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GAP SOLITONS IN PERIODIC SCHRÖDINGER LATTICE SYSTEM WITH NONLINEAR HOPPING

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Abstract. This article concerns the periodic discrete Schrödinger equation with nonlinear hopping on the infinite integer lattice. We obtain the existence of gap solitons by the linking theorem and concentration compactness method together with a periodic approximation technique. In addition, the behavior of such solutions is studied as $\alpha \to 0$. Notice that the nonlinear hopping can be sign changing.

1. Introduction

The discrete nonlinear Schrödinger equation (DNLS) is one of the most important inherently discrete models playing a crucial role in the modeling a wide variety of phenomena in many areas of science ranging from solid-state and condensed matter physics, and nonlinear optics to biology. For general reviews of such applications we refer to [2, 3, 4, 7] and references therein. Most publications in this area are devoted to spatially homogeneous DNLS, i.e., DNLS with complete (discrete) translation invariance. Majority of results obtained concerns spatially localized standing wave solution, often called breathers, in which case the evolutionary DNLS reduces to the stationary one. Methods commonly used in studying DNLS are based on perturbation analysis, dynamical system approach and numerical simulation. Also we point out that an exceptional DNLS, the so-called Ablowitz-Ladik equation, is a completely integrable system. On the other hand, Weinsten [16] made use of variational techniques, namely constrained minimization, to obtain the existence of standing waves and to study certain properties of such solutions, still in the fully translation invariant case. To the best of our knowledge, this is the first application of variational techniques in the context of DNLS.

Nevertheless spatially non-homogeneous DNLS is not less important than homogeneous one. Especially interesting is the case when the equation is periodic with respect to the spatial variable(s). Notice that we are still interesting in solutions confined in a finite region of the space. In [9, 10, 11, 13], the second author initiated the study of periodic DNLS with the aid of variational methods. Specifically, these papers make use of critical point theorems for smooth functionals in combination with periodic approximations and concentration compactness. This approach

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goes back to [8], where the continuous NLS is considered. Another line of research [12, 14, 17] is concerned with DNLS with infinitely growing potential.

Let us consider the time dependent DNLS with nonlinear hopping

$$i\dot{\psi}_n = -\Delta_d\psi_n + \varepsilon_n\psi_n - \beta_n f(\psi_n) - \alpha_n\psi_n(|\psi_{n+1}|^2 + |\psi_{n-1}|^2), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta_d\psi_n = \psi_{n+1} - 2\psi_n + \psi_{n-1}$ stands for the discrete one-dimensional Laplacian and $\{\varepsilon_n\}, \{\alpha_n\}, \{\beta_n\}$ are real sequences. The nonlinearity f is a gauge invariant complex-valued function of complex variable, *i.e.* $f(e^{i\omega}u) = e^{i\omega}f(u)$ for any $\omega \in \mathbb{R}$. We also suppose that $f(\mathbb{R}) \subset \mathbb{R}$. Because of the gauge invariance, this equation may possess localized in space standing wave solutions or, in other words, breathers. Karachalios et al. [5] consider equation (1.1) and its multi-dimensional version in the fully translation invariant case. Main results of that paper concern energy thresholds for such solutions. However, the existence of standing waves is obtained only in the case when the equation is restricted to a finite box, with the Dirichlet boundary conditions at the faces. This paper also discuss applications of DNLS with nonlinear hopping in physics and provides a number of relevant references. Recently, M. Cheng [1] has obtained a proof of existence of localized solitary waves on the entire lattice. His approach follows [8, 9, 13] and is based on the Nehari manifold argument combined with periodic approximations and concentration compactness.

Using the standing wave ansatz

$$\psi_n = e^{-i\omega t}u_n\,,$$

where $u_n \in \mathbb{R}$, we arrive at the equation

$$-\Delta_d u_n + \varepsilon_n u_n - \omega u_n = \beta_n f(u_n) + \alpha_n u_n(|u_{n+1}|^2 + |u_{n-1}|^2)$$

for the profile sequence $\{u_n\}$. It is convenient to introduce the operator

$$Lu_n = -\Delta_d u_n + \varepsilon_n u_n\,.$$

Then equation (1.1) becomes

$$Lu_n - \omega u_n = \beta_n f(u_n) + \alpha_n u_n(|u_{n+1}|^2 + |u_{n-1}|^2). \tag{1.2}$$

In the remaining part of the paper we study equation (1.2) in its own rights. Since we are looking for real solutions of this equation, we may and shall assume that the nonlinearity f is a real function on the real line. Throughout the paper we assume that the potential $\{\varepsilon_n\}$ and the coefficient sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are periodic. Notice that though we consider equation (1.2) on the one-dimensional lattice, our approach is not sensitive to the space dimension, and the results extend straightforwardly to multi-dimensional equations of this type.

Since the coefficients of L are periodic, L is a bounded self-adjoint operator in the space l^2. Moreover, its spectrum $\sigma(L)$ has the so-called band structure, i. e., is the union of a finite number of closed intervals which are called bands (see, e.g., [15]). Bands may touch, but in general, they are separated by open intervals called gaps. There exists a finite number of open intervals called spectral gaps. Semi-infinite intervals below and above the spectrum are also considered as gaps.

Under certain assumptions we prove the existence of nontrivial solutions in the space l^2 provided the frequency ω belongs to either finite spectral gap, or lies below the spectrum of L. Moreover, the solutions obtained are well-localized in the sense that they decay at infinity exponentially fast. Under certain extra assumptions we obtain the existence of *ground states*, i.e., solutions that minimize the energy among all nontrivial solutions. In view of application to solitary standing waves for equation (1.1), these solutions are called gap solitons. Notice that for frequencies below the spectrum we have two different existence results. Remarkably enough is that in our main result hopping coefficients α_n can be sign-changing, or even all negative. We do not know whether Assumption (v) below can be dropped or weakened in the case when the frequency belongs to a finite spectral gap even if the hopping term is positive. On the other hand, in the case when the frequency is above the spectrum we provide a simple nonexistence result.

Since equation (1.2) possesses an energy functional (see functional J in Section 2), our approach is variational. Following [8, 9], we employ periodic approximations in combination with classical Linking Theorem [18] and elementary concentration compactness argument to prove the existence of a nontrivial solution. Then we deduce the existence of *ground state solution*, i.e., a solution with minimum energy among all nontrivial solutions. Finally, we provide two simple general results on nonexistence of solutions and exponential decay, respectively.

2. Main results

Let us introduce our assumptions on the function f.

(i) The function f is continuous.

(ii) There exist $p > 2$ and $C > 0$ such that $|f(u)| \le C(1 + |u|^{p-1})$.

(iii) f is superlinear at 0, i.e., $f(u) = o(u)$ as $u \to 0$.

(iv) There exists $q > 2$ such that $0 < qF(u) \le f(u)u$ for all $u \ne 0$, where

$$F(u) = \int_0^u f(s)\,ds\,.$$

(v) There exists a constant $\mu > 0$ such that $F(u) \ge \mu u^4$ for all $u \in \mathbb{R}$.

Notice that Assumption (iv) is the standard Ambrosetti-Rabinowitz condition. Assumptions (ii) and (iii) imply that for any given $\epsilon > 0$, there exists $A(\epsilon) > 0$ such that

$$f(u)u \le \epsilon |u|^2 + A(\epsilon)|u|^p, \tag{2.1}$$

$$F(u) \le \frac{\epsilon}{2}|u|^2 + \frac{A(\epsilon)}{p}|u|^p. \tag{2.2}$$

Notice that the power nonlinearity

$$f(u) = |u|^{p-2}u\,, \quad p > 2\,,$$

satisfies Assumptions (i)–(iv) but (v) holds only when $p = 4$. On the other hand $f(u) = u^3 + u^5$ satisfies (i)–(v).

Now introduce the following notation. We set

$$\Lambda = \sup \sigma(L) \quad \text{and} \quad \lambda = \inf \sigma(L)\,.$$

These are the top and bottom of the spectrum of L. Also we set

$$\overline{\alpha} = \max_n \{\alpha_n\}\,, \quad \underline{\alpha} = \min_n \{\alpha_n\}\,,$$

$$\overline{\beta} = \max_n \{\beta_n\}\,, \quad \underline{\beta} = \min_n \{\beta_n\}\,.$$

The conjugate exponent to p is denoted by p':

$$\frac{1}{p} + \frac{1}{p'} = 1\,.$$

For convenience we formulate the following two alternative sets of assumptions.

(A1) Assume that $\omega \notin \sigma(L)$ and $\omega < \Lambda$, the function f satisfies (i)–(v), $\underline{\beta} > 0$ and $\underline{\alpha} > -\mu\underline{\beta}(q-2)$.

(A2) Assume that $\omega < \lambda$, f satisfies (i)–(iv), $\underline{\alpha} \geq 0$ and $\underline{\beta} > 0$.

Our main existence result is the following.

Theorem 2.1. *Suppose that either Assumption* (A1) *or* (A2) *is satisfied. Then* (1.2) *has a non-zero solution $u \in l^2$. Furthermore, the solution u decays exponentially fast at infinity, i.e.,*

$$|u_n| \leq Ce^{-\nu|n|}, \quad n \in \mathbb{Z}$$

for some constants $C > 0$ and $\nu > 0$. If, in addition, $\alpha_n \geq 0$ for all $n \in \mathbb{Z}$, then (1.2) *possesses a ground state solution in l^2.*

The next question we are interesting in is about zero hopping limit in (1.2). More precisely, we consider a sequence of equations

$$Lu_n - \omega u_n = \beta_n f(u_n) + \alpha_n^m u_n(|u_{n+1}|^2 + |u_{n-1}|^2)\,, \tag{2.3}$$

where $\lim_{m\to\infty} \alpha_n^m = 0$ for all n.

Theorem 2.2. *Suppose that* (2.3) *satisfies one of the Assumptions* (A1) *or* (A2) *for all m. Then, there exist nontrivial solutions $u^{(m)} \in l^2$ of* (2.3) *which converge to a nontrivial solution $u \in l^2$ of the equation*

$$Lu_n - \omega u_n = \beta_n f(u_n) \tag{2.4}$$

up to a passage to a subsequence and translations.

It follows from the proof that we can choose as $u^{(m)}$ the solution obtained in Proposition 5.1. If $\alpha_n^m \geq 0$, then $u^{(m)}$ can be a ground state solution. Notice that we do not know whether the limit solution in this case is a ground state of equation (2.4). On the other hand, the existence of ground state for equation (2.4) can be deduced from the results of [9, 10].

3. Variational setting

We denote by l^p the space of p-summable sequences and by $\|\cdot\|_p$ the norm on it. The space l^2 is shortly denoted by E and $\|\cdot\| = \|\cdot\|_2$. The inner product in E is denoted by $(\cdot,\cdot)$.

For any integer $k > 0$, we denote by E_k the space of all kN-periodic sequences. Let

$$\mathcal{P}_k = \big\{n \in \mathbb{Z} : -[\frac{kN}{2}] \leq n \leq kN - [\frac{kN}{2}] - 1\big\}.$$

On the space E_k we consider the norms

$$\|u\|_{(k,p)} = \Big(\sum_{n\in\mathcal{P}_k} |u_n|^p\Big)^{1/p}.$$

Since E_k is finite dimensional, all these norms are equivalent but not uniformly with respect to k. However,

$$\|u\|_{(k,q)} \leq \|u\|_{(k,p)}, \quad 1 \leq p \leq q \leq \infty.$$

We denote by $\|\cdot\|_k = \|\cdot\|_{(k,2)}$ the Euclidean norm on E_k and by $(\cdot,\cdot)_k$ the associated inner product. Note that, by the periodicity of the potential, L also acts as a self-adjoint operator in E_k. We denote this operator by L_k. Let us point out that $\sigma(L_k) \subset \sigma(L)$.

Let E^+ and E^- be the positive and negative spectral subspaces of the operator $L-\omega$ in E, respectively. Similarly, the positive and negative spectral subspaces of the operator $L_k-\omega$ in E_k are denoted by E_k^+ and E_k^-, respectively. Then we have orthogonal decompositions $E=E^+\oplus E^-$ and $E_k=E_k^+\oplus E_k^-$. Furthermore, let us introduce the orthogonal projectors $P^\pm$ (respectively, $P_k^\pm$) onto the subspaces $E^\pm$ (respectively, $E_k^\pm$). Let

$$\delta=\operatorname{dist}(\omega,\sigma(L))\,.$$

It is well-known that

$$\pm(Lu-\omega u,u)\ge\delta\|u\|^2,\ u\in E^\pm,$$
$$\pm(L_k u-\omega u,u)\ge\delta\|u\|_k^2,\ u\in E_k^\pm.$$

On the space E we introduce the functional

$$J(u)=\frac{1}{2}(Lu-\omega u,u)-\sum_{n\in\mathbb{Z}}\beta_nF(u_n)-\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{2}u_n^2u_{n+1}^2.$$

A straightforward calculation shows that

$$(J'(u),v)=(Lu-\omega u,v)-\sum_{n\in\mathbb{Z}}\beta_nf(u_n)v_n-\sum_{n\in\mathbb{Z}}\alpha_nu_n(u_{n+1}^2+u_{n-1}^2)v_n\,.$$

Therefore, critical points of J are solutions of (1.2) in the space E.

Since we will use periodic approximations to find l^2-solutions of (1.2), we introduce the functional

$$J_k(u)=\frac{1}{2}(L_ku-\omega u,u)_k-\sum_{n\in\mathcal{P}_k}\beta_nF(u_n)-\sum_{n\in\mathcal{P}_k}\frac{\alpha_n}{2}u_n^2u_{n+1}^2\,.$$

Its derivative is given by

$$(J_k'(u),v)=(L_ku-\omega u,v)_k-\sum_{n\in\mathcal{P}_k}\beta_nf(u_n)v_n-\sum_{n\in\mathcal{P}_k}\alpha_nu_n(u_{n+1}^2+u_{n-1}^2)v_n\,.$$

Critical points of J_k are solutions of equation (1.2) in E_k, i.e., kN-periodic solutions. It is easily seen that, under the assumptions imposed above, all critical values of the functionals J and J_k are nonnegative.

Now we derive some estimates for critical points and critical values of the functionals J and J_k.

Lemma 3.1. *Let $u^k\in E_k$ and $u\in E$ be critical points of J_k and J, respectively, with critical values $c=J(u)$ and $c_k=J_k(u^k)$. Then the following statements hold.*

(a) Under Assumption (A1) *there exists a constant $C>0$ independent of k and such that*

$$\|u\|\le C(c^{3/4}+c^{1/2}+c^{1/p'}),$$
$$\|u^k\|_k\le C(c_k^{3/4}+c_k^{1/2}+c_k^{1/p'})\,.$$

The constant C can be chosen independent of $\{\alpha_n\}$ if all of α_n, $n\in\mathbb{Z}$, belong to a compact subinterval of the interval $(-\mu\underline{\beta}(q-2),\infty)$.

(b) Under Assumption (A2) there exists a constant $C>0$ independent of k and $\{\alpha_n\}$, and such that

$$\|u\|\le Cc^{1/2}\quad\text{and}\quad\|u^k\|_k\le Cc_k^{1/2}\,.$$

Proof. We give the proof for the case of functional J, the other one being similar.

(a) By Assumption (iii), there exists a constant $C > 0$ such that

$$|u||f(u_n)| \geq C'|f(u_n)|^2$$

whenever $|u_n| < 1$. On the other hand, there exists a constant $\mu > 0$ such that

$$|u_n f(u_n)| = |f(u_n)||u_n|^{(p-1)(p'-1)} \geq \mu |f(u_n)|^{p'}$$

for $|u_n| \geq 1$. Note that

$$\underline{\alpha} > -(q-2)\underline{\beta}\mu.$$

Then there exists $\gamma > 0$ small enough such that

$$\underline{\alpha} + (1-\gamma)(q-2)\underline{\beta}\mu > 0.$$

Therefore, for any nontrivial critical point $u \in l^2$ of J,

$$\begin{aligned}
c &= J(u) - \frac{1}{2}(J'(u), u) \\
&= \sum_{n\in\mathbb{Z}} \beta_n \big[\frac{f(u_n)u_n}{2} - F(u_n)\big] + \sum_{n\in\mathbb{Z}} \frac{\alpha_n}{2} u_n^2 u_{n+1}^2 \\
&\geq (\frac{1}{2} - \frac{1}{q}) \sum_{n\in\mathbb{Z}} \beta_n f(u_n) u_n + \frac{\min\{\underline{\alpha}, 0\}}{2} \|u\|_4^4 \\
&= (\frac{1}{2} - \frac{1}{q}) \sum_{|u_n|\geq 1} \gamma\beta_n f(u_n)u_n + (\frac{1}{2} - \frac{1}{q}) \sum_{|u_n|<1} \gamma\beta_n f(u_n)u_n \\
&\quad + (\frac{1}{2} - \frac{1}{q}) \sum_{n\in\mathbb{Z}} (1-\gamma)\beta_n f(u_n)u_n + \frac{\min\{\underline{\alpha}, 0\}}{2} \|u\|_4^4 \\
&\geq (\frac{1}{2} - \frac{1}{q}) \sum_{|u_n|\geq 1} \gamma\underline{\beta}\mu |f(u_n)|^{p'} + (\frac{1}{2} - \frac{1}{q}) \sum_{|u_n|<1} \gamma\underline{\beta} C' |f(u_n)|^2 \\
&\quad + (\frac{1}{2} - \frac{1}{q}) \sum_{n\in\mathbb{Z}} (1-\gamma)\underline{\beta}\mu q |u_n|^q + \frac{\min\{\underline{\alpha}, 0\}}{2} \|u\|_4^4.
\end{aligned} \tag{3.1}$$

Let $u^{\pm} = P^{\pm}u$ be the orthogonal projection of u on $E^{\pm}$. Then

$$\begin{aligned}
0 &= (J'(u), u^+) \\
&= ((L-\omega)u^+, u^+) - \sum_{n\in\mathbb{Z}} \beta_n f(u_n) u_n^+ - \sum_{n\in\mathbb{Z}} \alpha_n u_n (|u_{n+1}|^2 + |u_{n-1}|^2) u_n^+
\end{aligned}$$

and, by the Hölder inequality, we obtain

$$\begin{aligned}
\delta \|u^+\|^2 &\leq \overline{\beta} \sum_{n\in\mathbb{Z}} |f(u_n)u_n^+| + 2\max\{|\overline{\alpha}|, |\underline{\alpha}|\} \|u^+\|_4 \|u\|_4^3 \\
&\leq 2\max\{|\overline{\alpha}|, |\underline{\alpha}|\} \|u^+\|_4 \|u\|_4^3 + \overline{\beta}(\sum_{|u_n|\geq 1} |f(u_n)|^{p'})^{1/p'} \|u^+\|_p \\
&\quad + \overline{\beta}(\sum_{|u_n|<1} |f(u_n)|^2)^{1/2} \|u^+\|.
\end{aligned}$$

Together with (3.1), this implies that

$$\|u^+\|^2 \leq \frac{2\max\{|\overline{\alpha}|, |\underline{\alpha}|\}}{\delta} \Big(\frac{2c}{\mu\underline{\beta}(q-2)(1-\gamma) + \min\{\underline{\alpha}, 0\}}\Big)^{3/4} \|u^+\|$$

$$+\frac{\overline{\beta}}{\delta}\Big[\Big(\frac{2cq}{\mu\gamma\underline{\beta}(q-2)}\Big)^{1/p'}+\Big(\frac{2cq}{C'\gamma\underline{\beta}(q-2)}\Big)^{1/2}\Big]\|u^+\|.$$

Similarly, we obtain

$$\begin{aligned}\|u^-\|^2 \le{}& \frac{2\max\{|\overline{\alpha}|,|\underline{\alpha}|\}}{\delta}\Big(\frac{2c}{\mu\underline{\beta}(q-2)(1-\gamma)+\min\{\underline{\alpha},0\}}\Big)^{3/4}\|u^-\|\\ &+\frac{\overline{\beta}}{\delta}\Big[\Big(\frac{2cq}{\mu\gamma\underline{\beta}(q-2)}\Big)^{1/p'}+\Big(\frac{2cq}{C'\gamma\underline{\beta}(q-2)}\Big)^{1/2}\Big]\|u^-\|.\end{aligned}$$

Since

$$\|u\|\ge\frac{1}{\sqrt{2}}(\|u^+\|+\|u^-\|)\,,$$

combining the last two inequalities we obtain that

$$\begin{aligned}\|u\|\le{}&\frac{2\sqrt{2}\max\{|\overline{\alpha}|,|\underline{\alpha}|\}}{\delta}\Big(\frac{2c}{\mu\underline{\beta}(q-2)(1-\gamma)+\min\{\underline{\alpha},0\}}\Big)^{3/4}\\ &+\frac{\overline{\beta}\sqrt{2}}{\delta}\Big[\Big(\frac{2cq}{\mu\gamma\underline{\beta}(q-2)}\Big)^{\frac{1/p'}{+}}\Big(\frac{2cq}{C'\gamma\underline{\beta}(q-2)}\Big)^{1/2}\Big]\end{aligned}$$

which implies the required.

(b) This case is simpler. We have that

$$\begin{aligned}c&=J(u)-\frac{1}{2}(J'(u),u)\\ &=\sum_{n\in\mathbb{Z}}\beta_n\Big[\frac{f(u_n)u_n}{2}-F(u_n)\Big]+\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{2}u_n^2u_{n+1}^2\\ &\ge(\frac{1}{2}-\frac{1}{q})\sum_{n\in\mathbb{Z}}\beta_nf(u_n)u_n+\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{2}u_n^2u_{n+1}^2\,.\end{aligned}$$

Since

$$\begin{aligned}0&=(J'(u),u)\\ &=((L-\omega)u,u)-\sum_{n\in\mathbb{Z}}\beta_nf(u_n)u_n-\sum_{n\in\mathbb{Z}}\alpha_nu_n(|u_{n+1}|^2+|u_{n-1}|^2)u_n\,,\end{aligned}$$

the last inequality yields

$$\delta\|u\|^2\le\sum_{n\in\mathbb{Z}}\beta_nf(u_n)u_n+2\sum_{n\in\mathbb{Z}}\alpha_nu_n^2u_{n+1}^2\le\frac{(6q-8)}{q-2}c\,.$$

This completes the proof of part (b). □

The next lemma provides a lower bound for nontrivial critical points.

Lemma 3.2. *Under the assumptions of Theorem 2.1 there exists a constant* $\kappa>0$ *independent of* k *and such that for all nontrivial critical points* $u^k\in E_k$ *of* J_k *and* $u\in E$ *of* J

$$\|u\|\ge\kappa\quad\text{and}\quad\|u^k\|_k\ge\kappa\,.$$

Furthermore, the constant κ *can be chosen independent of* $\{\alpha_n\}$ *provided that all* α_n, $n\in\mathbb{Z}$, *belong to a bounded interval.*

Proof. Assume (A1). As in Lemma 3.1, we have

$$\delta\|u^+\|^2 \le \overline{\beta}\sum_{n\in\mathbb{Z}}|f(u_n)u_n^+| + 2\max\{|\overline{\alpha}|,|\underline{\alpha}|\}\|u^+\|_4\|u\|_4^3,$$
$$\delta\|u^-\|^2 \le \overline{\beta}\sum_{n\in\mathbb{Z}}|f(u_n)u_n^-| + 2\max\{|\overline{\alpha}|,|\underline{\alpha}|\}\|u^-\|_4\|u\|_4^3\,.$$

Combining these two inequalities and making use of (2.1), we obtain

$$\delta - \overline{\beta}\epsilon\sqrt{2} \le \overline{\beta}A(\epsilon)\sqrt{2}\|u\|^{p-2} + 2\sqrt{2}\max\{|\overline{\alpha}|,|\underline{\alpha}|\}\|u\|^2\,.$$

Similar inequality holds for critical points of the kN-periodic problem with the same constants. Choosing ϵ small enough, we complete the proof. The case of Assumption (A2) is similar and simpler. □

Remark 3.3. By Lemmas 3.1 and 3.2, there exists a constant $\kappa_0 > 0$ independent of k such that all positive critical values of the functionals J_k and J belong to $[\kappa_0, \infty)$.

4. Periodic problem

We start with the Palais-Smale condition for the functional J_k. Recall that a sequence $\{v^j\}$ in E_k is a Palais-Smale sequence for J_k if the sequence $\{J_k(v^j)\}$ is bounded and $J_k'(v^j) \to 0$ as $j \to \infty$.

Lemma 4.1. *Under the assumptions of Theorem 2.1, the functional J_k satisfies the Palais-Smale condition, i.e., every Palais-Smale sequence contains a convergent subsequence.*

Proof. We prove the lemma under Assumption (A1), the other case being similar and simpler. Since the space E_k is finite dimensional, it is enough to show that every Palais-Smale sequence is bounded. Replacing L and ω by $L+\omega_0$ and $\omega+\omega_0$, respectively, we may and will assume that

$$(Lu,u)_k \ge \|u\|_k^2, \quad \forall u \in E_k\,,$$

and $\omega > 0$.

Let v^j be a Palais-Smale sequence at a level b, i.e., $J_k(v^j) \to b$ and $J_k'(v^j) \to 0$ as $j \to \infty$. Choose any $\lambda \in (1/4, 1/2)$. Then, for j large enough,

$$\begin{aligned}
&b + 1 + \lambda\|v^j\|_k \\
&\ge J_k(v^j) - \lambda(J_k'(v^j), v^j)_k \\
&= (\frac{1}{2}-\lambda)(Lv^j, v^j)_k - (\frac{1}{2}-\lambda)\omega\|v^j\|_k^2 + (2\lambda - \frac{1}{2})\sum_{n\in\mathcal{P}_k}\alpha_n v_n^{j2}v_{n+1}^{j2} \\
&\quad + \lambda\sum_{n\in\mathcal{P}_k}\beta_n f(v_n^j)v_n^j - \sum_{n\in\mathcal{P}_k}\beta_n F(v_n^j) \\
&\ge (\frac{1}{2}-\lambda)\|v^j\|_k^2 - (\frac{1}{2}-\lambda)\omega\|v^j\|_k^2 + \underline{\beta}(\lambda q - 1)\sum_{n\in\mathcal{P}_k}F(v_n^j) \\
&\quad + 2\min\{\underline{\alpha}, 0\}(\lambda - \frac{1}{4})\|v^j\|_{(k,4)}^4 \\
&\ge (\frac{1}{2}-\lambda)\|v^j\|_k^2 - (\frac{1}{2}-\lambda)\omega\|v^j\|_k^2 + \Big(\underline{\beta}\mu(\lambda q-1) + 2\min\{\underline{\alpha},0\}(\lambda-\frac{1}{4})\Big)\|v^j\|_{(k,4)}^4\,.
\end{aligned}$$

Since all norms of E_k are equivalent,

$$\begin{aligned} b+1 \geq -\lambda\|v^j\|_k + (\frac{1}{2}-\lambda)\|v^j\|_k^2 - (\frac{1}{2}-\lambda)\omega\|v^j\|_k^2 \\ + C\Big(\underline{\beta}\mu(\lambda q-1) + 2\min\{\underline{\alpha},0\}(\lambda-\frac{1}{4})\Big)\|v^j\|_k^4 \end{aligned}$$

for some constant $C>0$ depending on k. The coefficient in the forth term in the right hand side is positive and, hence, the sequence $\{v^j\}$ is bounded. □

Now we show that the functional J_k possess the so-called linking geometry (see, e.g., [18, Chapter 2]). Note that in the case when ω is below the spectrum of L, then $E_k^- = \{0\}$ and we can use its special case known as mountain pass geometry.

First we choose a unit vector $z^k \in E_k^+$ as follows. If the frequency ω is below the spectrum $\sigma(L)$, then $E_k^+ = E_k$ and we define z^k as follows

$$z_n^k = \begin{cases} 1 & \text{if } n = mkN,\ m\in\mathbb{Z}\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

In the case when ω belongs to a finite spectral gap, the choice of z^k is more delicate. Let S_k be the "periodization" operator that assigns to any sequence $u=\{u_n\}$ the kN-periodic sequence $S_k u$ defined by

$$(S_k u)_n = u_n, \quad n\in\mathcal{P}_k.$$

Let $z\in E^+$ be any unit vector. The results of [9, Appendix A] imply that $\|P_k^+ S_k z\|_k \to 1$. Hence, there exists $k_0\in\mathbb{N}$ such that for all $k\geq k_0$

$$z^k = \frac{P_k^+ S_k z}{\|P_k^+ S_k z\|_k}$$

is a well-defined unit vector in E_k^+. Moreover,

$$\|z^k\|_{(k,p)} \to \|z\|_{l^p} \tag{4.1}$$

for all $p\in[1,\infty)$. For $k<k_0$, we choose as z^k any unit vector in E_k^+.

Now let us introduce the sets

$$\begin{gathered} M = \{v = y + tz^k : y\in E_k^-,\, \|v\|_k \leq r_2 \text{ and } t\geq 0\},\\ S = \{v\in E_k^+ : \|v\|_k = r_1\}\,, \end{gathered}$$

where the constants r_1 and r_2 satisfying $0<r_1<r_2$ will be chosen later. The boundary of M is

$$M_0 = \partial M = \{v = y+tz^k : y\in E_k^-,\, \|v\|_k = r_2 \ \text{ and } \ t>0 \text{ or } \|y\|_k\leq r_2 \text{ and } t=0\}\,.$$

The following lemma is a more or less standard consequence of the fact that the combined local and hopping nonlinearity is superlinear at zero. However, we sketch its proof because we need an information on the dependence of constants obtained there on hopping parameters.

Lemma 4.2. *Under the assumptions of Theorem 2.1, suppose that r_1 satisfies*

$$\frac{A(\frac{\delta}{2\overline{\beta}})\overline{\beta} r_1^{p-2}}{p} + \frac{\max\{|\overline{\alpha}|,|\underline{\alpha}|\}r_1^2}{2} \leq \frac{\delta}{8}\,.$$

Then

$$J_k(v) \geq \frac{\delta r_1^2}{8}, \quad \forall v\in S.$$

Proof. Let $v \in S$. Using (2.2) with $\epsilon = \frac{\delta}{2\overline{\beta}}$, we have

$$\begin{aligned} J_k(v) &= \frac{1}{2}(L_k v - \omega v, v)_k - \sum_{n\in P_k} \beta_n F(v_n) - \sum_{n\in P_k} \frac{\alpha_n}{2} v_n^2 v_{n+1}^2 \\ &\geq \frac{\delta}{2}\|v\|_k^2 - \overline{\beta}(\frac{\epsilon}{2}\|v\|_k + \frac{A(\epsilon)}{p}\|v\|_{(k,p)}^p) - \sum_{n\in P_k} \frac{\alpha_n}{2} v_n^2 v_{n+1}^2 \\ &\geq \frac{\delta}{4}\|v\|_k^2 - \frac{A(\frac{\delta}{2\overline{\beta}})\overline{\beta}}{p}\|v\|_k^p - \frac{\max\{|\overline{\alpha}|, |\underline{\alpha}|\}}{2}\|v\|_k^4. \end{aligned}$$

This implies the required inequality. □

Lemma 4.3. *Under the assumptions of Theorem 2.1, there exist constants $K > 0$ and $r_2 > 0$ independent of k such that*

$$J_k(v) \leq K\,, \quad \forall v \in M$$

and $J_k(v) \leq 0$ for $v \in M_0$.

Proof. Suppose that Assumption (A1) holds. For $v = y + tz^k \in M$, we have

$$\begin{aligned} J_k(y+tz^k) &= \frac{1}{2}((L_k - \omega)y, y)_k + \frac{t^2}{2}((L_k - \omega)z^k, z^k)_k - \sum_{n\in\mathcal{P}_k} \beta_n F(y_n + tz_n^k) \\ &\quad - \sum_{n\in\mathcal{P}_k} \frac{\alpha_n}{2}(y_n + tz_n^k)^2(y_{n+1} + tz_{n+1})^2 \\ &\leq -\frac{\delta}{2}\|y\|_k^2 + \frac{t^2}{2}((L_k - \omega)z^k, z^k)_k - \Big(\underline{\beta}\mu + \frac{\min\{\underline{\alpha}, 0\}}{2}\Big)\|y + tz^k\|_{(k,4)}^4. \end{aligned}$$

Since the norm of any projector is ≥ 1, we have

$$\|y + tz^k\|_{(k,4)}^4 \geq \|tz^k\|_{(k,4)}^4$$

and, hence,

$$\begin{aligned} J_k(y+tz^k) &\leq -\frac{\delta}{2}\|y\|_k^2 + \frac{t^2}{2}((L_k - \omega)z^k, z^k)_k - \Big(\underline{\beta}\mu + \frac{\min\{\underline{\alpha}, 0\}}{2}\Big)\|tz^k\|_{(k,4)}^4 \\ &\leq \frac{t^2}{2}\|L_k - \omega\| - \Big(\underline{\beta}\mu + \frac{\min\{\underline{\alpha}, 0\}}{2}\Big)t^4\|z^k\|_{(k,4)}^4. \end{aligned}$$

By the definition of z^k and equation (4.1), there exist two positive constants K_1 and K_2 independent of k and such that

$$J_k(y + tz^k) \leq K_1 t^2 - K_2 t^4\,. \tag{4.2}$$

Hence, $J_k(v) \leq K$ for all $v \in M$, where $K > 0$ is independent of k.

Now suppose that $v = y + tz^k \in M_0$. Then $t^2 + \|y\|^2 = r_2^2$, and (4.1) yields

$$J_k(y + tz^k) \leq K_1 r_2^2 - K_2 r_2^4\,.$$

This implies immediately the second part of the lemma.

Uner Assumption (A2) the proof is similar. We only mention that instead of inequality (4.2) we have

$$J_k(tz^k) \leq K_1 t^2 - K_2 t^4 - K_3 t^q\,,$$

with positive constants K_i, $i = 1, 2, 3$. □

Remark 4.4. The constants K, r_1 and r_2 are independent not only on k but also on the hopping constants $\{\alpha_n\}$ if we assume that these constants belong to a compact subinterval of $(-\mu\underline{\beta}(q-2),\infty)$ in the case of Assumption (A1), or to a compact subinterval of $[0,\infty)$ in the case of Assumption (A2).

Now, we obtain the existence of periodic solutions.

Theorem 4.5. *Under the assumptions of Theorem 2.1, for every integer $k>0$ there exists a nontrivial critical point $u^{(k)}\in E_k$ of J_k such that $J_k(u^{(k)})\le K$ and $\|u^{(k)}\|_k\le K_0$, where K is the constant from Lemma 4.3, while $K_0>0$ is determined in terms of K according to Lemma 3.1.*

The above theorem follows immediately from Lemmas 4.1, 4.2, 4.3 and the Linking Theorem [18].

5. Proofs of main results

Theorem 2.1 is a straightforward consequence of the next two propositions and Proposition 6.2. In the first one we make the passage to the limit as the period of solutions obtained in Theorem 4.5 tends to infinity.

Proposition 5.1. *Under the assumptions of Theorem 2.1, there exists a nontrivial solution $u\in E$ of (1.2) such that $\|u\|\le K_0$, where K_0 is the constant from Theorem 4.5.*

Proof. Let $u^{(k)}$ be the k-periodic solution obtained in Theorem 4.5. Then

$$\|u^{(k)}\|_k\le K_0\,.$$

First we show that, along a subsequence, there exist $\delta_0>0$ and $b^k\in\mathbb{Z}$ such that $|u^{(k)}_{b^k}|\ge\delta_0$ for all k. Indeed, assume the contrary, *i.e.*, $\|u^{(k)}\|_{l^\infty}\to 0$. Making use of the following elementary inequality

$$\|v\|^p_{(k,p)}\le\|v\|^{p-2}_{l^\infty}\|v\|^2_k\,,$$

where $p>2$, we conclude that $\|u^{(k)}\|_{(k,p)}\to 0$ for all $p>2$. Under Assumption (A1), for any $\epsilon>0$, we have

$$\begin{aligned}
0<c_k&=J_k(u^{(k)})-\frac12(J_k'(u^{(k)}),u^{(k)})\\
&=\sum_{n\in\mathcal{P}_k}\beta_n[\frac{f(u^{(k)}_n)u^{(k)}_n}{2}-F(u^{(k)}_n)]+\sum_{n\in\mathcal{P}_k}\frac{\alpha_n}{2}(u^{(k)}_n)^2(u^{(k)}_{n+1})^2\\
&\le\frac{\overline{\beta}}{2}(\epsilon\|u^{(k)}\|^2_k+A(\epsilon)\|u\|^p_{(k,p)})+\frac{\max\{|\overline{\alpha}|,|\underline{\alpha}|\}}{2}\|u^{(k)}\|^4_{(k,4)}\to 0
\end{aligned}$$

Since $\epsilon>0$ can be chosen arbitrarily small, this contradicts Remark 3.3. The case of Assumption (A2) is similar.

From the N-periodicity, after integer translations and further passage to a subsequence we may assume that there exists an integer $b\in[0,N-1]$ such that $|u^{(k)}_b|\ge\delta_0>0$ and $u^{(k)}_n\to u_n$ for all $n\in\mathbb{Z}$. It is easily seen that $u\in E$. Furthermore, $|u_b|\ge\delta_0>0$ and, hence, $u\ne 0$. Since equation (1.2) possesses point-wise limits, u is a solution of (1.2). The inequality $\|u\|\le K_0$ follows immediately. □

Remark 5.2. In Proposition 5.1 we can not conclude that $J(u)\le K$, in general. However, this is so if $\alpha_n\ge 0$ for all $n\in\mathbb{Z}$.

Proposition 5.3. *In addition to the assumptions of Theorem 2.1, suppose that* $\alpha_n \geq 0$ *for all* $n \in \mathbb{Z}$. *Then* (1.2) *possesses a ground state solution in* E.

Proof. By Proposition 5.1 and Remark 3.3, the set $\mathcal{C}$ of all positive critical values of J is a non-empty subset of $[\kappa_0, \infty)$, $\kappa_0 > 0$. Let $c = \inf \mathcal{C} > 0$. If $c \in \mathcal{C}$, we are done. Otherwise, there exists a sequence $c^{(j)} = J(u^{(j)}) \in \mathcal{C}$ such that $c^{(j)} \to c$. By Lemmas 3.1 and 3.2, the norms $\|u^{(j)}\|$ are bounded below and above by positive constants. Arguing as in the proof of Proposition 5.1, we may assume that there exists an integer $b \in [0, N-1]$ such that, along a subsequence, $\|u^{(j)}\| \geq \delta_0 > 0$ and $u_n^{(j)} \to u_n$ for all $n \in \mathbb{Z}$. Then $0 \neq u \in E$ and u is a solution of equation (1.2). Since $u \in E$ is a nontrivial solution of (1.2), then $J(u) \geq c$.

On the other hand,

$$\begin{aligned} J(u^{(j)}) &= J(u^{(j)}) - \frac{1}{2}(J'(u^{(j)}), u^{(j)}) \\ &= \sum_{n\in\mathbb{Z}} \{\beta_n [\frac{f(u_n^{(j)})u_n^{(j)}}{2} - F(u_n^{(j)})] + \frac{\alpha_n}{2}(u_n^{(j)})^2 (u_{n+1}^{(j)})^2\} \end{aligned}$$

and

$$\begin{aligned} J(u) &= J(u) - \frac{1}{2}(J'(u), u) \\ &= \sum_{n\in\mathbb{Z}} \{\beta_n [\frac{f(u_n)u_n}{2} - F(u_n)] + \frac{\alpha_n}{2}(u_n)^2 (u_{n+1})^2\}\,. \end{aligned}$$

Notice that the summands in the right hand sides of the last two identities are non-negative. Now the discrete version of the Fatou's lemma implies immediately that $J(u) \leq c$. Hence, $J(u) = c$, and the proof is complete. □

Now we are ready to prove our second main result.

Proof of Theorem 2.2. Since $\alpha_n^m \to 0$ as $m \to \infty$ for all $n \in \mathbb{Z}$, the constant K_0 in Proposition 5.1 can be chosen independent of m. Denote by $u^{(m)} \in E$ the solution of equation (2.3) obtained in that proposition. Then $\|u^{(m)}\| \leq K_0$.

Using the same arguments as in the proofs of Propositions 5.1 and 5.3, we see that, after a passage to a subsequence and translations, $u^{(m)} \to u \neq 0$ point-wise. Moreover, $u \in E$ and solves equation (2.4). □

6. Additional results

First, we prove the following nonexistence result.

Proposition 6.1. *Let* $\underline{\alpha} > -2\underline{\beta}\mu$, $\underline{\beta} > 0$. *Assume that Assumptions* (i)–(iv) *hold and* $\omega > \Lambda$. *Then* (1.2) *has only the trivial solution in* l^2.

Proof. Let u be a critical point of J. Since $\omega > \Lambda$, then $E^+ = \{0\}$. Then,

$$\begin{aligned} 0 = (J'(u), u) &= (Lu - \omega u, u) - \sum_{n\in\mathbb{Z}} \beta_n f(u_n)u_n - 2\sum_{n\in\mathbb{Z}} \alpha_n u_n^2 u_{n+1}^2 \\ &\leq (Lu - \omega u, u) - (\underline{\beta} q\mu + 2\min\{\underline{\alpha}, 0\})\|u\|_4^4 \\ &\leq (Lu - \omega u, u) \\ &\leq -\delta \|u\|^2\,. \end{aligned}$$

This implies immediately that $u = 0$. □

Now we provide a sufficiently general result on exponential decay of solutions to equations of the form (1.2).

Proposition 6.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $f(u) = o(u)$ as $u \to 0$. Assume that the sequence $\{\varepsilon_n\}$ is N-periodic and $\omega \notin \sigma(L)$, while the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded. Then for any solution $u = \{u_n\} \in l^2$ there exist positive constants C and ν such that*

$$|u_n| \le Ce^{-\nu|n|}, \quad n \in \mathbb{Z}$$

Proof. Let us introduce the sequence h defined by

$$h_n = -\beta_n u_n^{-1} f(u_n) - \alpha_n(|u_{n+1}|^2 + |u_{n-1}|^2),$$

where, by definition, the first term in the right-hand side is 0 whenever $f(u_n) = 0$. Then

$$(L + h)u - \omega u = 0, \tag{6.1}$$

i.e., u is an eigenvector of the operator $L + h$ with eigenvalue ω. The assumptions of this proposition imply that $\lim_{|n|\to\infty} h_n = 0$. Now it is easily seen that, the multiplication operator by h is compact. Perturbations of such type do not change the essential spectrum, but may create eigenvalues of finite multiplicity outside (see, e.g., [6]). Hence,

$$\sigma_{ess}(L + h) = \sigma_{ess}(L)$$

and ω is of finite multiplicity. It is well-known that eigenvectors, with eigenvalue of finite multiplicity, of second order self-adjoint difference operators decay exponentially fast [15], and this completes the proof. □

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