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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLOCAL p-LAPLACIAN PROBLEMS

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Abstract. We study the existence and uniqueness of positive solutions to a class of nonlocal boundary-value problems involving the p-Laplacian. Our main tools are a variant of the Schaefer's fixed point theorem, an inequality which suitably handles the p-Laplacian operator, and a Sobolev embedding which is applicable to the bounded domain.

1. Introduction

We study the boundary-value problem

$$\begin{gathered} -M(\|u\|^p)\Delta_p u = f(x,u) \quad \text{in } D, \\ u = 0 \quad \text{on } \partial D. \end{gathered} \tag{1.1}$$

in which Δ_p denotes the p-Laplacian

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$$

and $\|\cdot\|$ denotes the norm in $W_0^{1,p}(D)$, $\|u\| = \big(\int_D |\nabla u|^p\,dx\big)^{1/p}$. As for the functions $M : [0,\infty) \to [0,\infty)$ and $f : D \times \mathbb{R} \to \mathbb{R}$, we shall refer to the following assumptions:

(A1) M is continuous, and $M(t) \geq m_0 > 0$, where m_0 is a constant. Moreover, the function:
$$\xi(t) := M(t)^{\frac{1}{p-1}}\, t^{1/p}$$
is invertible, and henceforth we let $q = \frac{p}{p-1}$.

(A2) Let $\hat{M}(t) := M(t^p)$. Then for a constant κ (to be defined by (2.4)), the function $\hat{M}$ is uniformly Hölder continuous with exponent $p-1$ in the interval $[0,\kappa)$. In other words
$$L := \sup_{t_1,t_2 \in [0,\kappa),\, t_1 \neq t_2} \frac{|\hat{M}(t_1) - \hat{M}(t_2)|}{|t_1 - t_2|^{p-1}} < \infty$$

The principal eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions on ∂D is defined as

$$\Lambda := \inf_{u \in W_0^{1,p}(D),\, u \neq 0} \frac{\int_D |\nabla u|^p\,dx}{\int_D |u|^p\,dx}. \tag{1.2}$$

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The eigenvalue Λ is positive, isolated, and simple [11]. We impose the following minimum condition on Λ:

(A3) $\hat{m} := m_0 - a/\Lambda > 0$, in which m_0 comes from (A1), the constant a is introduced in (A5), and Λ is the principal eigenvalue from (1.2).

(A4) The function f is a Carathéodory function, and for some $r \in (1, p^* - 1)$

$$|f(x,s)| \leq A(x)|s|^r + B(x), \quad \forall x \in D,\ \forall s \in \mathbb{R}, \tag{1.3}$$

in which:
- $A \in L^\infty(D)$ is a non-negative function
- $B \in L^{1+1/r}(D)$
- $p^* = \begin{cases} \frac{np}{n-p} & \text{if } 1 < p < n \\ \infty & \text{if } p \geq n. \end{cases}$

(A5) For some positive constants a and b,

$$sf(x,s) \leq a|s|^p + b|s|, \quad \text{a.e. } x \in D,\ \forall s \in \mathbb{R}.$$

(A6) $f(x,s) \geq 0$ a.e. $x \in D$, for all $s \geq 0$ and $f(x,0) > 0$, a.e. $x \in D$.

(A7) For a positive constant A,

$$(f(x,u) - f(x,v))(u - v) \leq A|u - v|^2, \quad \forall x \in D,\ u, v \in \mathbb{R}$$

Remark 1.1. Note that when $p \geq 2$, the condition (A2) is satisfied when M is a constant function, hence the boundary value problem (1.1) is no longer a nonlocal. Whence, even though the arguments to follow will hold for $p \geq 2$ but it is the case $1 < p < 2$ which is of interest.

Remark 1.2. Let us mention that a function M that satisfies the conditions (A1) and (A2) (these are the main conditions on M), for the case $p \in (1,2)$ is $M(t) = m_0 + t^\beta$, where $\beta \geq \frac{1}{q}$. On the other hand, any function $f(x,s)$ which is bounded and $\frac{\partial}{\partial s} f(x,s)$ is uniformly bounded in x satisfies (A4)–(A7).

The main results of this article are the following theorems.

Theorem 1.3. *Under assumptions* (A1)–(A7)*, the boundary value problem* (1.1) *has a positive solution.*

Theorem 1.4. *Suppose the conditions* (A1)–(A7) *are satisfied. Then* (1.1) *has a unique positive solution, provided that L is sufficiently small and m_0 is sufficiently large.*

The special case of problem (1.1) when $p = 2$ has been considered in [1], and [14]. In the former, the authors impose conditions on the functions M and f so that it is possible to settle the issue of existence of solutions via the Mountain Pass Theorem. However, in the latter the authors use a different set of conditions, and apply the Galerkin method to obtain their results (see also [2]).

Our paper is motivated by [14]. For the result of Theorem 1.3 regarding the existence of positive solutions, we apply a variant of the Schaefer's fixed point theorem coupled with a well known maximum principle. For the uniqueness result of Theorem 1.4, we use the ideas of [14]. In proving both existence and uniqueness of solutions we shall use an inequality which is particularly useful in dealing with the p-Laplace operator. See inequality (2.6) in Lemma 2.5.

Nonlocal problems have been used in modeling various physical phenomena, and the problem (1.1) which we have considered in this note is related to the steady state version of the Kirchhoff equation [12]

$$u_{tt} - M\Big(\int_D |\nabla u|^2\,dx\Big)\Delta u = f(x,t), \tag{1.4}$$

where the coefficient of the diffusion term depends on the unknown function $u(x,t)$ *globally*. It was the paper [13] by Lions that introduced an abstract setting for (1.4). Other relevant work are [3, 7, 9]. Some nonlocal problems in statistical mechanics are studied in [4, 5].

2. Preliminaries

This section contains the basic material that we need for proving Theorems 1.3 and 1.4. We begin with the following definition.

Definition 2.1. We say that $u \in W_0^{1,p}(D)$ is a solution of (1.1) if the following integral equation is satisfied:

$$M(\|u\|^p)\int_D |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \int_D vf(x,u)\,dx = 0, \quad \forall v \in W_0^{1,p}(D). \tag{2.1}$$

The convergence of the second integral in (2.1) follows from the following general result regarding Nemytskii mappings.

Lemma 2.2. *Let $g : D\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function and suppose that there is a constant $c > 0$, a function $l(x) \in L^\gamma(D)$ (where $1 \le \gamma \le \infty$) and $\tau > 0$ such that*

$$|g(x,s)| \le c|s|^\tau + l(x), \quad \forall x \in D,\ \forall s \in \mathbb{R}.$$

Then $N_g : L^{\gamma\tau}(D) \to L^\gamma(D)$ defined by $N_g(u)(x) = g(x,u(x))$ is continuous and bounded, i.e. it maps bounded sets into bounded sets.

For a proof of the above lemma, see [10, Theorem 2.3]. Let us review some basic facts regarding the problem

$$\begin{gathered} -\Delta_p u = h(x) \quad \text{in } D \\ u = 0 \quad \text{on } \partial D, \end{gathered} \tag{2.2}$$

where $h(x) \in L^{1+1/r}(D)$. It is well known, see for example [15], that (2.2) has a unique solution $u \in W_0^{1,p}(D)$ which is the unique minimizer of the strictly convex functional

$$\Phi(w) = \frac{1}{p}\int_D |\nabla w|^p\,dx - \int_D hw\,dx$$

relative to $w \in W_0^{1,p}(D)$. Therefore, the inverse mapping

$$(-\Delta_p)^{-1} : L^{1+1/r}(D) \to W_0^{1,p}(D)$$

which takes every $h \in L^{1+1/r}(D)$ to the unique solution of (2.2) is well-defined. It is straightforward to verify that

$$(-\Delta_p)^{-1}(\eta h) = \eta^{1/(p-1)}(-\Delta_p)^{-1}(h), \quad \forall h \in L^{1+1/r}(D), \eta > 0$$

and that the following inequality holds:

$$\|(-\Delta_p)^{-1}(h)\| \le C\|h\|_{1+1/r}^{1/(p-1)}, \quad \forall h \in L^{1+1/r}(D) \tag{2.3}$$

where C is a positive constant. Henceforth we shall use C as a generic symbol for the several constants which appear in various places in this document, whose values could be different.

Lemma 2.3. *Assume that $u \in W_0^{1,p}(D)$ is a solution of* (1.1)*, and that* (A1)–(A6) *hold. Then*

$$\|u\| \le \Big(\frac{b|D|^{1/q}}{\hat{m}\Lambda^{1/p}}\Big)^{q/p} =: \kappa, \tag{2.4}$$

where $q = \frac{p}{p-1}$. Here $|D|$ denotes the n-dimensional Lebesgue measure of D.

Proof. Setting $v = u$ in (2.1), assumption (A1) implies

$$\begin{aligned} m_0\|u\|^p \le M(\|u\|^p)\|u\|^p &= \int_D f(x,u)u\,dx \\ &\le a\int_D |u|^p\,dx + b\int_D |u|\,dx \quad \text{(by assumption (A5))} \\ &\le a\frac{\|u\|^p}{\Lambda} + b|D|^{1/q}\|u\|_p \quad \text{(by (1.2) and Hölder)} \\ &\le a\frac{\|u\|^p}{\Lambda} + \frac{b|D|^{1/q}\|u\|}{\Lambda^{1/p}} \quad \text{(again by (1.2))} \end{aligned} \tag{2.5}$$

From these inequalities, we infer that

$$\hat{m}\|u\|^p \le \frac{b|D|^{1/q}\|u\|}{\Lambda^{1/p}},$$

which in turn implies the desired estimate (2.4). □

We also need the following variant of the Schaefer's fixed point theorem, see for example [16], but we include the proof for the convenience of the reader.

Lemma 2.4. *Let X be a Banach space and assume that:*

- (a) *$\mathcal{P} \subseteq X$ is a non-empty, closed, and convex subset of X.*
- (b) *$T : \mathcal{P} \to \mathcal{P}$ is a strongly continuous mapping, i.e. T is continuous and for every bounded sequence $(u_n) \subseteq \mathcal{P}$, the image (Tu_n) has a strongly convergent subsequence.*
- (c) *The set $S = \{x \in \mathcal{P} \mid x = \lambda Tx,\ \text{for some } \lambda \in [0,1]\}$ is bounded.*

Then T has a fixed point, i.e. there exists $x \in \mathcal{P}$ such that $Tx = x$.

Proof. Consider the orthogonal projection $P : X \to \mathcal{P}$ of X on $\mathcal{P}$. This projection satisfies:

$$\forall x \in X : \quad \|Px - x\| = \inf_{m \in \mathcal{P}} \|x - m\|,$$

The mapping $T \circ P : X \to \mathcal{P} \subseteq X$ is clearly strongly continuous. Define $S' = \{x \in X : x = \lambda(T \circ P)x,\ \text{for some } \lambda \in [0,1]\}$. Hence, $S' \subseteq S$ and S' is bounded.

Now we can invoke the classical Schaefer's fixed point theorem, applied to $T \circ P$, and deduce that $T \circ P$ has a fixed point, say $x_0 \in X$. Thus:

$$\begin{aligned} x_0 = T(Px_0) &\Rightarrow x_0 \in \operatorname{range}(T) \\ &\Rightarrow x_0 \in \mathcal{P} \\ &\Rightarrow x_0 = Px_0 \\ &\Rightarrow x_0 = Tx_0 \end{aligned}$$

Thus, x_0 is a desired fixed point of T. □

We shall also need the following result, see for example [8] and [6]

Lemma 2.5. *For any vectors $X, Y \in \mathbb{R}^n$, the following inequalities hold:*

$$C_p\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle \geq \begin{cases} |X-Y|^p & \text{if } p \geq 2 \\ \frac{|X-Y|^2}{(|X|+|Y|)^{2-p}} & \text{if } 1 \leq p \leq 2, \end{cases} \tag{2.6}$$

in which $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$, and C_p is a constant depending on p.

The following lemma is elementary, so we omit its proof.

Lemma 2.6. *Let $t_n \xrightarrow{\mathbb{R}} t$ and $f_n \xrightarrow{L^p} f$. Then $t_n f_n \xrightarrow{L^p} tf$.*

Lemma 2.7. *If $f_n \xrightarrow{L^{1+1/r}} f$, then $(-\Delta_p)^{-1}(f_n) \xrightarrow{W_0^{1,p}} (-\Delta_p)^{-1}(f)$.*

Proof. Set $v_n = (-\Delta_p)^{-1}(f_n)$ and $v = (-\Delta_p)^{-1}(f)$. Thus:

$$\begin{gathered} -\Delta_p v_n = f_n \quad \text{in } D \\ v_n = 0 \quad \text{on } \partial D \end{gathered}$$

and

$$\begin{gathered} -\Delta_p v = f \quad \text{in } D \\ v = 0 \quad \text{on } \partial D \end{gathered}$$

The derivation of the following equation is then straightforward.

$$\begin{aligned} &\int_D (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v) \cdot (\nabla v_n - \nabla v)\, dx \\ &= \int_D (f_n - f)(v_n - v)\, dx. \end{aligned} \tag{2.7}$$

Applying the Hölder's inequality and the embedding $W_0^{1,p}(D) \to L^{1+1/r}(D)$, a bound on the integral on the right hand side of (2.7) is obtained as follows:

$$\begin{aligned} \int_D (f_n - f)(v_n - v)\, dx &\leq \|f_n - f\|_{1+1/r}\|v_n - v\|_{r+1} \\ &\leq C\|f_n - f\|_{1+1/r}\|v_n - v\|. \end{aligned} \tag{2.8}$$

Now we consider two cases:

Case $p \geq 2$. From Lemma 2.5 (setting $X = \nabla v_n$, $Y = \nabla v$), (2.7) and (2.8), we obtain

$$\|\nabla v_n - \nabla v\|_p^p \leq C\|f_n - f\|_{1+1/r}\|v_n - v\|,$$

hence $\|v_n - v\| \leq C\|f_n - f\|_{1+1/r}^{1/(p-1)}$. Thus, $v_n \to v$ in $W_0^{1,p}(D)$.

Case $1 \leq p \leq 2$. This case requires more work. We begin with the observation

$$\begin{aligned} &\|\nabla v_n - \nabla v\|_p^p \\ &= \int_D \frac{|\nabla v_n - \nabla v|^p}{(|\nabla v_n| + |\nabla v|)^{\frac{p(2-p)}{2}}} \, (|\nabla v| + |\nabla v|)^{\frac{p(2-p)}{2}}\, dx \\ &\leq \Big(\int_D \frac{|\nabla v_n - \nabla v|^2}{(|\nabla v_n| + |\nabla v|)^{2-p}}\, dx\Big)^{p/2} \Big(\int_D (|\nabla v_n| + |\nabla v|)^p\, dx\Big)^{(2-p)/2} \end{aligned} \tag{2.9}$$

This follows from Hölder's inequality which is applicable since $\frac{2}{p} \geq 1$. Applying inequality (2.3), we obtain $\|v_n\| \leq C\|f_n\|_{1+1/r}^{1/(p-1)}$ and $\|v\| \leq C\|f\|_{1+1/r}^{1/(p-1)}$. Since (f_n) is bounded in $L^{1+1/r}(D)$, we infer that $\max(\|v_n\|, \|v\|) \leq C$, for all $n \in \mathbb{N}$. Thus, from (2.9) we obtain

$$\|\nabla v_n - \nabla v\|_p^p \leq C\Big(\int_D \frac{|\nabla v_n - \nabla v|^2}{(|\nabla v_n| + |\nabla v|)^{2-p}}\Big)^{p/2} \tag{2.10}$$

Now, by setting $X = \nabla v_n$ and $Y = \nabla v$ in Lemma 2.5, together with (2.7), (2.8), and (2.10) we find that

$$\|v_n - v\|^p = \|\nabla v_n - \nabla\|_p^p \leq C\|f_n - f\|_{1+1/r}\|v_n - v\|,$$

This implies that $\|v_n - v\| \leq C\|f_n - f\|_{1+1/r}^{1/(p-1)}$. So, $v_n \to v$ in $W_0^{1,p}(D)$, as desired. The proof is complete. $\square$

3. Proofs of main theorems

To prove Theorem 1.3 we shall apply Lemma 2.4. To this end, we set $\mathcal{P} = L_+^{r+1}(D)$. Note that by Lemma 2.2 we have $\forall u \in \mathcal{P} : N_f(u) \in L^{1+1/r}(D)$. From assumption $(A6)$ we infer that $N_f(u)$ is non-negative. For every $u \in \mathcal{P}$, we define:

$$Tu = t^{1/p}\frac{v}{\|v\|},$$

in which $v := (-\Delta_p)^{-1}(N_f(u))$ and $t := \xi^{-1}(\|v\|)$. Observe that $w = Tu$ satisfies

$$\begin{gathered} -M(\|w\|^p)\Delta_p w = f(x,u) \quad \text{in } D \\ w = 0 \quad \text{on } \partial D. \end{gathered} \tag{3.1}$$

Since $Tu \in W_0^{1,p}(D)$, the embedding $W_0^{1,p}(D) \to L^{r+1}(D)$ implies $Tu \in L^{r+1}(D)$. Thus, by applying a classical maximum principle (see for example [17]) to (3.1), we deduce that w is positive, i.e. $Tu \in L_+^{r+1}(D)$.

The above discussion ensures that the mapping $T : \mathcal{P} \to \mathcal{P}$ is well defined. Note that if u is a fixed point of T, then u will be a solution of (1.1). The existence of such a fixed point will confirm the assertion of Theorem 1.3.

3.1. **Proof of Theorem 1.3.**
We just need to verify that the mapping T satisfies the hypotheses of Lemma 2.4.

Continuity. Let $(u_n) \subseteq \mathcal{P}$ be a sequence such that $u_n \to u$ in $L^{r+1}(D)$. Note that since $\mathcal{P}$ is closed then u must be non-negative. We need to show that $Tu_n \to Tu$ in $L^{r+1}(D)$. In view of the embedding $W_0^{1,p}(D) \to L^{r+1}(D)$, it suffices to show $Tu_n \to Tu$ in $W_0^{1,p}(D)$. To this end, we first recall Lemma 2.2 which ensures that $N_f(u_n) \to N_f(u)$ in $L^{1+1/r}(D)$. Whence, by Lemma 2.7:

$$(-\Delta_p)^{-1}(N_f(u_n)) \to (-\Delta_p)^{-1}(N_f(u)) \quad \text{in } W_0^{1,p}(D).$$

By the continuity of the norm we also have

$$\|(-\Delta_p)^{-1}(N_f(u_n))\| \to \|(-\Delta_p)^{-1}(N_f(u))\|.$$

On the other hand,

$$Tu_n = t_n^{1/p}\frac{(-\Delta_p)^{-1}(N_f(u_n))}{\|(-\Delta_p)^{-1}(N_f(u_n))\|},$$

in which $t_n = \xi^{-1}(\|(-\Delta_p)^{-1}(N_f(u_n))\|)$. Since ξ is continuous, we obtain

$$t_n \to t := \xi^{-1}(\|(-\Delta_p)^{-1}(N_f(u))\|).$$

Now we apply Lemma 2.6 to conclude that $Tu_n \to Tu$ in $W_0^{1,p}(D)$, as desired.

Compactness. Consider a bounded sequence $(u_n) \subseteq \mathcal{P}$. Setting $w_n = Tu_n$, we will have

$$\begin{gathered} -M(\|w_n\|^p)\Delta_p w_n = f(x,u_n) \quad \text{in } D \\ w_n = 0 \quad \text{on } \partial D. \end{gathered} \tag{3.2}$$

From (3.2) we obtain

$$M(\|w_n\|^p)\|w_n\|^p = \int_D f(x,u_n) w_n \, dx.$$

An application of Hölder's inequality then gives

$$M(\|w_n\|^p)\, \|w_n\|^p \le \|N_f(u_n)\|_{1+1/r} \|w_n\|_{r+1}. \tag{3.3}$$

The inequality (3.3), the embedding $W_0^{1,p}(D) \to L^{r+1}(D)$, and the assumption (A1) together lead to

$$m_0\|w_n\|^p \le C\|N_f(u_n)\|_{1+1/r}\|w_n\|,$$

Hence, we get $\|w_n\| \le C\|N_f(u_n)\|_{1+1/r}^{\frac{1}{p-1}}$. This, coupled with the boundedness of the operator N_f (see Lemma 2.2), implies that (w_n) is bounded in $W_0^{1,p}(D)$. So, there exists a subsequence $(w_{n_j}) \subseteq (w_n)$ such that $w_{n_j} \rightharpoonup w$ in $W_0^{1,p}(D)$, for some $w \in W_0^{1,p}(D)$. Since the embedding $W_0^{1,p}(D) \to L^{r+1}(D)$ is compact, we deduce that $w_{n_j} \to w$ in $L^{r+1}(D)$. This means that (Tu_{n_j}) is strongly convergent in $L^{r+1}(D)$ and as a result $T : \mathcal{P} \to \mathcal{P}$ is compact.

Boundedness of S. The final step is to prove the boundedness of the set

$$S = \{u \in \mathcal{P} : u = \lambda Tu, \text{ for some } \lambda \in [0,1]\}.$$

To that end, let us fix a $u \in S$ and assume that $u = \lambda Tu$ for some $\lambda \in [0,1]$. Thus, we must have

$$u = \lambda t^{1/p} \frac{(-\Delta_p)^{-1}(N_f(u))}{\|(-\Delta_p)^{-1}(N_f(u))\|},$$

where $t = \xi^{-1}(\|(-\Delta_p)^{-1}(N_f(u))\|)$. Since $\|u\| = \lambda t^{1/p}$ and assuming that $\lambda \neq 0$, then $t = \|u\|^p/\lambda^p$ and $M(\frac{\|u\|^p}{\lambda^p}) = M(t)$. So, we obtain

$$\begin{aligned} -M\Big(\frac{\|u\|^p}{\lambda^p}\Big)\Delta_p u &= \frac{M(t)\lambda^{p-1}t^{1/q}}{\|(-\Delta_p)^{-1}(N_f(u))\|^{p-1}}\ f(x,u) \\ &= \lambda^{p-1} f(x,u), \end{aligned} \tag{3.4}$$

where $q = \frac{p}{p-1}$. Since $u \in W_0^{1,p}(D)$, from (3.4), (A1), and (A5) one gets

$$\begin{aligned} m_0\|u\|^p \le M\Big(\frac{\|u\|^p}{\lambda^p}\Big)\|u\|^p &= \lambda^{p-1}\int_D f(x,u)u\,dx \\ &\le \lambda^{p-1}\Big(a\int_D |u|^p\,dx + b\int_D |u|\,dx\Big) \\ &\le a\frac{\|u\|^p}{\Lambda} + \frac{b|D|^{1/q}\|u\|}{\Lambda^{1/p}}. \end{aligned} \tag{3.5}$$

From (3.5) and (A3) we obtain $\|u\| \le \kappa$ (which was defined in (2.4)). Note that in case $\lambda = 0$, this last inequality trivially holds.

Finally, by invoking the embedding $W_0^{1,p}(D) \to L^{r+1}(D)$, we infer that $\|u\|_{r+1} \le C$. Whence, S is bounded, as desired. This completes the proof.

3.2. **Proof of Theorem 1.4.** The existence of solutions is guaranteed by Theorem 1.3. We prove uniqueness by contradiction. Let us assume that u_1 and u_2 are two solutions of (1.1), satisfying

$$\begin{gathered} -M(\|u_i\|^p)\Delta_p u_i = f(x,u_i) \quad \text{in } D \\ u_i = 0 \quad \text{on } \partial D, \end{gathered} \tag{3.6}$$

for $i = 1, 2$. From (3.6) we obtain

$$\begin{aligned} &\int_D (M(\|u_1\|^p)\|\nabla u_1\|^{p-2}\nabla u_1 - M(\|u_2\|^p)\|\nabla u_2\|^{p-2}\nabla u_2) \cdot \nabla w \, dx \\ &= \int_D (f(x,u_1) - f(x,u_2)) w \, dx. \end{aligned} \tag{3.7}$$

By rearranging terms, we obtain

$$\begin{aligned} &M(\|u_2\|^p) \int_D (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla w \, dx \\ &= (M(\|u_2\|^p) - M(\|u_1\|^p)) \int_D |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla w \, dx \\ &\quad + \int_D (f(x,u_1) - f(x,u_2)) w \, dx \\ &\le L\big|\,\|u_2\| - \|u_1\|\,\big|^{p-1}\|u_1\|^{p-1}\|w\| + A\|w\|_p^p, \end{aligned} \tag{3.8}$$

where we have used (A2) and (A7) in the last inequality. Note that

$$L|\,\|u_2\| - \|u_1\|\,|^{p-1}\,\|u_1\|^{p-1}\|w\| + A\|w\|_p^p \le \Big(L\kappa^{p-1} + \frac{A}{\Lambda}\Big)\|w\|^p. \tag{3.9}$$

On the other hand, using similar arguments as in the proof of Lemma 2.7, we obtain the estimate

$$\int_D (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla w \, dx \ge C\|w\|^p, \tag{3.10}$$

in which the constant C depends on κ if $p < 2$, otherwise it does not. From (3.8), (3.9), and (3.10) we obtain

$$m_0 C\|w\|^p \le \Big(L\kappa^{p-1} + \frac{A}{\Lambda}\Big)\|w\|^p. \tag{3.11}$$

Since $u_1 \ne u_2$, (3.11) implies

$$m_0 C - A\Lambda^{-p} \le L\kappa^{p-1}. \tag{3.12}$$

Now, if m_0 is large enough, and L is small enough as

$$m_0 > AC^{-1}\Lambda^{-p} \quad \text{and} \quad L < \frac{m_0 C - A\Lambda^{-p}}{\kappa^{p-1}}.$$

then we obtain the desired contradiction, and the proof is complete.

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