin{aligned} \mathcal{E}^\varepsilon(t) &= \mathcal{E}_{U-\tilde{U}^\varepsilon}(t) = \frac{1}{2} \int_{\mathbb{R}^3} |U_t - \tilde{U}_t^\varepsilon|^2 + c^2 |\nabla U - \nabla \tilde{U}^\varepsilon|^2 dx \\ &= \mathcal{E}_U(t) + \mathcal{E}_{\tilde{U}^\varepsilon}(t) - \int_{\mathbb{R}^3} U_t \tilde{U}_t^\varepsilon + c^2 \nabla U \cdot \nabla \tilde{U}^\varepsilon dx \\ &= \mathcal{E}_U(t) + \mathcal{E}_{U^\varepsilon}^\varepsilon(t) + \frac{\text{vol}(B_\varepsilon)}{2} |U_t^\varepsilon(t, \varepsilon)|^2 - U_t^\varepsilon(t, \varepsilon) \int_{|x|<\varepsilon} U_t(t, x) dx \\ &\quad - \int_{|x|>\varepsilon} U_t U_t^\varepsilon + c^2 \nabla U \cdot \nabla U^\varepsilon dx. \end{aligned}$$

Differentiating in time, applying Lemma 5.1, integrating by parts, and using the boundary condition $\partial_r U^\varepsilon(t, \varepsilon) \equiv 0$, then yield

$$\dot{\mathcal{E}}^\varepsilon(t) = \frac{d}{dt} \left[\frac{\text{vol}(B_\varepsilon)}{2} |U_t^\varepsilon(t, \varepsilon)|^2 - U_t^\varepsilon(t, \varepsilon) \int_{|x|<\varepsilon} U_t(t, x) dx \right] + c^2 \int_{|x|=\varepsilon} U_t^\varepsilon \partial_r U dS.$$

Integrating back up in time, and recalling that U^ε and U are radial, we obtain

$$\begin{aligned} \mathcal{E}^\varepsilon(T) &= \mathcal{E}^\varepsilon(0) + \left[\frac{\text{vol}(B_\varepsilon)}{2} |U_t^\varepsilon(t, \varepsilon)|^2 - U_t^\varepsilon(t, \varepsilon) \int_{|x| < \varepsilon} U_t(t, x) dx \right]_{t=0}^{t=T} \\ &\quad + c^2 \text{area}(B_\varepsilon) \int_0^T U_t^\varepsilon(t, \varepsilon) \partial_r U(t, \varepsilon) dt. \end{aligned} \quad (5.1)$$

Below we shall carefully estimate the terms on the right-hand side to show that $\mathcal{E}^\varepsilon(T) \rightarrow 0$ as $\varepsilon \downarrow 0$.

Before carrying out a similar representation of the 2nd order energy difference $\mathbb{E}^\varepsilon(t)$, we observe how $\mathcal{E}^\varepsilon(t)$ controls the L^2 -distance between U and U^ε . Setting

$$\mathcal{D}^\varepsilon(t) := \frac{1}{2} \int_{\mathbb{R}^3} |U(t, x) - \tilde{U}^\varepsilon(t, x)|^2 dx, \quad (5.2)$$

the Cauchy-Schwarz inequality gives

$$\dot{\mathcal{D}}^\varepsilon(t) \leq 2\mathcal{D}^\varepsilon(t)^{1/2} \mathcal{E}^\varepsilon(t)^{1/2},$$

such that

$$\mathcal{D}^\varepsilon(T) \lesssim \mathcal{D}^\varepsilon(0) + \int_0^T \mathcal{E}^\varepsilon(t) dt. \quad (5.3)$$

We now consider how $\mathbb{E}^\varepsilon(t)$ changes in time. Arguing as above, using Lemma 5.1 and the fact that $U_{x_i}^\varepsilon \equiv 0$ on B_ε , we have

$$\begin{aligned} \mathcal{E}_{\partial_i U - \partial_i \tilde{U}^\varepsilon}(t) &= \frac{1}{2} \int_{\mathbb{R}^3} |U_{x_i t} - \tilde{U}_{x_i t}^\varepsilon|^2 + c^2 |\nabla U_{x_i t} - \nabla \tilde{U}_{x_i t}^\varepsilon|^2 dx \\ &= \mathcal{E}_{\partial_i U}(t) + \mathcal{E}_{\partial_i \tilde{U}^\varepsilon}(t) - \int_{\mathbb{R}^3} U_{x_i t} \tilde{U}_{x_i t}^\varepsilon + c^2 \nabla U_{x_i} \cdot \nabla \tilde{U}_{x_i}^\varepsilon dx \\ &= \mathcal{E}_{\partial_i U}(0) + \mathcal{E}_{\partial_i U^\varepsilon}^\varepsilon(0) - \int_{|x| > \varepsilon} U_{x_i t} U_{x_i t}^\varepsilon + c^2 \nabla U_{x_i} \cdot \nabla U_{x_i}^\varepsilon dx. \end{aligned} \quad (5.4)$$

Differentiating in time and integrating by parts in the last integral, give

$$\dot{\mathcal{E}}_{\partial_i U - \partial_i \tilde{U}^\varepsilon}(t) = c^2 \int_{|x| = \varepsilon} (U_{x_i t}) (\partial_r U_{x_i}^\varepsilon) dS. \quad (5.5)$$

Observing that we have

$$\sum_{i=1}^3 U_{x_i t} \partial_r U_{x_i}^\varepsilon = (\partial_r U_t) (\partial_{rr} U^\varepsilon)$$

along $\{|x| = \varepsilon\}$ (recall that $\partial_r U^\varepsilon$ vanishes along $\{|x| = \varepsilon\}$), we obtain from (5.5) that

$$\mathbb{E}^\varepsilon(T) = \mathbb{E}^\varepsilon(0) + c^2 \text{area}(B_\varepsilon) \int_0^T (\partial_r U_t(t, \varepsilon)) (\partial_{rr} U^\varepsilon(t, \varepsilon)) dt. \quad (5.6)$$

To estimate $\mathcal{E}^\varepsilon(T)$ and $\mathbb{E}^\varepsilon(T)$, and hence also $\mathcal{D}^\varepsilon(T)$ according to (5.3), we employ the solution formulae (3.2) and (4.5).

5.2. Initial differences in energy. The details of estimating the initial differences of the first and second order energies, i.e. $\mathcal{E}^\varepsilon(0)$ and $\mathbb{E}^\varepsilon(0)$, were carried out in [2, Section 3.2] (and makes use of (4.8)). Translating to our present notation we have that

$$\mathcal{D}^\varepsilon(0) \lesssim \varepsilon^2 \|\Phi\|_{H^1(B_{2\varepsilon})}^2, \tag{5.7}$$

$$\mathcal{E}^\varepsilon(0) \lesssim \varepsilon^2 \|\Psi\|_{H^1(B_{2\varepsilon})}^2 + \|\Phi\|_{H^1(B_{2\varepsilon})}^2, \tag{5.8}$$

$$\mathbb{E}^\varepsilon(0) \lesssim \|\Psi\|_{H^1(B_{2\varepsilon})}^2 + \|\Phi\|_{H^2(B_{2\varepsilon})}^2. \tag{5.9}$$

5.3. Estimating growth of first order energy difference. According to (5.1), to estimate $\mathcal{E}^\varepsilon(T)$ we need to estimate the quantities U_t^ε and $\partial_r U$ along $|x| = \varepsilon$. For the remaining term involving $U_t(t, x)$ in (5.1) (for $|x| \leq \varepsilon$), it will suffice to employ an energy estimate that does not require formulae.

Before considering these terms in detail we record the following fact. For any $k \in \mathbb{R}$ and for any $t > 0$ let

$$Q_k^\varepsilon(t) := \frac{1}{\varepsilon^k} \left(e^{-\frac{ct+\varepsilon}{\varepsilon}} \int_\varepsilon^{ct+\varepsilon} \left[\frac{s\psi^\varepsilon(s)}{c} - \frac{s\varphi^\varepsilon(s)}{\varepsilon} \right] e^{s/\varepsilon} ds + (ct + \varepsilon)\varphi^\varepsilon(ct + \varepsilon) \right);$$

then

$$Q_k^\varepsilon(t) \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \tag{5.10}$$

To see this, integrate by parts in the φ^ε -term to get that

$$\begin{aligned} Q_k^\varepsilon(t) &= \frac{1}{c} \int_\varepsilon^{ct+\varepsilon} s\psi^\varepsilon(s)\varepsilon^{-k} e^{\frac{s-ct-\varepsilon}{\varepsilon}} ds + \varphi^\varepsilon(\varepsilon)\varepsilon^{1-k} e^{-\frac{ct}{\varepsilon}} \\ &\quad + \int_\varepsilon^{ct+\varepsilon} [\varphi^\varepsilon(s) + s\varphi^{\varepsilon'}(s)]\varepsilon^{-k} e^{\frac{s-ct-\varepsilon}{\varepsilon}} ds. \end{aligned}$$

Recalling (4.2)-(4.3) and using that φ and ψ are fixed, smooth functions, the Dominated Convergence Theorem yields $Q_k^\varepsilon(t) \rightarrow 0$ as $\varepsilon \downarrow 0$.

5.3.1. Estimating $U_t^\varepsilon(t, \varepsilon)$. According to (4.7) we have

$$U_t^\varepsilon(t, \varepsilon) = \frac{c}{\varepsilon} \left(\varphi^\varepsilon(ct + \varepsilon) + (ct + \varepsilon)\varphi^{\varepsilon'}(ct + \varepsilon) + \frac{(ct + \varepsilon)}{c}\psi^\varepsilon(ct + \varepsilon) \right) - cQ_2^\varepsilon(t).$$

As $Q_2^\varepsilon(t)$ tends to zero while φ^ε and ψ^ε remain bounded, we conclude that

$$|U_t^\varepsilon(t, \varepsilon)| \lesssim \frac{1}{\varepsilon} \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \tag{5.11}$$

5.3.2. Estimating $\partial_r U(t, \varepsilon)$. According to (3.2) we have

$$\partial_r U(t, \varepsilon) = \begin{cases} -\frac{1}{2\varepsilon^2} [(ct + \varepsilon)\varphi(ct + \varepsilon) - (ct - \varepsilon)\varphi(ct - \varepsilon)] \\ + \frac{1}{2\varepsilon} [\varphi(ct + \varepsilon) + (ct + \varepsilon)\varphi'(ct + \varepsilon) + \varphi(ct - \varepsilon) + (ct - \varepsilon)\varphi'(ct - \varepsilon)] \\ - \frac{1}{2c\varepsilon^2} \int_{ct-\varepsilon}^{ct+\varepsilon} s\psi(s) ds + \frac{1}{2c\varepsilon} [(ct + \varepsilon)\psi(ct + \varepsilon) + (ct - \varepsilon)\psi(ct - \varepsilon)] \\ \text{if } t \geq \frac{\varepsilon}{c} \\ -\frac{1}{2\varepsilon^2} [(\varepsilon + ct)\varphi(\varepsilon + ct) + (\varepsilon - ct)\varphi(\varepsilon - ct)] \\ + \frac{1}{2\varepsilon} [\varphi(\varepsilon + ct) + (\varepsilon + ct)\varphi'(\varepsilon + ct) + \varphi(\varepsilon - ct) + (\varepsilon - ct)\varphi'(\varepsilon - ct)] \\ - \frac{1}{2c\varepsilon^2} \int_{\varepsilon-ct}^{\varepsilon+ct} s\psi(s) ds + \frac{1}{2c\varepsilon} [(\varepsilon + ct)\psi(\varepsilon + ct) - (\varepsilon - ct)\psi(\varepsilon - ct)] \\ \text{if } t \leq \frac{\varepsilon}{c} \end{cases} \tag{5.12}$$

The terms for $t \geq \frac{\varepsilon}{c}$ are estimated by 2nd order Taylor expansion of $\varphi(ct \pm \varepsilon)$ and $\psi(ct \pm \varepsilon)$ about $\varepsilon = 0$. The terms for $t \leq \frac{\varepsilon}{c}$ are estimated by 2nd order Taylor expansion of φ and ψ about zero, and then using that $\varphi'(0) = \psi'(0) = 0$. (As observed earlier, this holds since φ and ψ are profile functions of the *smooth*, radial functions Φ and Ψ , respectively). These expansions are straightforward and we omit them. The end result is that the leading order terms in (5.12) cancel, leaving terms of size at most $O(\varepsilon)$. We thus have

$$|\partial_r U(t, \varepsilon)| \lesssim \varepsilon \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \quad (5.13)$$

(Note: this is actually obvious since we know that U is a smooth, radial solution satisfying $\partial_r U(t, 0) \equiv 0$ and with fixed data independent of ε .)

Finally, the Cauchy-Schwarz inequality and Lemma 5.1 give

$$\left| \int_{|x| < \varepsilon} U_t(t, x) dx \right| \lesssim \varepsilon^{3/2} \mathcal{E}_U(0)^{1/2}.$$

Applying this together with (5.11) and (5.13) in (5.1), we conclude that

$$\mathcal{E}^\varepsilon(T) \lesssim \mathcal{E}^\varepsilon(0) + \varepsilon^{1/2}. \quad (5.14)$$

5.4. Estimating growth of second order energy differences. Next, according to (5.6), to estimate $\mathbb{E}^\varepsilon(T)$, we need to estimate the quantities $\partial_r U_t$ and $\partial_{rr} U^\varepsilon$ along $|x| = \varepsilon$.

5.4.1. Estimating $\partial_r U_t(t, \varepsilon)$. By taking the time derivative of (5.12) and then Taylor expanding the various terms as outlined above, we deduce that

$$|\partial_r U_t(t, \varepsilon)| \lesssim \varepsilon \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \quad (5.15)$$

5.4.2. Estimating $\partial_{rr} U^\varepsilon(t, \varepsilon)$. This estimate again requires a direct, but rather long, calculation (which we omit), followed by a careful analysis of the resulting expression.

The first step is to calculate $\partial_{rr} U^\varepsilon(t, r)$ for $\varepsilon \leq r \leq ct + \varepsilon$, by using the first part of formula (4.5). A number of cancelations occur when the resulting expression is evaluated at $r = \varepsilon$, and we are left with

$$\begin{aligned} \partial_{rr} U^\varepsilon(t, \varepsilon) &= Q_3^\varepsilon(t) - \frac{1}{\varepsilon^2} [\varphi^\varepsilon + (ct - \varepsilon)\varphi^{\varepsilon'} - \varepsilon(ct + \varepsilon)\varphi^{\varepsilon''}] \\ &\quad - \frac{1}{c\varepsilon^2} [ct\psi^\varepsilon - \varepsilon(ct + \varepsilon)\psi^{\varepsilon'}], \end{aligned}$$

where φ^ε , ψ^ε , and their derivatives are evaluated at $ct + \varepsilon$. According to (4.2)-(4.3) we have that φ^ε , ψ^ε , and their first derivatives remain bounded independently of ε , while $\varphi^{\varepsilon''}$ is at most of order $\frac{1}{\varepsilon}$. Since $Q_3^\varepsilon(t) \rightarrow 0$ as $\varepsilon \downarrow 0$ by (5.10), we therefore have that

$$|\partial_{rr} U^\varepsilon(t, \varepsilon)| \lesssim \frac{1}{\varepsilon^2} \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \quad (5.16)$$

Finally, by using (5.15) and (5.16) in (5.6), we conclude that

$$\mathbb{E}^\varepsilon(T) \lesssim \mathbb{E}^\varepsilon(0) + \varepsilon. \quad (5.17)$$

5.5. Convergence of exterior Neumann solutions. According to the definitions of $\mathcal{D}^\varepsilon(t)$, $\mathcal{E}^\varepsilon(t)$, and $\mathbb{E}^\varepsilon(t)$, together with the estimates (5.3), (5.14), (5.17) we have

$$\begin{aligned} \|U(t) - \tilde{U}^\varepsilon(t)\|_{H^2(\mathbb{R}^3)}^2 &\lesssim \mathcal{D}^\varepsilon(t) + \mathcal{E}^\varepsilon(t) + \mathbb{E}^\varepsilon(t) \\ &\lesssim \mathcal{D}^\varepsilon(0) + \mathcal{E}^\varepsilon(0) + \mathbb{E}^\varepsilon(0) + \varepsilon^{1/2}, \end{aligned}$$

at any time $t \in [0, T]$. Applying the bounds (5.7), and (5.8), (5.9), we conclude that the (extended) Neumann solutions $\tilde{U}^\varepsilon(t)$ converge to the Cauchy solution $U(t)$ in $H^2(\mathbb{R}^3)$, uniformly on bounded time intervals, as $\varepsilon \downarrow 0$. Thanks to Proposition 3.2, this concludes the proof of part (i) of Theorem 3.1.

6. EXTERIOR DIRICHLET SOLUTIONS

In this and the next section U^ε refers to the exterior Dirichlet solutions; similarly for their extensions $\tilde{U}^\varepsilon(t, x)$.

For fixed initial data $\Phi, \Psi \in C_{c,\text{rad}}^\infty(\mathbb{R}^3)$ and any $\varepsilon > 0$ we shall derive a formula for the exterior, radial Dirichlet solution $U^\varepsilon(t, x)$, defined for $|x| \geq \varepsilon$ and satisfying $U^\varepsilon|_{r=\varepsilon} = 0$. We refer to U^ε as the *exterior Dirichlet solution* corresponding to the solution U of (2.1) with data Φ, Ψ . In Section (7) we will then estimate how it (really, its extension $\tilde{U}^\varepsilon(t, x)$ to all of \mathbb{R}^3) approximates the solution $U(t)$ in $H^1(\mathbb{R}^3)$ at fixed times.

To generate the exterior Dirichlet solution $U^\varepsilon(t, x)$ and its extension we use the following scheme. To smoothly approximate the original data (Φ, Ψ) with exterior Dirichlet data we fix a smooth, nondecreasing cutoff function $\chi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with

$$\chi \equiv 0 \quad \text{on } [0, 1], \quad \chi \equiv 1 \quad \text{on } [2, \infty). \quad (6.1)$$

Then, with φ and ψ as in (2.2) we define

$$\Phi^\varepsilon(x) \equiv \varphi^\varepsilon(|x|) := \chi\left(\frac{|x|}{\varepsilon}\right)\varphi(|x|), \quad (6.2)$$

$$\Psi^\varepsilon(x) \equiv \psi^\varepsilon(|x|) := \chi\left(\frac{|x|}{\varepsilon}\right)\psi(|x|). \quad (6.3)$$

We refer to $(\Phi^\varepsilon, \Psi^\varepsilon)$ as the *Dirichlet data* corresponding to the original Cauchy data (Φ, Ψ) for (2.1). Note that the Dirichlet data are actually defined on all of \mathbb{R}^3 , that they vanish identically on B_ε , and hence their restrictions to the exterior domain $\{x \in \mathbb{R}^3 : |x| \geq \varepsilon\}$ satisfy homogeneous Dirichlet conditions along $|x| = \varepsilon$.

The exterior Dirichlet solution $U^\varepsilon(t, x)$ is then the unique radial solution of the initial-boundary value problem

$$\begin{aligned} \square_{1+3}V &= 0 \quad \text{on } (0, T) \times \{|x| > \varepsilon\} \\ V(0, x) &= \Phi^\varepsilon(x) \quad \text{for } |x| > \varepsilon \\ V_t(0, x) &= \Psi^\varepsilon(x) \quad \text{for } |x| > \varepsilon \\ V(t, x) &= 0 \quad \text{along } |x| = \varepsilon \text{ for } t > 0. \end{aligned}$$

We next record the solution formula for the exterior, radial Dirichlet solution $U^\varepsilon(t, r)$ (which is simpler to derive than the formula for the exterior Neumann

solution):

$$U^\varepsilon(t, r) = \begin{cases} \frac{1}{2r}[(ct+r)\varphi^\varepsilon(ct+r) - (ct-r+2\varepsilon)\varphi^\varepsilon(ct-r+2\varepsilon)] \\ + \frac{1}{2cr} \int_{ct-r+2\varepsilon}^{ct+r} s\psi^\varepsilon(s) ds \\ \text{if } \varepsilon \leq r \leq ct + \varepsilon \\ \frac{1}{2r}[(r+ct)\varphi^\varepsilon(r+ct) + (r-ct)\varphi^\varepsilon(r-ct)] \\ + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi^\varepsilon(s) ds \\ \text{if } r \geq ct + \varepsilon. \end{cases} \tag{6.4}$$

We finally extend U^ε at each time to obtain an everywhere defined approximation of the Cauchy solution U . The natural choice is to extend U^ε continuously as zero on B_ε at each time

$$\tilde{U}^\varepsilon(t, x) = \begin{cases} 0 & \text{for } 0 \leq |x| \leq \varepsilon \\ U^\varepsilon(t, x) & \text{for } |x| \geq \varepsilon. \end{cases} \tag{6.5}$$

We note that

$$\tilde{U}^\varepsilon(0, x) = \Phi^\varepsilon(x), \quad \tilde{U}_t^\varepsilon(0, x) = \Psi^\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^3. \tag{6.6}$$

7. COMPARING THE CAUCHY AND EXTERIOR DIRICHLET SOLUTIONS

We proceed to estimating the H^1 -distance

$$\|U(t) - \tilde{U}^\varepsilon(t)\|_{H^1(\mathbb{R}^3)},$$

and show that it vanishes as $\varepsilon \downarrow 0$. As for exterior Neumann solutions we prefer to estimate this difference by estimating the first order energy

$$\mathcal{E}^\varepsilon(t) = \mathcal{E}_{U-\tilde{U}^\varepsilon}(t)$$

as defined in (5.1). This energy bounds the L^2 -norm of the gradient of the difference $U - \tilde{U}^\varepsilon$, and it also controls the L^2 -norm of $U - \tilde{U}^\varepsilon$ itself. The calculations for these estimates are similar to the ones for the Neumann case in Section (5.1), and will only be outlined.

First, a direct calculation similar to what was done above (using that the energies $\mathcal{E}_U(t)$ and $\mathcal{E}_{\tilde{U}^\varepsilon}(t)$ are both conserved in time), shows that

$$\mathcal{E}^\varepsilon(t) = \mathcal{E}_{\tilde{U}^\varepsilon}^\varepsilon(0) + \mathcal{E}_U(0) - \int_{|x|>\varepsilon} U_t U_t^\varepsilon + c^2 \nabla U^\varepsilon \cdot \nabla U \, dx.$$

Differentiating with respect to time, integrating by parts, and applying the Dirichlet condition for U^ε yield

$$\dot{\mathcal{E}}^\varepsilon(t) = c^2 \int_{|x|=\varepsilon} U_t(t, \varepsilon) \partial_r U^\varepsilon(t, \varepsilon) \, dS. \tag{7.1}$$

Also, with $\mathcal{D}^\varepsilon(t)$ defined as in (5.2), we have that (5.3) holds also in the case of Dirichlet boundary conditions.

To estimate the H^1 -distance between the Cauchy solution $U(t)$ and the exterior Dirichlet solution $\tilde{U}^\varepsilon(t)$, we proceed to provide bounds for the initial terms $\mathcal{D}^\varepsilon(0)$ and $\mathcal{E}^\varepsilon(0)$, as well as for the surface integral in (7.1). It is immediate to verify that

$$\mathcal{D}^\varepsilon(0) \lesssim \|\Phi^\varepsilon - \Phi\|_{L^2(\mathbb{R}^3)}^2 \leq \|\Phi\|_{L^2(B_{2\varepsilon})}^2,$$

and similarly that

$$\mathcal{E}^\varepsilon(0) \lesssim \|\Psi\|_{L^2(B_{2\varepsilon})}^2 + \|\nabla\Phi^\varepsilon - \nabla\Phi\|_{L^2(\mathbb{R}^3)}^2.$$

To bound the last term we recall the definition of Φ^ε in (6.2) to calculate that

$$\begin{aligned} \|\nabla\Phi^\varepsilon - \nabla\Phi\|_{L^2(\mathbb{R}^3)}^2 &\lesssim \int_{|x|<2\varepsilon} |\nabla\Phi|^2 dx + \frac{1}{\varepsilon^2} \int_{\varepsilon<|x|<2\varepsilon} |\Phi(x)|^2 dx \\ &\lesssim \|\nabla\Phi\|_{L^2(B_{2\varepsilon})}^2 + \int_{\varepsilon<|x|<2\varepsilon} \frac{|\Phi(x)|^2}{|x|^2} dx. \end{aligned} \tag{7.2}$$

As Φ belongs to $H^1(\mathbb{R}^3)$, Hardy’s inequality (as formulated in [5, Lemma 17.1]) shows that $\frac{|\Phi(x)|^2}{|x|^2}$ belongs to $L^1(\mathbb{R}^3)$, so that the Dominated Convergence Theorem yields

$$\|\nabla\Phi^\varepsilon - \nabla\Phi\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

We have thus established that

$$\mathcal{D}^\varepsilon(0) \rightarrow 0 \quad \text{and} \quad \mathcal{E}^\varepsilon(0) \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \tag{7.3}$$

7.0.1. *Estimating $\partial_t U(t, \varepsilon)$.* According to (3.2) we have

$$\partial_t U(t, \varepsilon) = \begin{cases} \begin{aligned} &\frac{c}{2\varepsilon} [\varphi(ct + \varepsilon) + (ct + \varepsilon)\varphi'(ct + \varepsilon) - \varphi(ct - \varepsilon) - (ct - \varepsilon)\varphi'(ct - \varepsilon)] \\ &+ \frac{1}{2\varepsilon} [(ct + \varepsilon)\psi(ct + \varepsilon) - (ct - \varepsilon)\psi(ct - \varepsilon)] \end{aligned} & \text{if } t \geq \frac{\varepsilon}{c} \\ \begin{aligned} &\frac{c}{2\varepsilon} [\varphi(\varepsilon + ct) + (\varepsilon + ct)\varphi'(\varepsilon + ct) - \varphi(\varepsilon - ct) - (\varepsilon - ct)\varphi'(\varepsilon - ct)] \\ &+ \frac{1}{2\varepsilon} [(\varepsilon + ct)\psi(\varepsilon + ct) + (\varepsilon - ct)\psi(\varepsilon - ct)] \end{aligned} & \text{if } t \leq \frac{\varepsilon}{c}. \end{cases} \tag{7.4}$$

We estimate the terms for $t \geq \frac{\varepsilon}{c}$ by 2nd order Taylor expansion of $\varphi(ct \pm \varepsilon)$ and $\psi(ct \pm \varepsilon)$ about $\varepsilon = 0$. The terms for $t \leq \frac{\varepsilon}{c}$ are estimated by 2nd order Taylor expansion of φ and ψ about zero, and then using that $\varphi'(0) = \psi'(0) = 0$. These expansions are straightforward and are omitted. The result is that the leading order term in (7.4) for all times is $O(1)$. We thus have that

$$|U_t(t, \varepsilon)| \lesssim 1 \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \tag{7.5}$$

7.0.2. *Estimating $\partial_r U^\varepsilon(t, \varepsilon)$.* We first calculate $\partial_r U^\varepsilon(t, r)$ for $\varepsilon \leq r \leq ct + \varepsilon$ by using the first part of formula (6.4). Evaluating at $r = \varepsilon$ gives that

$$\partial_r U^\varepsilon(t, \varepsilon) = \frac{1}{\varepsilon} [\varphi^\varepsilon(ct + \varepsilon) + (ct + \varepsilon)\varphi^{\varepsilon'}(ct + \varepsilon)] + \frac{(ct + \varepsilon)}{c\varepsilon} \psi^\varepsilon(ct + \varepsilon).$$

Recalling the definitions of φ^ε and ψ^ε in (6.2)-(6.3), and splitting the calculations into $t \gtrsim \frac{\varepsilon}{c}$, we obtain

$$|\partial_r U^\varepsilon(t, \varepsilon)| \lesssim \frac{1}{\varepsilon} \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0. \tag{7.6}$$

7.1. Convergence of exterior Dirichlet solutions. By using (7.5) and (7.6) in (7.1) we obtain that

$$|\dot{\mathcal{E}}^\varepsilon(t)| \lesssim \varepsilon \quad \text{for all } t \in [0, T] \text{ as } \varepsilon \downarrow 0,$$

such that (7.3)₂ gives

$$\mathcal{E}^\varepsilon(t) \rightarrow 0 \quad \text{uniformly for } t \in [0, T] \text{ as } \varepsilon \downarrow 0.$$

Finally, recalling that (5.3) also holds in the Dirichlet case, we have that (7.3)₁ yields

$$\mathcal{D}^\varepsilon(t) \rightarrow 0 \quad \text{uniformly for } t \in [0, T] \text{ as } \varepsilon \downarrow 0,$$

as well. We thus conclude that

$$\|U(t) - \tilde{U}^\varepsilon(t)\|_{H^1(\mathbb{R}^3)} \lesssim \mathcal{D}^\varepsilon(t) + \mathcal{E}^\varepsilon(t) \rightarrow 0 \quad \text{uniformly for } t \in [0, T] \text{ as } \varepsilon \downarrow 0.$$

Thanks to Proposition 3.2, this concludes the proof of part (ii) of Theorem 3.1.

Acknowledgments. Jenssen was supported by the National Science Foundation under Grant DMS-1311353. Tsikkou was supported by the WVU ADVANCE Sponsorship Program.

REFERENCES

- [1] Lipman Bers, Fritz John, Martin Schechter; *Partial differential equations*, American Mathematical Society, Providence, R.I., With supplements by Lars Gårding and A. N. Milgram; With a preface by A. S. Householder; Reprint of the 1964 original; Lectures in Applied Mathematics, 3A.
- [2] Helge Kristian Jenssen, Charis Tsikkou; *Radial solutions to the cauchy problem for $\square_{1+3}u = 0$ as limits of exterior solutions*, Submitted, available at <https://arxiv.org/abs/1512.02297>.
- [3] Jeffrey Rauch; *Partial differential equations*, Graduate Texts in Mathematics, vol. 128, Springer-Verlag, New York.
- [4] Sigmund Selberg; *Lecture notes, math 632, pde*, Johns Hopkins University.
- [5] Luc Tartar; *An introduction to sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana, vol. 3, Springer, Berlin; UMI, Bologna.

HELGE KRISTIAN JENSSEN

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, STATE COLLEGE, PA 16802, USA

E-mail address: jenssen@math.psu.edu

CHARIS TSIKKOU

DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506, USA

E-mail address: tsikkou@math.wvu.edu