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BOUNDARY-VALUE PROBLEMS FOR WAVE EQUATIONS WITH DATA ON THE WHOLE BOUNDARY

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Abstract. In this article we propose a new formulation of boundary-value problem for a one-dimensional wave equation in a rectangular domain in which boundary conditions are given on the whole boundary. We prove the well-posedness of boundary-value problem in the classical and generalized senses. To substantiate the well-posedness of this problem it is necessary to have an effective representation of the general solution of the problem. In this direction we obtain a convenient representation of the general solution for the wave equation in a rectangular domain based on d'Alembert classical formula. The constructed general solution automatically satisfies the boundary conditions by a spatial variable. Further, by setting different boundary conditions according to temporary variable, we get some functional or functional-differential equations. Thus, the proof of the well-posedness of the formulated problem is reduced to question of the existence and uniqueness of solutions of the corresponding functional equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain, bounded by following lines: $AB : t = 0,\ 0 \le x \le \ell$; $BC : x = \ell,\ 0 \le t \le T$; $CD : t = T,\ 0 \le x \le \ell$; and $AD : x = 0,\ 0 \le t \le T$.

We consider a nonhomogeneous wave equation in Ω,

$$u_{tt} - u_{xx} = f(x,t). \tag{1.1}$$

It is well known that the Dirichlet problem for the wave equation (1.1) in a rectangular domain is ill-posed [4]. Specifically, in case of our domain Ω it is easy to see that the homogeneous equation (1.1) with Dirichlet conditions

$$u\big|_{AB\cup BC\cup AD} = 0, \tag{1.2}$$

$$u\big|_{CD} = 0 \tag{1.3}$$

has countable number of nontrivial solutions of the form $u_{mn}(x,t) = \sin\frac{m\pi x}{\ell}\sin\frac{n\pi t}{T}$, $m, n = 1, 2, \dots$, when the conditions $n\ell = mT$ hold.

The Dirichlet problem for a wave equation is one of the most difficult models of mathematical physics. The wave equation describes almost all types of small vibrations in distributional mechanical systems such as longitudinal sound vibrations in

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gas, fluid, solids; transverse waves in strings and etc. Components of electromagnetic vectors and potentials, and hence many electromagnetic phenomena (from quasistatics to optics) in some extent are explained by properties of solutions of wave equation.

Hadamard [5], Huber [6] for the first time noted nonuniqueness of solution of the Dirichlet problem for a wave equation. Bourgin and Duffin [3] considered Dirichlet problem for the one-dimensional equation (1.1) in a rectangle $\{0 \leq t \leq T,\ 0 \leq x \leq \ell\}$. By using Laplace transformation, they showed that if the number T/ℓ is irrational, then there is the uniqueness of the solution of the problem in the class of continuously differentiable functions with the second derivatives integrable according to Lebesgue.

There are many works that were dedicated to study Dirichlet problem for the string equation (see [12]). Arnold's survey [1] and Berezanskii's [2, Chap. IV] monograph give more detailed discussion of papers related to this topic. These papers show that the homogeneous Dirichlet problem has nontrivial solutions, if the ratio T/ℓ of the sides of the rectangle $\{0 \leq t \leq T,\ 0 \leq x \leq \ell\}$ (in which the solution of the Dirichlet problem for the string equation is sought) is a rational number. By using the method of separation of variables, the solution of inhomogeneous Dirichlet problem is constructed. In this process, small denominators which are hampering the series' convergence arise [13]. If the ratio T/ℓ of the sides is an algebraic number of degree $n \geq 2$ or an irrational number with bounded element, then, for sufficiently smooth boundary data (functions), the constructed series' convergence can be proved for the class of smooth solutions of the string equation.

In [11] the existence and uniqueness of generalized solution for a second-order hyperbolic equation with integral conditions in a rectangle are proved.

In [8] the uniqueness of solution of initial-boundary value problem for a one-dimensional wave equation is proved and it is shown that this solution coincides with the wave potential.

In [14]-[15] it is proved the well-posedness of boundary value problems for a one-dimensional wave equation in a rectangular domain in case when boundary conditions are given on the whole boundary of domain.

Also we note that lately interest has increased to the research of classical initial-boundary problems for a wave equation in rectangular domains in connection with problems of the optimization of boundary control of string vibrations (see [7, 9, 10]).

In this article, we prove the well-posedness of the problem for a one-dimensional wave equation in a rectangular domain in case when boundary conditions are given on the whole boundary of domain which generalizes results of [14]-[15].

2. Representation of solution of the first initial-boundary value problem

Hereafter, we will assume that $\ell/T \geq 2$.

Problem 1. Find a solution of equation (1.1) in Ω with the initial conditions

$$u(x,0) = \tau(x), \quad 0 \leq x \leq \ell, \tag{2.1}$$

$$u_t(x,0) = \nu(x), \quad 0 \leq x \leq \ell \tag{2.2}$$

and with boundary conditions

$$u(0,t) = 0, \quad u(\ell,t) = 0, \quad 0 \leq t \leq \frac{\ell}{2}. \tag{2.3}$$

Problem 1 is a classical first initial-boundary value problem. The solution of the Cauchy problem for (1.1) with initial conditions (2.1) and (2.2) exists and is unique. But it is uniquely defined not in all Ω, but only in its part $\Omega_1 = \{(x,t) : (x,t) \in \Omega, t \le x \le \ell - t\}$. And in the domain $\Omega \setminus \Omega_1$ the solution is not uniquely defined from the data of Cauchy (2.1), (2.2). It is uniquely defined only by using boundary conditions of considered problems.

Let $u(x,t)$ be a solution of Problem 1. We introduce a new function $\widetilde{u}(x,t)$ defined in $\widetilde{\Omega}$, containing initial domain Ω: $\widetilde{\Omega} = \{(x,t) : 0 \le t \le \frac{\ell}{2}, t - \frac{\ell}{2} \le x \le \frac{3\ell}{2} - t\}$.

The function $\widetilde{u}(x,t)$ is given by the formula

$$\tilde{u}(x,t) = \begin{cases} -u(-x,t), & -\frac{\ell}{2} \le x \le 0; \\ u(x,t), & 0 \le x \le \ell; \\ -u(2\ell - x, t), & \ell \le x \le \frac{3\ell}{2}. \end{cases} \tag{2.4}$$

Taking into account the boundary condition (2.3), it is easy to see that the function $\widetilde{u}(x,t)$ is continuous and continuously differentiable at the transition lines $x = 0$ and $x = \ell$. Since the function $u(x,t)$ is smooth in Ω, then the function $\widetilde{u}(x,t)$ is smooth in $\widetilde{\Omega}$.

Let us find an equation in which the function $\widetilde{u}(x,t)$ satisfies that equation in $\widetilde{\Omega}$. By direct calculation, it is easy to see that this function satisfies the following nonhomogeneous wave equation in $\widetilde{\Omega}$

$$\widetilde{u}_{tt} - \widetilde{u}_{xx} = \widetilde{f}(x,t), \tag{2.5}$$

where

$$\widetilde{f}(x,t) = \begin{cases} -f(-x,t), & -\frac{\ell}{2} \le x \le 0; \\ f(x,t), & 0 \le x \le \ell; \\ -f(2\ell - x, t), & \ell \le x \le \frac{3\ell}{2}. \end{cases} \tag{2.6}$$

From the initial conditions (2.1), (2.2), taking into account (2.4), we obtain initial conditions for the function $\widetilde{u}(x,t)$ in $\widetilde{\Omega}$:

$$\widetilde{u}(x,0) = \widetilde{\tau}(x),\ 0 \le x \le \ell, \tag{2.7}$$

$$\widetilde{u}_t(x,0) = \widetilde{\nu}(x),\ 0 \le x \le \ell, \tag{2.8}$$

where functions $\widetilde{\tau}(x)$ and $\widetilde{\nu}(x)$ are given by the equalities

$$\widetilde{\tau}(x) = \begin{cases} -\tau(-x), & -\frac{\ell}{2} \le x \le 0; \\ \tau(x), & 0 \le x \le \ell; \\ -\tau(2\ell - x), & \ell \le x \le \frac{3\ell}{2}. \end{cases} \tag{2.9}$$

$$\widetilde{\nu}(x) = \begin{cases} -\nu(-x), & -\frac{\ell}{2} \le x \le 0; \\ \nu(x), & 0 \le x \le \ell; \\ -\nu(2\ell - x), & \ell \le x \le \frac{3\ell}{2}. \end{cases} \tag{2.10}$$

In $\widetilde{\Omega}$ the solution of the Cauchy problem (2.5), (2.7), (2.8) exists, is unique and expressed by the classical formula of d'Alambert:

$$\widetilde{u}(x,t) = \frac{\widetilde{\tau}(x+t) + \widetilde{\tau}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{\nu}(\xi) d\xi + \frac{1}{2}\int_0^t \int_{x-t+\eta}^{x+t-\eta} \widetilde{f}(\xi,\eta) d\xi\, d\eta. \tag{2.11}$$

By direct calculation, it is easy to check that the function $\widetilde{u}(x,t)$ satisfies equation (2.5) and the initial conditions (2.7) and (2.8).

Now we show that by (2.9), (2.10), and taking into account (2.6), the function $\widetilde{u}(x,t)$ satisfies the boundary condition (2.3) of Problem 1.

We calculate

$$\widetilde{u}(0,t)=\frac{\widetilde{\tau}(t)+\widetilde{\tau}(-t)}{2}+\frac{1}{2}\int_{-t}^{t}\widetilde{\nu}(\xi)d\xi+\frac{1}{2}\int_{0}^{t}\int_{-t+\eta}^{t-\eta}\widetilde{f}(\xi,\eta)d\xi\,d\eta. \tag{2.12}$$

By (2.9) it is easy to obtain

$$\frac{\widetilde{\tau}(t)+\widetilde{\tau}(-t)}{2}=\frac{\tau(t)-\tau(t)}{2}=0. \tag{2.13}$$

From (2.10) by a simple change of variables in the integral we obtain

$$\begin{aligned}\frac{1}{2}\int_{-t}^{t}\widetilde{\nu}(\xi)d\xi&=\frac{1}{2}\int_{-t}^{0}\widetilde{\nu}(\xi)d\xi+\frac{1}{2}\int_{0}^{t}\widetilde{\nu}(\xi)d\xi\\&=\frac{1}{2}\int_{t}^{0}\nu(\xi)d\xi+\frac{1}{2}\int_{0}^{t}\nu(\xi)d\xi=0.\end{aligned} \tag{2.14}$$

In the third summand in (2.12) we shall make obvious change of variables. Since $0\le t-\eta\le\frac{\ell}{2},-\frac{\ell}{2}\le\eta-t\le 0$, then we obtain

$$\begin{aligned}&\frac{1}{2}\int_{0}^{t}\int_{-t+\eta}^{t-\eta}\widetilde{f}(\xi,\eta)d\xi\,d\eta\\&=\frac{1}{2}\int_{0}^{t}\int_{-t+\eta}^{0}\widetilde{f}(\xi,\eta)d\xi\,d\eta+\frac{1}{2}\int_{0}^{t}\int_{0}^{t-\eta}\widetilde{f}(\xi,\eta)d\xi\,d\eta\\&=\frac{1}{2}\int_{0}^{t}\int_{t-\eta}^{0}f(\xi,\eta)d\xi\,d\eta+\frac{1}{2}\int_{0}^{t}\int_{0}^{t-\eta}f(\xi,\eta)d\xi\,d\eta=0.\end{aligned} \tag{2.15}$$

Summing in (2.13)-(2.15), we obtain from (2.12), that $\widetilde{u}(0,t)=0$. That is, the first boundary condition of (2.3) is fulfilled.

Similarly we check the fulfilling of the second boundary condition from (2.3).

Hence, the formula (2.11) gives the solution of Problem 1. Let us write its solution in Ω by functions f,τ,ν. For that, we substitute values $\widetilde{f},\widetilde{\tau},\widetilde{\nu}$ into formula (2.11) expressed by formulas (2.6),(2.9) and (2.10).

Let us introduce notation: $\Omega_1=\{(x,t):(x,t)\in\Omega,t<x<\ell-t\}$, $\Omega_2=\{(x,t):(x,t)\in\Omega,x<t\}$ and $\Omega_3=\{(x,t):(x,t)\in\Omega,x+t>\ell\}$. Then by direct calculation we obtain representation of solution of Problem 1.

In Ω_2:

$$\begin{aligned}\widetilde{u}(x,t)&=\frac{\tau(x+t)-\tau(t-x)}{2}+\frac{1}{2}\int_{t-x}^{x+t}\nu(\xi)d\xi\\&\quad+\frac{1}{2}\int_{0}^{t-x}\int_{t-x-\eta}^{x+t-\eta}f(\xi,\eta)d\xi\,d\eta+\frac{1}{2}\int_{t-x}^{t}\int_{x-t+\eta}^{x+t-\eta}f(\xi,\eta)d\xi\,d\eta.\end{aligned} \tag{2.16}$$

In Ω_1:

$$\widetilde{u}(x,t)=\frac{\tau(x+t)+\tau(x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\nu(\xi)d\xi+\frac{1}{2}\int_{0}^{t}\int_{x-t+\eta}^{x+t-\eta}f(\xi,\eta)d\xi\,d\eta. \tag{2.17}$$

In Ω_3:

$$
\begin{aligned}
&\widetilde{u}(x,t)\\
&= \frac{\tau(x-t)-\tau(2\ell-x-t)}{2} + \frac{1}{2}\int_{x-t}^{2\ell-x-t} \nu(\xi)d\xi\\
&\quad + \frac{1}{2}\int_0^{x+t-\ell}\int_{x-t+\eta}^{2\ell-x-t+\eta} f(\xi,\eta)d\xi\,d\eta + \frac{1}{2}\int_{x+t-\ell}^{t}\int_{x-t+\eta}^{x+t-\eta} f(\xi,\eta)d\xi\,d\eta.
\end{aligned} \tag{2.18}
$$

3. Main result and its proof

Let $E=(T,T)$, $F=(\ell-T,T)$ be points on a boundary CD.

Problem 2. Find a solution of equation (1.1), satisfying the boundary condition (1.2) and conditions on the boundary CD:

$$u_t\big|_{DE}=0, \tag{3.1}$$

$$\alpha u_x+\beta u_t\big|_{EF}=0, \tag{3.2}$$

$$u\big|_{CF}=0, \tag{3.3}$$

where α and β are real numbers.

As usual, we say the function $u\in L_2(\Omega)$ is a *strong solution* of Problem 2, if there exists the sequence of functions $u_n\in W_2^2(\Omega)$, satisfying boundary conditions of Problem 2 such that u_n and Lu_n converge in $L_2(\Omega)$ to u and f respectively.

Let us denote by A the matrix

$$
\begin{pmatrix}
\beta-\alpha & 0 & \alpha+\beta & 0 & \dots & \dots & \dots & 0 & 0 & 0\\
0 & \beta-\alpha & 0 & \alpha+\beta & 0 & \dots & \dots & 0 & 0 & 0\\
0 & 0 & \beta-\alpha & 0 & \alpha+\beta & 0 & \dots & 0 & 0 & 0\\
\vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\
0 & \dots & \dots & 0 & \dots & \dots & \dots & 0 & \dots & 0\\
0 & 0 & 0 & \dots & 0 & \dots & \dots & . & \dots & 0\\
0 & 0 & \dots & \dots & \dots & 0 & \beta-\alpha & 0 & \alpha+\beta & 0\\
0 & 0 & \dots & \dots & \dots & & 0 & \beta-\alpha & 0 & \alpha+\beta\\
0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & -1 & -1\\
-1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0
\end{pmatrix}
$$

and assume that $\det A\neq 0$. We denote the inverse of a matrix A by $B=A^{-1}$ and elements of a matrix B by $\{b_{ij}\}_{i,j=\overline{1,n}}$.

The closest even and odd numbers to n (including n) are denoted by p and q respectively. Let K be a broken line in Ω, consisting of segments of the characteristics $x=2iT+t$, $i=0,1,\dots,[\frac{n-1}{2}]$ and $x=2jT-t$, $j=1,\dots,[\frac{n}{2}]$. Here $[z]$ denotes the integer part of z.

Also, let us introduce some notation:

$$
\begin{gathered}
\Phi_1(x)=-F'_{1t}(x,T),\quad 0\le x\le T,\\
\Phi_{2j}(x)=-\alpha F'_{2x}(2jT-x,T)-\beta F'_{2t}(2jT-x,T),\\
j=1,2,\dots,[\frac{n-2}{2}]+1, 2j<n, 0\le x\le T,\\
\Phi_{2j+1}(x)=-\alpha F'_{2x}(2jT+x,T)-\beta F'_{2t}(2jT+x,T),\\
j=1,2,\dots,[\frac{n-2}{2}], 0\le x\le T,
\end{gathered}
$$

$$\Phi_n(x) = \begin{cases} -F'_{3x}(nT - x, T), & \text{if } n \text{ is even} \\ -F'_{3x}((n-1)T + x, T), & \text{if } n \text{ is odd}, \end{cases} \quad 0 \le x \le T,$$

$$F_1(x,t) = \int_0^{t-x} \int_{t-x-\eta}^{x+t-\eta} f(\xi,\eta) d\xi\, d\eta + \int_{t-x}^{t} \int_{x-t+\eta}^{x+t-\eta} f(\xi,\eta) d\xi\, d\eta,$$

$$F_2(x,t) = \int_0^{t} \int_{x-t+\eta}^{x+t-\eta} f(\xi,\eta) d\xi, d\eta,$$

$$F_3(x,t) = \int_0^{x+t-\ell} \int_{x-t+\eta}^{2\ell-x-t+\eta} f(\xi,\eta) d\xi\, d\eta + \int_{x+t-\ell}^{t} \int_{x-t+\eta}^{x+t-\eta} f(\xi,\eta) d\xi\, d\eta.$$

Theorem 3.1. *Let $\ell/T = n \ge 2$ be a positive integer. A solution of the Problem 2 is unique, if and only if*

$$\alpha(\alpha+\beta)(\alpha-\beta) \neq 0\,. \tag{3.4}$$

If this (3.4) holds, then:

(a) For all functions $f \in L_2(\Omega)$ Problem 2 have a unique strong solution. This solution belongs to the class $u \in W_2^1(\Omega) \bigcap C(\overline{\Omega})$ and satisfies the estimate

$$\|u\|_{W_2^1(\Omega)} \le C\|f\|_{L_2(\Omega)}. \tag{3.5}$$

(b) If $f \in C^1(\overline{\Omega})$, then the strong solution of Problem 2 belongs to the class $u \in C^2(\overline{\Omega}\backslash K) \bigcap C(\overline{\Omega})$.

(c) If $f \in C^1(\overline{\Omega})$, then the strong solution of Problem 2 is classical, i.e. $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$, if and only if conditions (3.6) and (3.7) hold:

$$\begin{gathered} \begin{pmatrix} b_{21} - (-1)^i b_{11} & b_{22} - (-1)^i b_{12} & \dots & b_{2n} - (-1)^i b_{1n} \\ b_{41} - (-1)^i b_{31} & b_{42} - (-1)^i b_{32} & \dots & b_{4n} - (-1)^i b_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-21} - (-1)^i b_{p-31} & b_{p-22} - (-1)^i b_{p-32} & \dots & b_{p-2n} - (-1)^i b_{p-3n} \\ b_{p1} - (-1)^i b_{p-11} & b_{p2} - (-1)^i b_{p-12} & \dots & b_{pn} - (-1)^i b_{p-1n} \end{pmatrix} \\ \times \begin{pmatrix} \Phi_1^{(i)}(0) \\ \Phi_2^{(i)}(0) \\ \vdots \\ \Phi_n^{(i)}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 0, 1; \end{gathered} \tag{3.6}$$

$$\begin{gathered} \begin{pmatrix} b_{31} - (-1)^i b_{21} & b_{32} - (-1)^i b_{22} & \dots & b_{3n} - (-1)^i b_{2n} \\ b_{51} - (-1)^i b_{41} & b_{52} - (-1)^i b_{42} & \dots & b_{5n} - (-1)^i b_{4n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{q-21} - (-1)^i b_{q-31} & b_{q-22} - (-1)^i b_{q-32} & \dots & b_{q-2n} - (-1)^i b_{q-3n} \\ b_{q1} - (-1)^i b_{q-11} & b_{q2} - (-1)^i b_{q-12} & \dots & b_{qn} - (-1)^i b_{q-1n} \end{pmatrix} \\ \times \begin{pmatrix} \Phi_1^{(i)}(T) \\ \Phi_2^{(i)}(T) \\ \vdots \\ \Phi_n^{(i)}(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 0, 1. \end{gathered} \tag{3.7}$$

This solution is stable in the norm of $C^1(\overline{\Omega})$.

Proof. Since $T \le \ell/2$, we use the representation of solution (2.16)–(2.18) in this proof. Taking into account (1.2), we obtain $u\big|_{AB} = \tau(x) = 0$, then substitute the representation of solution (2.16) in the boundary condition (3.1):

$$\nu(x+T) - \nu(T-x) + F'_{1t}(x,T) = 0, \quad 0 \le x \le T. \tag{3.8}$$

Now substitute the representation of solution (2.17) into the boundary condition (3.2). Then

$$(\alpha+\beta)\nu(x+T) + (\beta-\alpha)\nu(x-T) + \alpha F'_{2x}(x,T) + \beta F'_{2t}(x,T) = 0, \tag{3.9}$$

for $T \le x \le T(n-1)$. In equation (3.9) for each $n-2$ segments $[Ti, T(i+1)], i = \overline{1, n-2}$ we make change of $x = T(i+1) - y$, $0 \le y \le T$,

$$\begin{aligned}(\alpha+\beta)&\nu(T(i+2)-y) + (\beta-\alpha)\nu(Ti-y) + \alpha F'_{2x}(T(i+1)-y,T)\\ &+ \beta F'_{2t}(T(i+1)-y,T) = 0, \quad 0 \le y \le T,\ i = \overline{1, n-2}.\end{aligned} \tag{3.10}$$

Now we substitute the representation of solution (2.18) into the boundary condition (3.3). Then

$$\int_{x-T}^{2\ell-x-T} \nu(\xi)d\xi + F_3(x,T) = 0, \quad T(n-1) \le x \le Tn.$$

We take a derivative with respect to x, then we have

$$-\nu(2\ell-x-T) - \nu(x-T) + F'_{3x}(x,T) = 0, \quad T(n-1) \le x \le Tn. \tag{3.11}$$

Then we make change of variable $x = \ell - y$, $0 \le y \le T$,

$$-\nu(\ell+y-T) - \nu(\ell-y-T) + F'_{3x}(\ell-y,T) = 0, \quad 0 \le y \le T. \tag{3.12}$$

We have n nonhomogeneous equations. Now we show that the number of unknown functions in equations (3.8), (3.10) and (3.12) equals n. For that we consider 2 cases:

Case 1. Let $n = 2m, m \in \mathbb{Z}^+$. Then in (3.10) for even numbers $i = 2k$, $k = \overline{1, m-1}$ we make change of variable $y = T - z$,

$$\begin{aligned}(\alpha+\beta)&\nu((2k+1)T+z) + (\beta-\alpha)\nu((2k-1)T+z) + \alpha F'_{2x}(2kT+z,T)\\ &+ \beta F'_{2t}(2kT+z,T) = 0, \quad 0 \le z \le T,\ k = \overline{1, m-1}.\end{aligned} \tag{3.13}$$

Here it is easy to see that the number of unknown functions in equations (3.8), (3.10) (for odd numbers $i = 2k-1$, $k = \overline{1, m-1}$), (3.12) and (3.13) equals n.

Case 2. Let $n = 2m+1, m \in \mathbb{Z}^+$. Then in (3.10) for even $i = 2k$, $k = \overline{1, m-1}$ we make change of $y = T - z$,

$$\begin{aligned}(\alpha+\beta)&\nu((2k+1)T+z) + (\beta-\alpha)\nu((2k-1)T+z) + \alpha F'_{2x}(2kT+z,T)\\ &+ \beta F'_{2t}(2kT+z,T) = 0, \quad 0 \le z \le T,\ k = \overline{1, m-1}.\end{aligned} \tag{3.14}$$

Then in (3.12) we make change of of variable $y = T - z$,

$$-\nu((2m+1)T-z) - \nu((2m-1)T+z) + F'_{3x}(2mT+z,T) = 0, quad 0 \le z \le T. \tag{3.15}$$

Here it is easy to see that the number of unknown functions in equations (3.8), (3.10) (for odd numbers $i = 2k-1$, $k = \overline{1, m}$), (3.14) and (3.15) equals n.

Thus, we have n nonhomogeneous equations for n unknown functions $\nu(x_i)$, $T(i-1) \le x_i \le Ti$, $i = \overline{1, n}$. The existence and uniqueness of the solution of Problem 2 are equivalent to the existence and uniqueness of the functions $\nu(x_i)$, $T(i-1) \le x_i \le Ti$, $i = \overline{1, n}$, satisfying equations (3.8), (3.10) and (3.12).

Also the existence and uniqueness of the functions $\nu(x_i)$, $T(i-1) \le x_i \le Ti$, $i = \overline{1,n}$ are equivalent to the following:

$$0 \neq \det A = \begin{cases} 2(\beta-\alpha)^{2k-1}(\alpha+\beta)^{2k-1}, & \text{if } n = 4k; \\ -2\alpha(\beta-\alpha)^{2k-2}(\alpha+\beta)^{2k-2}, & \text{if } n = 4k-1; \\ -2(\beta-\alpha)^{2k-2}(\alpha+\beta)^{2k-2}, & \text{if } n = 4k-2; \\ 2\alpha(\beta-\alpha)^{2k-3}(\alpha+\beta)^{2k-3}, & \text{if } n = 4k-3 \end{cases} \quad k = 1, 2, \dots.$$

In case the $\alpha(\alpha+\beta)(\alpha-\beta) \neq 0$ we can see that $\det A \neq 0$. Thus, we have proved the existence and uniqueness of the solution of Problem 2 when condition (3.4) holds.

Now we show the stability according to the norm of $C^1(\overline{\Omega})$. By (3.6), (3.7) and equations (3.8), (3.10), (3.12), we obtain

$$\nu(iT-0) = \nu(iT+0), \quad i = 1, 2, \dots, n-1; \tag{3.16}$$

$$\nu'(iT-0) = \nu'(iT+0), \quad i = 1, 2, \dots, n-1. \tag{3.17}$$

Therefore the solution of Problem 2 is stable according to the norm of $C^1(\overline{\Omega})$.

From the existence and uniqueness of the classical solution of Problem 2 by standard methods we obtain existence and uniqueness of the strong solution of Problem 2.

From the representation of the solution of the problem it is easy to note that the strong solution depends only on $\nu(x)$ and $f(x,t)$. Since $\det A \neq 0$, then from equations (3.8), (3.10) and (3.12) it is seen that the function $\nu(x)$ depends only on $f(x,t)$. Then

$$\begin{aligned} \|u\|_{W_2^1(\Omega)} &\le C_1\|\nu(x)\|_{L_2(0,\ell)} + C_2\|f\|_{L_2(\Omega)} = C_1\|Bf\|_{L_2(0,\ell)} + C_2\|f\|_{L_2(\Omega)} \\ &\le C_3|B| \times \|f\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)}. \end{aligned}$$

□

We note that in [12] it is proved the well-posedness of problem for (1.1) with boundary condition (1.2) and with conditions on the boundary CD:

$$u_t\big|_{DE} = 0, \quad u\big|_{CE} = 0. \tag{3.18}$$

From the condition (3.4) follows that the case $\alpha = 0$ in (3.2) leads to ill-posedness of Problem 2. Thus, unlike problem (1.1), (1.2), (3.18) the problem with boundary conditions

$$u_t\big|_{DF} = 0, \quad u\big|_{CF} = 0$$

is ill-posed.

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