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ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS GOVERNED BY SUBDIFFERENTIALS

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Abstract. Sufficient conditions are given to ensure the existence and eventually the uniqueness of bounded and almost periodic solutions of evolution equations governed by subdifferential operators.

1. Introduction

Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces, with $\mathcal{V}$ densely and compactly embedded in $\mathcal{H}$ which will be referred as the compactness condition. We consider an evolution equation of the form

$$u'(t) + \partial\varphi^t(u(t)) + B(t, u(t)) \ni f(t) \quad \text{in } \mathcal{H}\ t \geq s,\ s \in \mathbb{R} \tag{1.1}$$

where $f \in L^2_{\rm loc}([s,\infty];\mathcal{H})$, $\partial\varphi^t$ is the subdifferential of a time dependent proper lower semicontinuous (lsc) convex function φ^t on $\mathcal{H}$, $B(t,\cdot)$ is a multivalued operator from a subset $D(B(t,\cdot)) \subset \mathcal{H}$ into $\mathcal{H}$ for each $t \in \mathbb{R}$. The Cauchy problem for (1.1) with the initial value $u_0 \in \mathcal{H}$ is

$$\begin{gathered} u'(t) + \partial\varphi^t(u(t)) + B(t, u(t)) \ni f(t) \quad \text{in } \mathcal{H}\ t \geq s \\ u(s) = u_0. \end{gathered} \tag{1.2}$$

We use some classical results on subdifferentials of time dependent convex functions in a Hilbert space ; we refer to [4], [5] or [8] for the definitions and known results.

Denote by $\mathcal{A}$ the totality of operator functions $A(t)(\cdot) = \partial\varphi^t(\cdot) + B(t,\cdot), t \in \mathbb{R}$ satisfying

$$(v_1^* - v_2^*, v_1 - v_2) \geq \gamma(t)|v_1 - v_2|^2_{\mathcal{H}}, v_i^* \in A(t)v_i. \tag{1.3}$$

We shall prove an existence theorem for bounded solutions under a compactness condition, and an existence-uniqueness theorem for bounded solutions assuming (1.3), f and γ being in the Stepanov space. After we discuss whether the boundedness of a solution to (1.1) on the whole line, implies its almost periodicity or the existence of an almost periodic solution, via compactness arguments combined with monotonicity methods.

Let $C_b(W)$ be the space of continuous and bounded functions on the real line with values in Banach space W and denote by $|\cdot|_W$ the norm for W.

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Definition 1.1. A function x belonging to $C_b(W)$ is called Bochner almost periodic if the family of its shifts $x^\tau(\cdot) = x(. + \tau)(\tau \in \mathbb{R})$ is precompact in $C_b(W)$. The set of almost periodic functions is denoted by $\dot{C}_b(W)$.

The Stepanov space of index $r \geq 1, S^r(J;W)$ will play an important role in the sequel.

$$S^r(J;W) = \{f \in L^r(J;W), \sup_{t\in J} \int_0^1 |f(t+\tau)|_W^r d\tau < \infty\}.$$

If $W = \mathbb{R}$, we denote this space by$S^r(J)$.

Definition 1.2. A function f is said to be almost periodic in the sense of Stepanov $(S^r)-$ a.p., if for every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}_\varepsilon$ such that for all $\tau \in \{\tau\}_\varepsilon$, we have

$$\sup_{t\in\mathbb{R}} \int_0^1 |f(t+\tau+\eta) - f(t+\eta)|_W^r d\eta \leq \varepsilon.$$

Before stating our main results, we give a metric topology on Φ the set of all proper lsc convex functions on $\mathcal{H}$, and precise some preliminary results. For $\varphi, \psi \in \Phi$, we define for each $r \geq 0$,

$$\rho_r(\varphi,\psi) := \begin{cases} 0 & \text{if } L_\varphi(r) = \emptyset \\ \sup_{z\in L_\varphi(r)} \inf_{y\in D(\psi)}\{\max\{|y-z|, |\psi(y) - \varphi(z)|\} & \text{if } L_\varphi(r) \neq \emptyset, \end{cases} \tag{1.4}$$

where $D(f)$ stands for the domain of f and put

$$L_\varphi(r) = \{z \in D(\varphi); |z| \leq r, \varphi(z) \leq r\}.$$

We define the functional $\zeta_r(\cdot,\cdot)$ on $\Phi \times \Phi$ as follows

$$\zeta_r(\varphi,\psi) = \rho_r(\varphi,\psi) + \rho_r(\psi,\varphi); \varphi, \psi \in \Phi.$$

We say that $\{\varphi_n\}$ is a Cauchy sequence for the metric topology of Φ if

$$\zeta_r(\varphi_n, \varphi_m) \to 0 \quad \text{as } n, m \to \infty.$$

Note that Φ is not complete. However, $\Phi_r = \{\varphi \in \Phi; L_\varphi(r) \neq \emptyset\}$ is a complete subset of Φ for this topology. Following [4], the function $t \to \varphi^t$ is Φ-almost periodic if from any sequence $\{t_n\}$, one can select a subsequence $\{t_{n'}\}$ of $\{t_n\}$ such that $\varphi^{t+t_{n'}}$ converges in Φ uniformly in $t \in \mathbb{R}$.

We use the notion of weak solution to (1.1) introduced in [8].

Definition 1.3. (i) A function $u : [t_0, t_1] \to \mathcal{H}$ is called a solution of (1.1) on $[t_0, t_1]$ if: $u \in L^p([t_0,t_1];\mathcal{H}) \cap C([t_0,t_1];\mathcal{H})$, $u' \in L^2([t_0,t_1];\mathcal{H})$, $\varphi^{(\cdot)}u \in L^1(t_0,t_1), p \geq 1$ and (1.1) is satisfied.

(ii) A function $u : \mathbb{R} \to \mathcal{H}$ is called a weak solution of (1.1) on $\mathbb{R}$, if the restriction of u to every compact K of $\mathbb{R}$ is a solution of (1.1) on K in the above sense.

To prove the existence of solutions to (1.1), we use the following conditions:

(A1) $\varphi^t(z) \geq c_0|z|_\mathcal{V}^p - \gamma_1(t)$ for all $z \in \mathcal{V}$ and $t \in \mathbb{R}$.

(A2) (Smoothness of φ^t in t) There are functions $\alpha \in W_{loc}^{1,2}(\mathbb{R})$ and $\beta \in W_{loc}^{1,1}(\mathbb{R})$ satisfying

$$S(\alpha,\beta) = \sup_{t\in\mathbb{R}} |\alpha'|_{L^2(t,t+1)} + \sup_{t\in\mathbb{R}} |\beta'|_{L^1(t,t+1)} < \infty\,;$$

for each $s, t \in \mathbb{R}$ and $z \in D(\varphi^s)$, there exists an element $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_{\mathcal{H}} \leq |\alpha(t) - \alpha(s)|(1 + \varphi^s(z))^{\frac{1}{p}},$$
$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |\beta(t) - \beta(s)|(1 + \varphi^s(z)).$$

(A3) B is an operator from $D(B(t,\cdot)) \subset \mathcal{H}$ into $\mathcal{H}$ such that $D(\varphi^t) \subset D(B(t,\cdot))$ and

$$(-\partial\varphi^t(u(t)) - B(t,u(t)), u)_{\mathcal{H}} + c_1\varphi^t(u) + c(t)|u(t)|^2_{\mathcal{H}} \leq \gamma_2(t), t \in \mathbb{R}.$$

We assume that

$$m_t c = \liminf_{t_1 - t_0 \to \infty} \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} c(s)ds > 0.$$

(A4) $|||B(t,u)|||^2_{\mathcal{H}} \leq c_2\varphi^t(u) + \gamma_3(t)$ for all $u \in D(\varphi^t)$, where $|||B(t,u)|||_{\mathcal{H}} = \sup\{|b|_{\mathcal{H}}; b \in B(t,u)\}$ are valid, in these inequalities $2 \leq p < \infty$ c_0, c_1 and c_2 are positive constants; $\gamma_1, \gamma_2, \gamma_3 \in S^1([t_0, +\infty[)$.

(A5) $B(t,\cdot)$ is measurable in the sense: For each function $u \in C([0,T];\mathcal{H})$ such that $\frac{du(t)}{dt} \in L^2([0,T];\mathcal{H})$ and there exists a function $g(t) \in L^2([0,T];\mathcal{H})$ with $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0,T]$, there exists an $\mathcal{H}$-valued measurable function $b(t) \in B(t,u(t))$ for a.e $t \in [0,T]$.

(A6) $B(t,\cdot)$ is demiclosed in the following sense: If $u_n \to u$ in $C([0,T];\mathcal{H})$, $g_n \to g$ weakly in $L^2([0,T];\mathcal{H})$ with $g_n(t) \in \partial\varphi^t(u(t))$, $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0,T]$, and if $b_n \to b$ weakly in $L^2([0,T];\mathcal{H})$ with $b_n(t) \in B(t,u_n(t))$ for a.e. $t \in [0,T]$, then $b(t) \in B(t,u(t))$ holds a.e.

A slight modification of the results in [8] ensure the existence of a weak solution given by Definition 1.3. In all the cases in which this construction is possible, u is given by the formula $u(t+s) = E_f(t,s)u(s), (E_f(t,s))_{t\in\mathbb{R}, s\in\mathbb{R}^+}$ is called a process evolved in [3], see also the references therein.

This article is organized as follows: in section 2, we present the main results. In section 3, we introduce some preliminaries. The proof of the main results are given in section 4. In the last section, we show the applicability of our abstract theorems to the heat equation and its variants in domains with moving boundaries.

2. Main results

Conditions for the existence and uniqueness of bounded and almost periodic solutions to (1.1) are obtained. Our main results extend previous works [4, 6, 7] and are stated as follows:

Theorem 2.1. *Let $f \in S^2(J;\mathcal{H})$. Under the compactness condition,* (1.1) *has at least one solution $\mathcal{H}$-bounded on the whole line.*

Theorem 2.2. *Let $f \in S^{p'}(J;\mathcal{H})$, $A(t) \in \mathcal{A}$ and the function γ appearing in* (1.3) *satisfy the inequality $m_t\gamma > 0$. Then the inclusion* (1.1) *has a unique bounded solution.*

The following result is an easy consequence of Theorem 2.2.

Corollary 2.3. *Suppose moreover that $f(t), \varphi^t$ and $B(t,\cdot)$ are periodic with the same period, the existence and uniqueness of a periodic solution to* (1.1) *is straightforward.*

Remark 2.4. It is important to note that if in inequality (1.3) $\gamma(t) \geq 0$ a.e and $m_t\gamma > 0$, the existence and uniqueness of a bounded solution do not require any compactness condition.

Theorem 2.5. *For a positive fixed number A, we denote by Ψ_A the set of all families $\{\varphi^t\}$ of almost periodic functions satisfying the properties* (A1) *and* (A2) *with $\beta' S^1$-almost periodic such that $S(\alpha,\beta) \leq A$. Let $\{\varphi^t\} \in \Psi_A \cap \Phi_R$ and $\partial\varphi^t(\cdot) + B(t,\cdot)$, $t \in \mathbb{R}$ be monotone. Suppose that $f(t)$ is Stepanov-almost periodic and $B(t,\cdot) \in \mathcal{B}$, the set of multivalued almost periodic operators B. There is at least one solution u^* of* (1.1) *which belongs to $\dot{C}(\mathcal{H})$.*

A new element beyond many previous works is a substitution of the requirement of monotonicity of the global operator $\partial\varphi^t(\cdot) + B(t,\cdot)$ by a weaker assumption of the type of semi-boundedness from below, ensuring the existence of almost periodic solutions to (1.1). The proof is similar enough to that given in [6], so it will be omitted.

Theorem 2.6. *Let $f(t)$ be Stepanov almost periodic and the map $t \to \partial\varphi^t(\cdot) + B(t,\cdot)$ from $\mathbb{R}$ into $\mathcal{H}$ be almost periodic. Assume also that the function γ appearing in* (1.3) *satisfies the inequality $m_t\gamma > 0$. Then the inclusion* (1.1) *has a unique $\mathcal{H}$-almost periodic solution.*

Another method used for studying almost periodicity, is based on the process $(E_f(t,s))_{t\in\mathbb{R},s\in\mathbb{R}^+}$. In this way the methods apply to a much larger class of equations than merely the equation (1.1) itself.

Theorem 2.7. *Let $(E_f(t,s))_{t\in\mathbb{R},s\in\mathbb{R}^+}$ be the process generated by* (1.1) *on a closed convex subset of $\mathcal{H}$. We assume that $(E_f(t,s))$ is T-periodic ($T > 0$) that is,*

$$\forall t \in \mathbb{R}, \forall s \in \mathbb{R}^+, \quad E_f(t+T,s) = E_f(t,s),$$

and that for some $M \geq 1$ we have

$$|E_f(s,t)x - E_f(s,t)y| \leq M|x-y|.$$

Let u be any solution of (1.1)*. Under the compactness condition, there exists an almost periodic solution $v(t)$ such that $|u(t) - v(t)|_{\mathcal{H}} \to 0$ as $t \to \infty$.*

Remark 2.8. Theorem 2.7 is used to treat (1.1) when $f(t), \varphi^t$ and $B(t,\cdot)$ are periodic with the same period.

In the time-independent case of φ^t, one has the following result.

Theorem 2.9. *Assume that $\varphi^t = \varphi$ and $f = 0$. Assume also that $\partial\varphi$ and B satisfy monotonicity types conditions of the form*

$$(\partial\varphi(x) - \partial\varphi(y), x-y) \geq \phi(|x-y|_{\mathcal{H}}), \quad \forall x, y \in D(A)$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that, $\phi(r) > 0$ for $r > 0$.

$$(B(t,x) - B(t,y), x-y) \geq -\psi(|x-y|_{\mathcal{H}}), t \in \mathbb{R}, x, y \in \mathcal{H}$$

the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous satisfying $\psi(0) = 0$.

We suppose that the map $t \to B(t,\cdot)$ from $\mathbb{R}$ into $\mathcal{H}$ is almost periodic for every fixed $x \in \mathcal{H}$, uniformly on every bounded set of $\mathcal{H}$. We assume finally that

$$\limsup_{r\to\infty} \frac{\psi(r) - \phi(r)}{r} < 0$$

while the largest root $r(\varepsilon)$ of the equation $\psi(r)-\phi(r)+\varepsilon r=0$ verifies the condition $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. There exists an almost periodic solution $u:\mathbb{R}\to\mathcal{H}$ of (1.1).

3. Preliminaries

We start with a closedness result stated in [10] which describes the convergence of the solutions of the variational inequality related to (φ_n^t, B_n, f_n) to that of (φ^t, B, f).

Lemma 3.1. *Let J be a compact interval in $\mathbb{R}$ and $\{\varphi^t\}_{t\in J}$ be a family of continuous mappings from J into Φ. Assume that $\varphi_n^{(\cdot)}$ is continuous from J into Φ, and for all $t\in J$ $\varphi_n^{(t)}$ converges to $\varphi^{(t)}$ in Φ (when $n\to\infty$), where $\varphi^{(t)}$ is a family in Φ and suppose that $B_n(t_n,z_n)\to B(t,z)$ weakly in $\mathcal{H}$ (when $n\to\infty$). Let $f,f_n\in L^2_{\rm loc}(\mathbb{R};\mathcal{H})$, u_n be the solution of $u_n'(t)+\partial\varphi^t(u_n(t))+B(t,u_n(t))\ni f_n(t)$ on a fixed interval $J=[t_0,t_1]$. Suppose that $f_n\to f$ in $L^2(J;\mathcal{H})$, and (u_n) is a Cauchy sequence in $C(J;\mathcal{H})$, then the limit function is a solution of (1.1) on J.*

The following lemma will be useful [6].

Lemma 3.2. *Let an absolutely continuous and bounded function $\psi(t)$ satisfy a.e. the differential inequality*

$$\psi'(t)+b(t)\psi(t)\le\phi(t),$$

where $\phi\in S^1(\mathbb{R}), b\in S^1((\mathbb{R})$, and

$$m_t b=\liminf_{t_1-t_0\to\infty}\frac{1}{t_1-t_0}\int_{t_0}^{t_1} b(s)ds>0.$$

Then

$$\psi(t)\le c(|b|_{S^1(\mathbb{R})}+\frac{1}{m_t b})|\phi|_{S^1(\mathbb{R})}.$$

where $c(\cdot)$ is an increasing function on $(0,\infty)$.

The next proposition establishes the boundedness on the half line of any solution in the $\mathcal{H}$-norm.

Proposition 3.3. *Let $\{\varphi^t\}_{0\le t<\infty}$ satisfy (A1) and (A2), B satisfy (A3). Assume that $f\in S^{p'}([t_0,\infty);\mathcal{H}), p'\ge 2$ and $\varphi^t(0)=0$. For each solution of (1.1) on $[t_0,\infty)$, we have*

$$\sup_{t\ge t_0}|u(t)|_{\mathcal{H}}, sup_{t\ge t_0}\int_t^{t+1}\varphi^\tau(u(\tau))d\tau\le M_0,$$

where M_0 depends on $|f|_{S^{p'}([t_0,\infty);\mathcal{H})}$.

Proof. Multiply both sides of: $u'(\tau)+\partial\varphi^\tau(u(\tau))+B(\tau,u(\tau))\ni f(\tau)$ by $u(\tau)$ and make use of Hölder and Young inequalities, we obtain:

$$\frac{d|u(\tau)|^2_{\mathcal{H}}}{d\tau}+\theta|u(\tau)|^p_{\mathcal{V}}+2c(\tau)|u(\tau)|^2_{\mathcal{H}}\le\delta|f(\tau)|^{p'}_{\mathcal{H}}+\gamma(\tau), \tag{3.1}$$

where θ,δ depend on c_0,c_1 and $\gamma\in S^1([t_0,\infty))$ depend on γ_1,γ_2.

The first inequality is a consequence of Lemma 3.2. The second assertion holds since,

$$|u(t+1)|^2_{\mathcal{H}}+\theta\int_t^{t+1}\varphi^\tau(u(\tau))d\tau\le|u(t)|^2_{\mathcal{H}}+\delta\int_t^{t+1}|f(\tau)|^{p'}_{\mathcal{H}}d\tau+\int_t^{t+1}|\gamma(\tau)|d\tau.$$

□

The following lemma is due to Yamada [9].

Lemma 3.4. *Let* $\{\varphi^t\}_{0\leq t<\infty}$ *satisfy* (A1) *and* (A2) *and* $u(t)$ *be a strongly absolutely continuous function from* $[0,T]$ *to* $\mathcal{H}$*, where* T *is an arbitrarily fixed positive number. Then the mapping* $t\to\varphi^t(u(t))$ *is absolutely continuous on* $[0,T]$ *and we have*

$$\begin{aligned}&\frac{d\varphi^t(u(t))}{dt}-(\frac{d(u(t))}{dt},\partial\varphi^t(u(t)))_{\mathcal{H}}\\&\leq|\beta'(t)|\{1+\varphi^t(u(t))\}+|\alpha'(t)|||\partial\varphi^t(u(t))||\{1+(\varphi^t(u(t)))\}^{\frac{1}{p}}.\end{aligned}$$

The next proposition establishes the boundedness on the half line of any solution in the $\mathcal{V}$-norm.

Proposition 3.5. *Let* $\{\varphi^t\}_{0\leq t<\infty}$ *satisfy* (A1) *and* (A2)*,* B *satisfy* (A3) *and* (A4)*. Suppose moreover that* $f\in S^2([t_0,\infty);\mathcal{H})$*. Then any solution* $u(t)(t\geq t_0)$ *of* (1.1) *satisfies the estimate*

$$\sup_{t\geq t_0}|u(t)|_{\mathcal{V}}\leq\max\{|u(t_0)|_{\mathcal{V}},l\},\sup_{t\geq t_0}\int_t^{t+1}|u'(\tau)|^2_{\mathcal{H}}d\tau\leq M_1,$$

where the constant l *does not depend on the solution, and* M_1 *depends on* $|u(t_0)|_{\mathcal{V}}$*.*

Proof. Multiply both sides of (1.1) by $u'(t)$. From Lemma 3.4, we obtain

$$\begin{aligned}&\frac{d}{d\tau}\varphi^\tau(u(\tau))+a_1|u'(\tau)|^2_{\mathcal{H}}\\&\leq a_2|f(\tau)|^2_{\mathcal{H}}+|\beta'(\tau)|\{1+\varphi^\tau(u(\tau))\}\\&\quad+|\alpha'(\tau)|||\partial\varphi^\tau(u(\tau))||\{1+(\varphi^\tau(u(\tau)))^{\frac{1}{p}}\}+a_3\varphi^\tau(u(\tau))+a_4|\gamma_3(\tau)|\end{aligned}\tag{3.2}$$

a.e. $\tau\geq t_0$. Using Young inequality we obtain

$$\begin{aligned}&\frac{d}{d\tau}\varphi^\tau(u(\tau))+\nu|u'(\tau)|^2_{\mathcal{H}}\\&\leq\mu|f(\tau)|^2_{\mathcal{H}}+\theta|\beta'(\tau)|\{1+\varphi^\tau(u(\tau))\}\\&\quad+\theta|\alpha'(\tau)|^2\{1+(\varphi^\tau(u(\tau)))\}+\zeta\varphi^\tau(u(\tau))+\delta|\gamma_3(\tau)|\quad;\text{a.e. }\tau\geq t_0,\end{aligned}\tag{3.3}$$

Note that all the constants appearing in the previous inequality are positive. We put $A(\tau)=\int_{t_0}^\tau(|\alpha'(\sigma)|^2+|\beta'(\sigma)|)d\sigma$ and multiply both sides of this inequality by $e^{-A(\tau)}$,

$$\begin{aligned}&\frac{d}{d\tau}[e^{-A(\tau)}\varphi^\tau(u(\tau))]+\nu e^{-A(\tau)}|u'(\tau)|^2_{\mathcal{V}^*}\\&\leq\mu e^{-A(\tau)}|f(\tau)|^2_{\mathcal{H}}+\theta\{|\alpha'(\tau)|^2+|\beta'(\tau)|\}e^{-A(\tau)}+\{\zeta\varphi^\tau(u(\tau))+\delta|\gamma_3(\tau)|\}e^{-A(\tau)}\end{aligned}\tag{3.4}$$

a.e. $\tau\geq t_0$. Integrating (3.4) over $[s,t]$, $t>s\geq t_0$ yields

$$\begin{aligned}&e^{-A(t)}\varphi^t(u(t))-e^{-A(s)}\varphi^s(u(s))+\nu\int_s^te^{-A(\tau)}|u'(\tau)|^2_{\mathcal{H}}d\tau\\&\leq\int_s^te^{-A(\tau)}\{\mu|f(\tau)|^2_{\mathcal{H}}+\theta[|\alpha'(\tau)|^2+|\beta'(\tau)|]\}d\tau\\&\quad+\int_s^t\{\zeta\varphi^\tau(u(\tau))+\delta|\gamma_3(\tau)|\}e^{-A(\tau)}d\tau,\end{aligned}\tag{3.5}$$

where ζ and δ depend on c_2. Assume that the half-line contains an interval $\Delta = [t_1 - 1, t_1]$ such that

$$\Gamma = |u(t_1)|_{\mathcal{V}} = \max_{t\in\Delta} |u(t)|_{\mathcal{V}}.$$

From Lemma 3.3 follows the boundedness of $\int_{\Delta} \varphi^{\tau}(u(\tau))d\tau$. Then we can find a t_2 in Δ such that $\varphi^{t_2}(u(t_2))$ is bounded. Now if in (3.5) we take $s = t_2$ and $t = t_1$, we conclude that Γ is bounded by some constant l_1. Hence, we obtain the inequality

$$\sup_{t\geq t_0} |u(t)|_{\mathcal{V}} \leq \max\{ \max_{t_0\leq t\leq 1+t_0} \{|u(t)|_{\mathcal{V}}\}, l_1\}.$$

Take $s = t_3 \in [t_0, 1 + t_0]$ so that $\varphi^{t_3}(u(t_3))$ is bounded, then estimating the expression $\max_{t_0\leq t\leq 1+t_0}\{|u(t)|_{\mathcal{V}}\}$ with the help of (3.5) again, we show that the first estimate holds. We multiply both sides of (3.3) by $e^{-A(\tau)}(\tau - s + 1)$, $\tau \geq s \geq t_0$, it follows that for a.e. $\tau \geq s$,

$$\begin{aligned}
&\frac{d}{d\tau}\{e^{-A(\tau)}(\tau - s + 1)\varphi^{\tau}(u(\tau))\} + \nu(\tau - s + 1)e^{-A(\tau)}|u'(\tau)|^2_{\mathcal{H}} \\
&\leq e^{-A(\tau)}(\tau - s + 1)[\mu|f(\tau)|^2_{\mathcal{H}} + \theta[|\alpha'(\tau)|^2 + |\beta'(\tau)|] + \zeta\varphi^{\tau}(u(\tau)) + \delta|\gamma_3(\tau)|].
\end{aligned} \tag{3.6}$$

Integrating (3.6) over $[s, s + 1]$ we obtain

$$\begin{aligned}
&\int_s^{s+1} |u'(\tau)|^2_{\mathcal{H}} d\tau \\
&\leq Ce^{A(s+1)-A(s)} \int_s^{s+1} \{|f(\tau)|^2_{\mathcal{H}} + |\alpha'(\tau)|^2 + |\beta'(\tau)| + \varphi^{\tau}(u(\tau)) + |\gamma_3(\tau)|\}\, d\tau,
\end{aligned}$$

for some positive constant C, which completes the proof. □

4. Proofs of main results

Proof of Theorem 2.1. Let (s_n) be a decreasing sequence of real numbers, $s_n \to -\infty$, and let u_n be the weak solution on $[s_n, \infty[$ of

$$u_n'(t) + \partial\varphi^t(u_n(t)) + B(t, u_n(t)) \ni f(t)\ ;\ u_n(s_n) = 0.$$

In view of the estimates in Proposition 3.5 and according to Ascoli's theorem, there is a subsequence $\{u_{n_k}\}$ which converges uniformly on every compact interval in $\mathbb{R}$ as $k \to \infty$ (as a $\mathcal{H}$-valued function). If $u(t)$ ($t \in \mathbb{R}$) is a limit function, thanks to Lemma 3.1 it must be a solution of equation (1.1). In the monotone case, let $z(t)(t \geq t_0)$ be the solution with the initial condition $z(t_0) = u(t_0)$. Since, $|z(t) - u_n(t)|_{\mathcal{H}} \leq |z(t_0) - u_n(t_0)|_{\mathcal{H}} \to 0$, we have $u(t) = z(t)(t \geq t_0)$. □

Proof of Theorem 2.2. The existence part is proved as in Theorem 2.1, here we prove the uniqueness of the bounded solution under the condition $m_t\gamma > 0$. Let y_1 and y_2 be two solutions of the inclusion (1.1), $\psi_0(t) = |y_1(t)-y_2(t)|^2_{\mathcal{H}}$. The function ψ_0 is bounded on $\mathbb{R}$ and $\psi_0'(t) \leq -\gamma(t)\psi_0(t)$. In view of Lemma 3.2, $\psi_0(t) \leq 0$ which implied the desired inequality $y_1 = y_2$. □

Proof of Remark 2.4. Consider now the case when $\gamma(t) \geq 0$ and $m_t\gamma > 0$ and the inclusion $\mathcal{V} \subset \mathcal{H}$ may not be compact. Let y_n be a sequence of solutions of (1.1) satisfying the conditions $y_n(-n) = y_n(n)$, $||y_n, L^{\infty}((-n, n); \mathcal{H})|| \leq r$. If $m > n$, the function $\psi(t) = |y_n(t) - y_m(t)|^2_{\mathcal{H}}(-n \le t \le n)$ satisfies the relations $\psi(-n) \le r^2$, $\psi'(t) \le -\gamma(t)\psi(t)$ from which the bound

$$\psi(t) \le -\int_n^t 2\gamma(s)ds$$

follows in a standard manner. Since $\gamma(t) \ge 0$ and $m_t\gamma > 0$, the same bound implies that the sequence y_n is a Cauchy sequence in $C(J, \mathcal{H})$. Now the existence of a bounded solution for the inclusion (1.1) is proved in the same manner as in the case of the compact inclusion $\mathcal{V} \subset \mathcal{H}$. □

To prove Theorem 2.5, we need the following lemma [5].

Lemma 4.1. *Let u and v be solutions of* (1.1) *on* $[t_0, \infty)$ *such that* $|u(t)-v(t)|_{\mathcal{H}} = d$ *for all* $t \ge t_0$ *where* d *is a nonnegative constant. Then for each* $\lambda \in (0,1)$, $\lambda u + (1-\lambda)v$ *becomes a solution of* (1.1) *on* $[t_0, \infty)$.

Proof of Theorem 2.5. An almost periodic solution to (1.1) is selected by the minimax principle. Denote by, $\mathcal{K}$ the set of all solutions v to (1.1) on $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} |v(t)|_{\mathcal{H}} \le \sup_{t\ge t_0} |u(t)|_{\mathcal{H}} \text{ and } v(t) \in F \text{ for all } t \in \mathbb{R}.$$

$\mathcal{K}$ is non empty. Further we put $\mu = inf_{v\in\mathcal{K}} I(v)$, where $I(v) = sup_{t\in\mathbb{R}}|v(t)|_{\mathcal{H}}$. Since I is lsc with respect to the pointwise convergence, $\mathcal{K}$ is being closed for this topology, the following statement is straightforward.

Proposition 4.2. *There is at least one element* $u^* \in \mathcal{K}$ *such that* $\mu = I(u^*)$.

We use Lemma 4.1 to obtain the following result.

Proposition 4.3. *There is at most one element* $v \in \mathcal{K}$ *such that* $\mu = I(v)$.

We state that u^* is $\mathcal{H}$-almost periodic, the proof of Theorem 2.5 will be complete. We proceed by contradiction. Suppose that u^* is not $\mathcal{H}$-almost periodic on $\mathbb{R}$. Then there exists a sequence t_n such that $u_n = u^*(t + t_n)$ does not contain any Cauchy subsequence in $L^\infty(\mathbb{R}; \mathcal{H})$. By the almost periodicity of φ^t, f and B, we may assume that

$$f(t - t_n) \to m(t) \quad \text{in } S^2(\mathbb{R}; \mathcal{V}),$$
$$\varphi^{t-t_n} \to \psi^t \quad \text{in } \Phi \text{ uniformly on } \mathbb{R},$$
$$B(t - t_n) \to C(t) \quad \text{in } \mathcal{B}$$

and $u^i(t - t_n) \to v^i$ in $\mathcal{H}$ uniformly on each compact interval in $\mathbb{R}$, for $i = 1, 2$ for some $\mathcal{V}-$ almost periodic function m on $\mathbb{R}$, C in $\mathcal{B}$ and applying the Bochner criterion to Stepanov almost periodic functions, we establish that Ψ_A is stable in the sense, ψ^t in B_A. Clearly ψ^t is Φ-almost periodic on $\mathbb{R}$, and $\psi^t \in \Phi_R$ for all $t \in \mathbb{R}$. Since u_n contains no Cauchy sequence in $L^\infty(\mathbb{R}; \mathcal{H})$, there are a sequence $\theta_k \in \mathbb{R}$, two subsequences t_{n_k} and t_{m_k} of t_n and $\varepsilon_0 > 0$ such that:

$$|u^*(\theta_k + t_{n_k}) - u^*(\theta_k + t_{m_k})|_{\mathcal{H}} \ge \varepsilon_0 \quad \text{for all } k.$$

Moreover we may assume that

$$m(t + \theta_k) \to \tilde{m}(t) \quad \text{in } S^2(\mathbb{R}; \mathcal{V}),$$
$$C(t + \theta_k) \to \tilde{C}(t) \quad \text{in } \mathcal{B},$$
$$\psi^{t+\theta_k} \to \hat{\psi}^t \quad \text{in } \Phi \text{ uniformly on } \mathbb{R},$$

when $k \to \infty$, for some $\mathcal{V}$-almost periodic function $\tilde{m}$, $\tilde{C}$ in $\mathcal{B}$ and a certain element $\hat{\psi}^t$ of B_A, such that $\hat{\psi}^t$ is Φ-almost periodic on $\mathbb{R}$. Then it is easy to see that

$$f(t+\theta_k+t_{n_k}) \to \tilde{m}(t) \quad \text{and} \quad f(t+\theta_k+t_{m_k}) \to \tilde{m}(t) \quad \text{in } S^2(\mathbb{R};\mathcal{V}),$$
$$B(t+\theta_k+t_{n_k}) \to \tilde{C}(t) \quad \text{and} \quad B(t+\theta_k+t_{m_k}) \to \tilde{C}(t) \quad \text{in } \mathcal{B},$$
$$\varphi^{t+\theta_k+t_{n_k}} \to \hat{\psi}^t \quad \text{and} \quad \varphi^{t+\theta_k+t_{m_k}} \to \hat{\psi}^t \quad \text{in } \Phi, \text{ uniformly on } \mathbb{R}$$

when $k \to \infty$. Taking Proposition 3.5 into account, we may assume that

$$u^*(t+\theta_k+t_{n_k}) \to w_1(t) \quad \text{and} \quad u^*(t+\theta_k+t_{m_k}) \to w_2(t)$$

in $\mathcal{H}$ uniformly on each compact interval in $\mathbb{R}$ when $k \to \infty$, w_1 and w_2 are in $C(\mathbb{R};\mathcal{H})$; we observe that: $|w_1(0) - w_2(0)|_{\mathcal{H}} \geq \varepsilon_0$, where w_1 and w_2 are solutions of (1.1) on $\mathbb{R}$ satisfying

$$w_1(t), w_2(t) \in F \quad \text{for all } t \in \mathbb{R},$$
$$\mu = I(w_1) = I(w_2).$$

Finally we choose a sequence $\tau_k \to \infty$ so that

$$\tilde{m}(t-\tau_k) \to f(t) \quad \text{in } S^2(\mathbb{R};\mathcal{V}),$$
$$\tilde{C}(t-\tau_k) \to B(t) \quad \text{in } \mathcal{B},$$
$$\hat{\psi}^{t-\tau_k} \to \varphi^t \quad \text{in } \Phi, \text{ uniformly on } \mathbb{R},$$

and $w_i(t-\tau_k) \to w_i^*(t)$ in $\mathcal{H}$ uniformly on each compact interval in $\mathbb{R}$, for some $w_i^* \in \mathcal{K}$ $(i = 1, 2)$. Since $I(w_1^*) = I(w_2^*)$ it follows that $w_1^* = w_2^*$. On the other hand,

$$\begin{aligned} 0 = |w_1(t) - w_2(t)|_{\mathcal{H}} &= lim_{k\to\infty} |w_1(t-\tau_k) - w_2(t-\tau_k)|_{\mathcal{H}} \\ &\geq |w_1(0) - w_2(0)|_{\mathcal{H}} \geq \varepsilon_0, \end{aligned}$$

which is a contradiction. □

Proof of Theorem 2.7. By Proposition 3.5, any solution u of (1.1) has a precompact trajectory, then we apply the result of [3]. □

Proof of Theorem 2.9. Theorems 2.1 and 2.2 ensure the existence and eventually the uniqueness of a bounded solution. By [2], this solution is actually almost periodic. □

5. Further comments and applications

Let $A(t)$ be a maximal monotone operator, we recall that $J_\lambda(t) = (I + \lambda A(t))^{-1}$ for $\lambda > 0$ is the resolvent of $A(t)$ and the Yosida approximation $A_\lambda(t) = \frac{I - J_\lambda(t)}{\lambda}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$; in the special case $A(t) = \partial\varphi^t$, the function defined by $\varphi^t_\lambda(x) = \varphi^t(J_\lambda(t)x) + \frac{1}{2\lambda}|x - J_\lambda(t)x|^2$ is Fréchet differentiable and $A_\lambda(t) = \partial\varphi^t_\lambda$.

In [1], the authors consider the inclusion $\frac{du}{dt} + A(t)u + B(t)u \ni 0$ in a Banach space W (more general setting) where $\overline{D(A(t))} = D(B(t)) = W$, $B(t)$ is assumed to be Lipschitzian and almost periodic, $A(t)$ is in a class $\mathcal{A}(\omega(t))$ defined by

$$|(x + \lambda A(x)) - (y + \lambda A(y))|_W \geq (1 - \lambda\omega(t))|x - y|_W.$$

They assume that $\limsup_{|t|\to\infty} \omega(t) < 0$ and $\sup_{t\in\mathbb{R}} \omega(t) < \infty$, the almost periodic dependence of the operator $A(t)$ is traduced in terms of its Yosida approximant.

By a fixed point method, they show that the perturbed equation has exactly one almost periodic solution $u(t)$.

We shall exemplify the applicability of our abstract theorems to heat equations.

Example 5.1. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\Gamma = \partial\Omega$. If $b(t)$ is an almost periodic function and $m_t b > 0$, one could establish the existence and uniqueness of bounded and almost periodic solutions of the problem

$$\begin{gathered} u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial x_i}\big) + b(t)u = f(x,t), \quad x \in \Omega,\ t \in \mathbb{R}, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \in \mathbb{R}. \end{gathered} \tag{5.1}$$

Example 5.2. The heat equations in bounded regions with almost periodic moving boundaries. Let $Q(t)$ be a bounded domain in $\mathbb{R}^n_x$ with smooth boundary $\Gamma(t)$ for each t. Put $Q(r,s) = \bigcup_{r<t<s}(Q(t) \times t)$.

- $Q(t)$ is almost periodic.
- For each t, the boundary $\Gamma(t)$ of $Q(t)$ is a $(n-1)$ dimensional sufficiently smooth manifold (say, of class C^3).
- Q is covered by m slices $Q(s_i,t_i)(i = 1,2,\dots,m)$ such that for each slice $Q(s_i,t_i)$ is mapped onto the cylindrical domain $Q(s) \times (s_i,t_i)$ by a diffeomorphism Y_i which is of class C^3 up to the boundary and preserves the time coordinate t.

Let $2 + \alpha > p$ and the following condition be satisfied,

$$\begin{gathered} -1 < \alpha < \infty \quad \text{if } n \leq p \\ -1 < \alpha < \frac{np}{2(n-p)-1} \quad \text{if } n > p. \end{gathered} \tag{5.2}$$

Our abstract framework can deal with the following problem,

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) = \Delta_p u + |u|^{\alpha}u + f(x,t)\} \quad \text{in } Q \\ u(x,t) = 0 \quad \text{on } \Gamma, \end{gathered} \tag{5.3}$$

where Δ_p is the nonlinear Laplace operator defined by

$$\Delta_p = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}), \quad p \geq 2.$$

Ôtani [8] also studies the Navier-Stokes equations in regions with moving boundaries.

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