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EXISTENCE OF THREE SOLUTIONS FOR HIGHER ORDER BVP WITH PARAMETERS VIA MORSE THEORY

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Abstract. We prove the existence of at least three solutions to a general Lidstone problem using the Morse Theory.

1. Introduction

In 1929, Lidstone introduced a generalization of the Taylor series. It approximates a given function in the neighbourhood of two points instead of one. Thus, the initial research was devoted to description of a maximal set of functions that could be expressed as a Lidstone series. Those considerations led to so-called functions of exponential type and next to general Lidstone boundary value problem (BVP) (comp. [1, 8, 9]), examined today in various configurations. Motivated by many papers (see for example ([5], [16]), in which authors studied the existence of multiple solutions to Lidstone BVP, and being inspired by ideas given by Chang [3, 4], we have decided to study the case of existence of at least three solutions to the BVP that is being described. The crucial results of our research are presented in this paper.

Coming to the point, we shall consider a special case of a general Lidstone BVP for a nonlinear ordinary differential equation of the $2k$-th order studied earlier in [11]:

$$\begin{gathered} x^{(2k)} - \sum_{j=1}^{k} \lambda_j x^{(2k-2j)} = f(t, x, x'', \dots, x^{(2k-2)}), \\ x^{(2j)}(0) = 0 = x^{(2j)}(1), \quad j = 0, 1, \dots, k-1. \end{gathered} \tag{1.1}$$

It is a natural generalization of the beam equation with fixed ends (comp. [10]). The topological methods used in the mentioned papers admit arbitrary parameters $\lambda := (\lambda_1, \dots, \lambda_k)$ but need very restrictive conditions on an asymptotic behaviour of the nonlinear term f. The problem (1.1) is equivalent to the fixed point problem for completely continuous operator if the linear homogeneous problem ($f = 0$) has only the trivial solution. In [10], it has been shown that it is for λ which does not belong to a sequence of $(k-1)$-dimensional hyperplanes; the paper contains mainly the case when λ sits in this sequence – the resonant case.

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Here, we study a special nonlinear case of the general Lidstone problem, that takes the form

$$\begin{gathered} (-1)^k x^{(2k)} + \sum_{j=1}^{k} \lambda_j x^{(2k-2j)} = (-1)^{i-1} f(t, x^{(2i-2)}) \\ x^{(2j)}(0) = x^{(2j)}(1) = 0, \quad j = 0, \dots, k-1, \end{gathered} \tag{1.2}$$

with fixed parameters $k \geq 2$ and $i \in \{1, \dots, k-1\}$. Note that continuous f depends on one of the even order derivatives only. Then, the problem is equivalent to looking for a critical point of a C^1-functional on a Hilbert space. We have considered it in [14] and we have found assumptions that guarantee the existence of at least one solution. Furthermore, in [13] one of the authors has answered a question concerning existence of infinitely many solutions to (1.2). After joining those results a natural problem about intermediate case arose. Actually, this is the goal of the paper to fill this hiatus, what means to formulate the assumptions that lead us to the conclusion of existence no fewer than three solutions.

For a vector of real coefficients $\lambda = (\lambda_1, \dots, \lambda_n)$, let us define

$$\Lambda(n) := \sum_{j=1}^{k} (-1)^{k-j} \lambda_j (n^2\pi^2)^{k-j},$$

$H_n := \{\lambda \in \mathbb{R}^k : \Lambda(n) + (n^2\pi^2)^k = 0\}$ for $n = 1, 2, \dots$. Then the linear homogeneous problem has nontrivial solutions if and only if

$$\lambda \in \sigma^k := \bigcup_{n=1}^{\infty} H_n.$$

We shall assume that $\lambda \in \Delta_+$, where

$$\Delta_+ := \cap_{n=1}^{\infty} \{\lambda : \Lambda(n) \geq 0\}.$$

This set is nonempty and even large since it contains all λ such that $(-1)^j \lambda_j \geq 0$ for any j if k is even and $(-1)^j \lambda_j \leq 0$ if k is odd. Obviously, $\Delta_+ \cap \sigma^k = \emptyset$. It can be proved that (1.2) is equivalent (comp. [14]) to

$$x^{(2i-2)}(t) = \int_0^1 \mathcal{H}_i(t,s) f(s, x^{(2i-2)}(s))\, ds,$$

where

$$\mathcal{H}_i(t,s) := 2 \sum_{n=1}^{\infty} \frac{(n^2\pi^2)^{i-1}}{\Lambda(n) + (n^2\pi^2)^k} \sin(n\pi s) \sin(n\pi t).$$

The last formula is obtained by using the spectral theory of compact self-adjoint operators in Hilbert space. Hence, one should find fixed points of an operator which is a superposition of the Nemytskiĭ operator $\mathbf{f}$ defined by f and the linear integral operator $\mathbf{H}$ with the kernel $\mathcal{H}_i$. Denote this superposition as $T : C[0,1] \to C[0,1]$. The spectrum of H is composed of eigenvalues

$$\mu_n^2 := \frac{(n^2\pi^2)^{i-1}}{\Lambda(n) + (n^2\pi^2)^k}, \quad n = 1, 2, \dots$$

and the limit of this sequence – 0 which belongs to the continuous spectrum. We can define H by the same formula on the Hilbert space $L^2(0,1)$. The assumption $\lambda \in \Delta_+$ is necessary to get $H \geq 0$ that enables us to define a square root $S :$

$L^2(0,1) \to L^2(0,1)$ – a nonnegative linear operator such that $S^2 = H$. From the spectral theory (comp. [7]), we know that S is an integral operator with the kernel

$$\mathcal{S}(t,s) := \sum_{n=1}^{\infty} \frac{2(n\pi)^{i-1}}{\sqrt{\Lambda(n) + (n^2\pi^2)^k}} \sin(n\pi s)\sin(n\pi t).$$

In [13], it is proved that fixed points of T are exactly critical points of the functional $\varphi : L^2(0,1) \to \mathbb{R}$,

$$\varphi(y) := \frac{1}{2}\|y\|^2 - \int_0^1 F(t, Sy(t))\, dt,$$

since

$$\varphi'(y) \cdot u = \langle y, u \rangle - \langle \mathbf{f}(Sy), Su \rangle,$$

where $F(t,w) = \int_0^w f(t,u)\, du$, $\langle \cdot, \cdot \rangle$ stands for the scalar product in $L^2(0,1)$, and S is self-adjoint.

In [14], we proved the following result.

Theorem 1.1. *Assume that $\lambda \in \Delta_+$ and let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the following conditions*

(i) $\limsup_{u \to 0} \frac{f(t,u)}{u} < \mu_1^{-2}$,
(ii) $\liminf_{u \to +\infty} \frac{f(t,u)}{u} > \mu_1^{-2}$
(iii) *there exist $k \in [0, 1/2)$ and $N > 0$, such that for $\|w\| \geq N$, we have*

$$F(t,w) \leq kwf(t,w),$$

uniformly with respect to $t \in [0,1]$.

Then problem (1.2) possesses at least one nonzero solution.

If $f(t,\cdot)$ is odd for any t and the limit in condition (ii) equals $+\infty$, then in [13] it has been proved that φ has infinitely many critical points which are solutions of (1.2). We used slightly different notations in both papers but one can see that

$$\mu_1^2 = \Big(\frac{\Lambda(1) + \pi^{2k}}{\pi^{2i-2}}\Big)^{-1} = \|H\|.$$

Observe that condition (i) implies $f(t,0) \equiv 0$ and we get the null solution.

2. Main results

Now, we can formulate the main result.

Theorem 2.1. *Let f be continuous, and assume the conditions:*

(C1) *there exist an integer $m \geq 1$ and the limits, for any t, such that*

$$\frac{\partial f}{\partial x}(t,0) := \lim_{u \to 0} \frac{f(t,u)}{u} \in (\mu_m^{-2}, \mu_{m+1}^{-2});$$

(C2) *the infinite system of linear equations*

$$c_j = 2\mu_j \sum_{n=1}^{\infty} \mu_n \int_0^1 \frac{\partial f}{\partial x}(t,0) \sin(n\pi t)\sin(j\pi t)\, dt\; c_n, \quad j = 1, 2, \dots,$$

has only trivial solutions: $c_j = 0$, $j = 1, 2, \dots$.

(C3) *there exist $0 < \alpha < \mu_1/2$, $\beta \in \mathbb{R}$ such that $F(t,u) \leq \alpha u^2 + \beta$ for any t, u.*

Then (1.2) has at least three solutions.

Proof. We apply [4, Thm. 5.4] (first slightly weaker result appears in [3]) which gives at least three critical points for C^1-functional ψ defined on the Hilbert space, if ψ is bounded below, it satisfies Palais-Smale condition and has a nondegenerate critical point different from the argument of its minimum with finite Morse index. Here, $L^2(0,1)$ is the Hilbert space, φ is the functional and the null solution is a critical point with finite Morse index.

Condition (C3) leads to the estimate

$$\varphi(y) \geq \frac{1}{2}\|y\|^2 - \int_0^1 (\alpha(Sy)(t)^2 + \beta)\,dt \geq \big(\frac{1}{2} - \alpha\mu_1^2\big)\|y\|^2 - \beta$$

that implies values of φ are bounded from below by a quadratic function with finite minimum. It gives that φ is bounded from below and coercive. Let (y_n) be a Palais-Smale sequence, i.e. $|\varphi(y_n)| \leq M$ and $\varphi'(y_n) = y_n - S(\mathbf{f}(S(y_n))) \to 0$. Since S is compact, $S(\mathbf{f}(S(y_n)))$ has a convergent subsequence and the Palais-Smale condition is satisfied.

Let us observe that $\varphi'(0) = 0$ and there exists the second derivative of this functional at 0:

$$\varphi''(0)(u,v) = \langle u,v\rangle - \int_0^1 \frac{\partial f}{\partial x}(t,0)\cdot Su(t)\cdot Sv(t)\,dt = \langle u,v\rangle - \langle \frac{\partial f}{\partial x}(\cdot,0)\cdot Su, Sv\rangle.$$

This bilinear functional on $L^2(0,1)$ is symmetric and continuous which means that it defines a self-adjoint operator L on $L^2(0,1)$ given by $\varphi''(0)(u,v) = \langle Lu, v\rangle$. It follows that

$$Lu = u - S\Big(\frac{\partial f}{\partial x}(\cdot,0)\cdot Su\Big).$$

If we denote by $e_n(t) := \sqrt{2}\cdot\sin(n\pi t)$ for $n = 1,2,\dots$, then it is a complete orthonormal basis of eigenfunctions of S corresponding eigenvalues μ_n. Let H_- denotes a subspace spanned by e_n with $n \leq m$ and H_+ its orthogonal complement. We shall show that L is one-to-one. If it is not the case, there exists $u = \sum_n c_n e_n \neq 0$, $Lu = 0$ and then

$$u = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \mu_j c_n \mu_n \int_0^1 \frac{\partial f}{\partial x}(s,0)e_j(s)e_n(s)\,ds\cdot e_j. \tag{2.1}$$

that implies the system from (C2).

Since $L = I - S_1$ where $S_1 u = S(\frac{\partial f}{\partial x}(\cdot,0)\cdot Su)$ is compact, then the range of L is closed with codimension 0. It follows that L is an isomorphism of $L^2(0,1)$ onto itself and 0 is a nondegenerate critical point of φ. We shall show that

$$\varphi''(0)(x,x) \leq -C\|x\|^2, \quad x \in H_-,$$
$$\varphi''(0)(x,x) \geq C\|x\|^2, \quad x \in H_+,$$

for some positive constant C.

From (C1) there exists $\xi > 0$ such that

$$\mu_m^{-2} + \xi \leq \frac{\partial f}{\partial x}(t,0) \leq \mu_{m+1}^{-2} - \xi, \quad t \in [0,1].$$

Then

$$\varphi''(0)(x,x) = \|x\|^2 - \int_0^1 \frac{\partial f}{\partial x}(t,0)|Sx(t)|^2\,dt$$

and, for $x \in H_-$,

$$\varphi''(0)(x,x) \le \|x\|^2 - (\mu_m^{-2} + \xi)\|Sx\|^2 \le -\xi\mu_m^2\|x\|^2$$

and similarly, for $x \in H_+$,

$$\varphi''(0)(x,x) \ge \|x\|^2 - (\mu_{m+1}^{-2} - \xi)\|Sx\|^2 \ge \xi\mu_{m+1}^2\|x\|^2.$$

This means that $L|H_-$ is negative and $L|H_+$ is positive and the Morse index of the critical point 0 equals m so it is finite.

It remains to show that there exists x such that $\varphi(x) < 0 = \varphi(0)$. But from $(C1)$ one can take $\xi > 0$ and $\varepsilon > 0$ such that $f(t,u) \ge (\mu_m^{-2} + \xi)u$ for any t and $|u| \le \varepsilon$. It follows that $F(t,u) \ge \frac{1}{2}(\mu_m^{-2} + \xi)u^2$ for $|u| \le \varepsilon$. Then take $0 \ne y \in H_-$ such that $\sup_t |Sy(t)| \le \varepsilon$ and get

$$\varphi(y) = \frac{1}{2}\|y\|^2 - \int_0^1 F(t, Sy(t))\,dt \le -\frac{\xi\mu_m^2}{2}\|y\|^2 < 0.$$

The proof is complete. □

Remarks. If $\frac{\partial f}{\partial x}(t,0)$ does not depend on $t \in [0,1]$, then condition (C2) is satisfied since

$$\int_0^1 \frac{\partial f}{\partial x}(t,0)\sin(n\pi s)\sin(j\pi s)\,ds = 0$$

for $n \ne j$. In other cases, condition (C2) can be verified by a finite algorithm. The condition is equivalent to:

$$1 \notin \sigma(S_1)$$

and this operator is compact selfadjoint, hence its eigenvalues can be obtained by the Courant-Hilbert method (see [7]), and the sequence of eigenvalues of S_1 tends to 0,therefore only finite number of them can be greater then 1. The main result can be obtained (essentially with the same proof) by using [2, Corollary 3] where only the Leray-Schauder degree is applied.

Corollary 2.2. *The equation describing a shape of a beam freely supported on both ends:*

$$u^{(4)} - \lambda_1 u'' + \lambda_2 u = f(t,u), \quad u(0) = u(1) = u''(0) = u''(1) = 0$$

with $\lambda_{1,2} \ge 0$ has at least three solutions if there exists $m \in \mathbb{N}$ such that

$$\lambda_2 + \lambda_1 m^2\pi^2 + m^4\pi^4 < \frac{\partial f}{\partial x}(t,0) < \lambda_2 + \lambda_1(m+1)^2\pi^2 + (m+1)^4\pi^4$$

for all t and (C2), (C3) *hold.*

A similar problem with $\lambda_1 = 0$, λ_2 depending on t and $f(t,u) = h(t)$ was recently studied in [6].

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