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HIGHER ORDER BOUNDARY VALUE PROBLEMS AT RESONANCE ON AN UNBOUNDED INTERVAL

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The original version is attached at on pages 10-18

Abstract. The aim of this paper is the solvability of a class of higher order differential equations with initial conditions and an integral boundary condition on the half line. Using coincidence degree theory by Mawhin and constructing suitable operators, we prove the existence of solutions for the posed resonance boundary value problems.

1. Introduction

In this article, we are concerned with the existence of solutions of the higher-order ordinary differential equation

$$x^{(n)}(t) = f(t, x(t)), \quad t \in (0, \infty), \tag{1.1}$$

with the integral boundary value conditions

$$x^{(i)}(0) = 0,\ i = 0, 1, \dots, n-2, \quad x^{(n-1)}(\infty) = \frac{n!}{\xi^n} \int_0^\xi x(t)dt\,, \tag{1.2}$$

where $n \geq 3$ is an integer, $\xi > 0$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying certain conditions.

A boundary value problem (BVP for short) is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be formulated as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, as is known, the coincidence degree theory of Mawhin [19] has played an important role in dealing with the existence of solutions for these problems. For more recent results, we refer the reader to [3, 5, 6, 6, 8, 9, 14, 20, 22, 24, 25] and the references therein.

Moreover boundary value problems on the half line arise in many applications in physics such that in modeling the unsteady flow of a gas through semi-infinite porous media, in plasma physics, in determining the electrical potential in an isolated neutral atom, or in combustion theory. For an extensive literature of results

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as regards boundary value problems on unbounded domains, we refer the reader to the monograph by Agarwal and O'Regan [1].

Recently, there have been many works concerning the existence of solutions for the boundary value problems on the half-line. For instance see [2, 4, 10, 11, 12, 13, 15, 16, 17, 18, 21, 23] and the references therein. By the way, much of work on the existence of solutions for the boundary value problems on unbounded domains involves second or third-order differential equations.

However, for the resonance case, there is no work done for the higher-order boundary value problems with integral boundary conditions on the half-line, such as BVP (1.1)-(1.2).

The remaining part of this paper is organized as follows. We present in Section 2 some notations and basic results involved in the reformulation of the problem. In Section 3, we give the main theorem and some lemmas, then we will show that the proof of the main theorem is an immediate consequence of these lemmas and the coincidence degree of Mawhin.

2. Preliminaries

For the convenience of the readers, we recall some notation and two theorems which will be used later.

Let X, Y be two real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a linear operator which is Fredholm map of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Then $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if the map $QN(\overline{\Omega})$ is bounded and $K_P(I - QN) : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([19]). *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1. *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$.*
2. *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$.*
3. *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Since the Arzelá-Ascoli theorem fails in the noncompact interval case, we use the following result in order to show that $K_P(I - QN) : \overline{\Omega} \to X$ is compact.

Theorem 2.2 ([1]). *Let F be a subset of $C_\infty = \{y \in C([0, +\infty)), \lim_{t\to\infty} y(t) \text{ exists}\}$ that is equipped with the norm $\|y\|_\infty = \sup_{t\in[0,+\infty)} |y(t)|$. Then F is relatively compact if the following conditions hold:*

1. *F is bounded in X.*
2. *The functions belonging to F are equi-continuous on any compact subinterval of $[0, \infty)$.*
3. *The functions from F are equi-convergent at $+\infty$.*

Let

$$X = \big\{x \in C^{n-1}[0, +\infty),\ \lim_{t\to\infty} e^{-t}|x^{(i)}(t)| \text{ exist},\ 0 \le i \le n-1\big\}$$

endowed with the norm $\|x\| = \max_{0\le i\le n-1}\left(\sup_{t\in[0,+\infty)} e^{-t}|x^{(i)}(t)|\right)$. Then X is a Banach space.

Lemma 2.3. *Let $M \subset X$, then M is relatively compact in X if the following conditions hold:*

1. *M is bounded in X.*
2. *The family $V^i = \{y_i : y_i(t) = e^{-t}x^{(i)}(t), t \ge 0, x \in M\}$ is equicontinuous on any compact subinterval of $[0,+\infty)$ for $i = 0,\dots,n-1$.*
3. *The family $V^i = \{y_i : y_i(t) = e^{-t}x^{(i)}(t),\ t \ge 0,\ x \in M\}$ is equiconvergent at ∞ for $i = 0,\dots,n-1$.*

Proof. Let $(x_k)_k$ be a sequence in M and set $y_{i,k}(t) = e^{-t}x_k^{(i)}(t)$. Since the set V^i, for every $i = 0,\dots,n-1$ is relatively compact in C_∞ (see Theorem 2.2), then from the sequence $(y_{0,k})_k \subset V^0$, we can extract a subsequence denoted also by $(y_{0,k})_k$, that converges to y_0^* in C_∞. Set $x_0^*(t) = e^t y_0^*(t)$, then

$$\lim_{k\to\infty}\sup_{t\in[0,\infty)} e^{-t}|x_k(t) - x_0^*(t)| = 0.$$

Now let $y_{1,k}(t) = e^{-t}x_k'(t)$ then the sequence $(y_{1,k})_k \subset V^1$ and we can extract from it a subsequence denoted also by $(y_{1,k})_k$, that converges to y_1^* in C_∞. Set $x_1^*(t) = e^t y_1^*(t)$, then $\lim_{k\to\infty}\sup_{t\in[0,\infty)} e^{-t}|x_k'(t)-x_1^*(t)| = 0$, from the fact that the convergence is uniform on $[0,T]$, $T > 0$, we get that x_0^* is differentiable on $[0,+\infty)$ and $x_1^* = (x_0^*)'$. Reasoning the same way, we obtain $x_i^* = (x_0^*)^{(i)}$, $i = 0,\dots,n-1$, and $\lim_{k\to\infty}\|x_k - x_0^*\| = 0$. Then M is relatively compact. □

Let $Y = L^1[0,+\infty)$ with norm $\|y\|_1 = \int_0^{+\infty}|y(t)|dt$. Denote $AC_{\rm loc}[0,+\infty)$ the space of locally absolutely continuous functions on the interval $[0,+\infty)$. Define the operator $L : \operatorname{dom} L \subset X \to Y$ by $Lx = x^{(n)}$, where

$$\begin{gathered}\operatorname{dom} L = \Big\{x \in X, x^{(n-1)} \in AC_{\rm loc}[0,+\infty), x^{(i)}(0) = 0,\ i = \overline{0,n-2}\\ x^{(n-1)}(\infty) = \frac{n!}{\xi^n}\int_0^\xi x(t)dt, x^{(n)} \in Y\Big\} \subset X,\end{gathered}$$

then L maps $\operatorname{dom} L$ into Y. Let $N : X \to Y$ be the operator $Nx(t) = f(t,x(t))$, $t \in [0,+\infty)$, then (1.1)-(1.2) can be written as $Lx = Nx$.

3. Main results

We can now state our results on the existence of a solution for (1.1)-(1.2).

Theorem 3.1. *Assume that the following conditions are satisfied:*

- (H1) *There exists functions $\alpha,\beta \in L^1[0,\infty)$, such that for all $x \in \mathbb{R}$ and $t \in [0,\infty)$,*
$$|f(t,x)| \le e^{-t}\alpha(t)|x| + \beta(t). \tag{3.1}$$
- (H2) *There exists a constant $M > 0$, such that for $x \in \operatorname{dom} L$, if $|x^{(n-1)}(t)| > M$, for all $t \in [0,\infty)$, then*
$$\int_0^\infty f(s,x(s))ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s,x(s))ds \ne 0. \tag{3.2}$$

(H3) *There exists a constant $M^* > 0$, such that for any $x(t) = c_0 t^{n-1} \in \ker L$ with $|c_0| > M^*/(n-1)!$, either*

$$c_0\Big[\int_0^\infty f(s, c_0 s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1})ds\Big] < 0, \tag{3.3}$$

or

$$c_0\Big[\int_0^\infty f(s, c_0 s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1})ds\Big] > 0. \tag{3.4}$$

Then (1.1)-(1.2)*, has at least one solution in X, provided*

$$1 - 2M_n\|\alpha\|_1 > 0, \tag{3.5}$$

where $M_n = \max_{0\le i\le n-1}\big(\sup_{t\in[0,\infty)} e^{-t}t^{n-1-i}\big)$.

To prove Theorem 3.1, we need to prove some Lemmas.

Lemma 3.2. *The operator $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero. Furthermore, the linear projector operator $Q : Y \to Y$ can be defined by*

$$Qy(t) = ae^{-t}\Big[\int_0^\infty y(s)ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds\Big],$$

where

$$\frac{1}{a} = 1 - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n e^{-s}ds = 1 - \sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!\xi^k} + (-1)^n n!\frac{e^{-\xi}}{\xi^n}$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be written as

$$K_p y(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)ds, \quad y \in \operatorname{Im} L.$$

Furthermore,

$$\|K_p y\| \le M_n\|y\|_1, \quad \textit{for every } y \in \operatorname{Im} L. \tag{3.6}$$

Proof. It is clear that

$$\ker L = \big\{x \in \operatorname{dom} L : x = ct^{n-1},\ c \in \mathbb{R},\ t \in [0,\infty)\big\}.$$

Now we show that

$$\operatorname{Im} L = \big\{y \in Y : \int_0^\infty y(s)ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds = 0\big\}. \tag{3.7}$$

The problem

$$x^{(n)}(t) = y(t) \tag{3.8}$$

has a solution $x(t)$ that satisfies the conditions $x^{(i)}(0) = 0$, for $i = 0, 1, \dots, n-2$, and $x^{(n-1)}(\infty) = \frac{n!}{\xi^n}\int_0^\xi x(t)dt$ if and only if

$$\int_0^\infty y(s)ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds = 0. \tag{3.9}$$

In fact from (3.8) and the boundary conditions (1.2) we have

$$\begin{aligned} x(t) &= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)ds + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1}t^{n-1} \\ &= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)ds + ct^{n-1}. \end{aligned}$$

From $x^{(n-1)}(\infty) = \frac{n!}{\xi^n}\int_0^\xi x(t)dt$, we obtain

$$\int_0^\infty y(s)ds = \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds.$$

On the other hand, if (3.9) holds, setting

$$x(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)ds + ct^{n-1}$$

where c is an arbitrary constant, then $x(t)$ is a solution of (3.8). Hence (3.7) holds. Setting

$$Ry = \int_0^\infty y(s)ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds,$$

define $Qy(t) = ae^{-t}Ry$, it is clear that $\dim \operatorname{Im} Q = 1$. We have

$$\begin{aligned} Q^2y = Q(Qy) &= ae^{-t}(a.Ry)\Big(\int_0^\infty e^{-s}ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n e^{-s}ds\Big) \\ &= ae^{-t}Ry = Qy, \end{aligned}$$

which implies the operator Q is a projector. Furthermore, $\operatorname{Im} L = \ker Q$.

Let $y = (y - Qy) + Qy$, where $y - Qy \in \ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $\ker Q = \operatorname{Im} L$ and $Q^2y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then, we have $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus $\dim\ker L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 1$, this means that L is a Fredholm operator of index zero. Now we define a projector P from X to X by setting

$$Px(t) = \frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1}.$$

Then the generalized inverse $K_P : \operatorname{Im} L \operatorname{dom} L \cap \ker P$ of L can be written as

$$K_p y(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)ds.$$

Obviously, $\operatorname{Im} P = \ker L$ and $P^2x = Px$. It follows from $x = (x - Px) + Px$ that $X = \ker P + \ker L$. By simple calculation, we obtain that $\ker L \cap \ker P = \{0\}$. Hence $X = \ker L \oplus \ker P$.

From the definitions of P and K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = (K_p y(t))^{(n)} = y(t),$$

and for $x \in \operatorname{dom} L \cap \ker P$, we know that

$$\begin{aligned} (K_pL)x(t) = (K_p)x^{(n)}(t) &= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x^{(n)}(s)ds \\ &= x(t) - [x(0) + x'(0)t + \dots + \frac{x^{(n-2)}(0)}{(n-2)!}t^{n-2} + \frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1}]. \end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \ker P$, $x^{(i)}(0) = 0$, for $i = 0, 1, \dots, n-2$, and $Px = 0$, thus

$$(K_pL)x(t) = x(t).$$

This shows that $K_p = (L|_{\operatorname{dom} L\cap \ker P})^{-1}$. From the definition of K_p, we have for $i = 0, \dots, n-1$,

$$e^{-t}|(K_p y)^{(i)}(t)| \le \frac{e^{-t}}{(n-1-i)!}\int_0^t (t-s)^{n-1-i}|y(s)|ds \le M_n\|y\|_1,$$

which leads to

$$\|K_p y\| = \max_{0\le i\le n-1}\Big(\sup_{t\in[0,\infty)} e^{-t}|(K_p y)^i(t)|\Big) \le M_n\|y\|_1.$$

This completes the proof. □

Lemma 3.3. *Let $\Omega_1 = \{x \in \operatorname{dom} L\setminus \ker L : Lx = \lambda Nx$ for some $\lambda \in [0,1]\}$. Then Ω_1 is bounded.*

Proof. Suppose that $x \in \Omega_1$, and $Lx = \lambda Nx$. Thus $\lambda \ne 0$ and $QNx = 0$, so that

$$\int_0^\infty f(s,x(s))ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s,x(s))ds = 0.$$

Thus, by condition (H2), there exists $t_0 \in \mathbb{R}_+$, such that $|x^{(n-1)}(t_0)| \le M$. It follows from the absolute continuity of $x^{(n-1)}$ that

$$|x^{(n-1)}(0)| = \big|x^{(n-1)}(t_0) - \int_0^{t_0} x^{(n)}(s)ds\big|,$$

then, we have

$$|x^{(n-1)}(0)| \le M + \int_0^\infty |Lx(s)|ds \le M + \int_0^\infty |Nx(s)|ds = M + \|Nx\|_1. \tag{3.10}$$

Again for $x \in \Omega_1$ and $x \in \operatorname{dom} L\setminus \ker L$, we have $(I-P)x \in \operatorname{dom} L \cap \ker P$ and $LPx = 0$; thus from Lemma 3.2,

$$\begin{aligned} \|(I-P)x\| &= \|K_p L(I-P)x\| \\ &\le M_n\|L(I-P)x\|_1 \\ &= M_n\|Lx\|_1 \le M_n\|Nx\|_1. \end{aligned}\tag{3.11}$$

So

$$\|x\| \le \|Px\| + \|(I-P)x\| = M_n|x^{(n-1)}(0)| + M_n\|Nx\|_1, \tag{3.12}$$

again from (3.10) and (3.11), (3.12) becomes

$$\|x\| \le M_n M + M_n\|Nx\|_1 + M_n\|Nx\|_1 \le M_n M + 2M_n\|Nx\|_1. \tag{3.13}$$

On the other hand by (3.1) we have

$$\|Nx\|_1 = \int_0^\infty |f(s,x(s))|ds \le \|x\|\|\alpha\|_1 + \|\beta\|_1. \tag{3.14}$$

Therefore, (3.13) and (3.14), it yield

$$\|x\| \le M_n M + 2M_n\|x\|\|\alpha\|_1 + 2M_n\|\beta\|_1;$$

since $1 - 2M_n\|\alpha\|_1 > 0$, we obtain

$$\|x\| \le \frac{M_n M}{1 - 2M_n\|\alpha\|_1} + \frac{2M_n\|\beta\|_1}{1 - 2M_n\|\alpha\|_1}.$$

So Ω_1 is bounded. □

Lemma 3.4. *The set $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ is bounded.*

Proof. Let $x \in \Omega_2$. Then $x \in \ker L$ implies $x(t) = ct^{n-1}$, $c \in \mathbb{R}$, and $QNx = 0$; therefore

$$\int_0^\infty f(s, cs^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, cs^{n-1})ds = 0.$$

From condition (H2), there exists $t_1 \in \mathbb{R}_+$, such as $|x^{(n-1)}(t_1)| \le M$. We have $(n-1)!|c| \le M$ so $|c| \le \frac{M}{(n-1)!}$. On the other hand

$$\|x\| = |c| \max_{0 \le i \le n-1}\Big(\sup_{t \in [0,\infty)} e^{-t}(t^{n-1})^{(i)}\Big) \le MM_n < \infty,$$

so Ω_2 is bounded. $\square$

Lemma 3.5. *Suppose that the first part of Condition* (H3) *holds. Let*

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(ct^{n-1}) = ce^{-t}$, for all $c \in \mathbb{R}$ $t \ge 0$. Then Ω_3 is bounded.

Proof. In fact $x_0 \in \Omega_3$, means that $x_0 \in \ker L$ i.e. $x_0(t) = c_0 t^{n-1}$ and $\lambda Jx_0 = (1-\lambda)QNx_0$. Then we obtain

$$\lambda c_0 = (1-\lambda)a\Big(\int_0^\infty f(s, c_0 s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1})ds\Big).$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*/(n-1)!$, in view of (3.3) one has

$$\lambda c_0^2 = (1-\lambda)ac_0\Big(\int_0^\infty f(s, c_0 s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1})ds\Big) < 0,$$

which contradicts the fact that $\lambda c_0^2 \ge 0$. So $|c_0| \le M^*/(n-1)!$, moreover

$$\|x_0\| = |c_0| \max_{0 \le i \le n-1}\Big(\sup_{t \in [0,\infty)} e^{-t}|(t^{n-1})^{(i)}|\Big) \le M^* M_n.$$

Therefore Ω_3 is bounded. $\square$

Lemma 3.6. *Suppose that the second part of Condition* (H3) *holds. Let*

$$\Omega_3 = \{x \in \ker L : \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(ct^{n-1}) = ce^{-t}$, for all $c \in \mathbb{R}$, $t \ge 0$. Then Ω_3 is bounded, here J is as in Lemma 3.5.

Proof. Similar to the above argument, we can verify that Ω_3 is bounded. $\square$

Lemma 3.7. *Suppose that Ω is an open bounded subset of X such that* $\operatorname{dom}(L) \cap \overline{\Omega} \neq \emptyset$. *Then N is L-compact on $\overline{\Omega}$.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Without loss of generality, we may assume that $\Omega = B(0, r)$, then for any $x \in \overline{\Omega}$, $\|x\| \le r$. For $x \in \overline{\Omega}$, and by condition (3.1), we obtain

$$\begin{aligned}
|QNx| &\le ae^{-t}\Big[\int_0^\infty |f(s, x(s))|ds + \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n |f(s, x(s))|ds\Big] \\
&\le ae^{-t}\Big[\int_0^\infty e^{-s}\alpha(s)|x(s)| + \beta(s)ds \\
&\quad + \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n (e^{-s}\alpha(s)|x(s)| + \beta(s))ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le ae^{-t}\Big[r\int_0^\infty \alpha(s)ds+\int_0^\infty \beta(s)ds+r\int_0^\xi \alpha(s)ds+\int_0^\xi \beta(s)ds\Big]\\
&\le ae^{-t}[2r\|\alpha\|_1+2\|\beta\|_1]\\
&\le 2a[r\|\alpha\|_1+\|\beta\|_1];
\end{aligned}
$$

thus,

$$
\|QNx\|_1\le 2a[r\|\alpha\|_1+\|\beta\|_1], \tag{3.15}
$$

which implies that $QN(\overline{\Omega})$ is bounded. Next, we show that $K_P(I-Q)N(\overline{\Omega})$ is compact, for this we use Lemma 2.3 . Let $x\in\overline{\Omega}$, by (3.1) we have

$$
\|Nx\|_1=\int_0^\infty |fs,x(s)|ds\le [r\|\alpha\|_1+\|\beta\|_1]; \tag{3.16}
$$

on the other hand, from the definition of K_P and together with (3.6), (3.15) and (3.16) one obtain

$$
\begin{aligned}
\|K_P(I-Q)Nx\| &\le M_n\|(I-Q)Nx\|_1\le M_n[\|Nx\|_1+\|QNx\|_1]\\
&\le M_n[r(1+2a)\|\alpha\|_1+(1+2a)\|\beta\|_1].
\end{aligned}
$$

It follows that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

Let us prove that T is equicontinuous. For any $x\in\overline{\Omega}$ and any $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $T\in[0,\infty)$, we have for $0\le i\le n-2$:

$$
\begin{aligned}
&\big|e^{-t_1}(K_P(I-Q)Nx)^{(i)}(t_1)-e^{-t_2}(K_P(I-Q)Nx)^{(i)}(t_2)\big|\\
&=\big|\int_{t_1}^{t_2}[e^{-s}(K_P(I-Q)Nx)^{(i)}(s)]'ds\big|\\
&=\big|\int_{t_1}^{t_2}[-e^{-s}(K_P(I-Q)Nx)^{(i)}(s)+e^{-s}(K_P(I-Q)Nx)^{(i+1)}(s)]ds\big|\\
&\le 2(t_2-t_1)\|K_P(I-Q)Nx\|\\
&\le 2(t_2-t_1)M_n[r(1+2a)\|\alpha\|_1+(1+2a)\|\beta\|_1]\to 0,\quad \text{as } t_1\to t_2.
\end{aligned}
$$

For $i=n-1$, we obtain

$$
\begin{aligned}
&\big|e^{-t_1}(K_P(I-Q)Nx)^{(n-1)}(t_1)-e^{-t_2}(K_P(I-Q)Nx)^{(n-1)}(t_2)\big|\\
&=\big|e^{-t_1}\int_0^{t_1}(I-Q)Nx(s)ds-e^{-t_2}\int_0^{t_2}(I-Q)Nx(s)ds\big|\\
&\le \int_0^{t_1}(e^{-t_1}-e^{-t_2})|(I-Q)Nx(s)|ds+\int_{t_1}^{t_2}e^{-t_2}|(I-Q)Nx(s)|ds\\
&\le (t_2-t_1)\int_0^{t_1}|(I-Q)Nx(s)|ds+\int_{t_1}^{t_2}|(I-Q)Nx(s)|ds\to 0,
\end{aligned}
$$

as $t_1\to t_2$. So $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous on every compact subinterval of $[0,\infty)$. In addition, we claim that $K_P(I-Q)N(\overline{\Omega})$ is equiconvergent at infinity. In fact, for $x\in\overline{\Omega}$, $i=0,\dots,n-1$, we have

$$
\begin{aligned}
&\big|e^{-t}(K_p(I-Q)Nx)^{(i)}(t)\big|\\
&\le \frac{e^{-t}}{(n-1-i)!}\int_0^t (t-s)^{n-1-i}|(I-Q)Nx(s)|ds\\
&\le e^{-t}t^{n-1-i}\int_0^t |(I-Q)Nx(s)|ds\le e^{-t}t^{n-1-i}\|(I-Q)Nx\|_1
\end{aligned}
$$

$$\leq e^{-t}t^{n-1-i}[\|Nx\|_1 + \|QNx\|_1] \leq e^{-t}t^{n-1-i}(1+2a)[r\|\alpha\|_1 + \|\beta\|_1],$$

thus, $\lim_{t\to\infty} e^{-t}(K_p(I-Q)Nx)^{(i)}(t) = 0$, for every $i = 0, \dots, n-1$, which means that $K_P(I-Q)N(\overline{\Omega})$ is equiconvergent at infinity. □

Now we are able to give the proof of Theorem 3.1, which is an immediate consequence of Theorem 2.1 and the above lemmas.

Proof of Theorem 3.1. We shall prove that all conditions of Theorem 2.1 are satisfied. Set Ω to be an open bounded subset of X such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω we have

(i) $Lx \neq \lambda Nx$ pour tout $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;
(ii) $Nx \notin \operatorname{Im} L$ pour tout $x \in \ker L \cap \partial\Omega$.

At last we prove that condition (iii) of Theorem 2.1 is satisfied. To this end, let

$$H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$$

By the definition of Ω we know that $\overline{\Omega}_3 \subset \Omega$, thus $H(x,\lambda) \neq 0$ for every $x \in \ker L \cap \partial\Omega$. Then, by the homotopy property of degree, we obtain

$$\begin{aligned}\deg(QN|_{\ker L}, \Omega\cap\ker L, 0) &= \deg(H(\cdot,0), \Omega\cap\ker L, 0)\\ &= \deg(H(\cdot,1), \Omega\cap\ker L, 0)\\ &= \deg(\pm J, \Omega\cap\ker L, 0) \neq 0.\end{aligned}$$

So, the third assumption of Theorem 2.1 is fulfilled and $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e. (1.1)-(1.2) has at least one solution in X. The prove is complete. □

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Next, for record keeping we include the original article that has several mistakes.

11. Introduction

In this paper, we are concerned with the existence of solutions of the higher-order ordinary differential equation

$$x^{(n)}(t) = f(t, x(t)), \quad t \in (0, \infty), \tag{11.1}$$

with the integral boundary value conditions

$$x^{(i)}(0) = 0,\ i = 0, 1, \dots, n-2, \quad x^{(n-1)}(\infty) = \frac{n!}{\xi^n}\int_0^{\xi} x(t)dt\,, \tag{11.2}$$

where $n \geq 3$ is an integer, $\xi > 0$ and $f \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$.

A boundary value problem (BVP for short) is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be formulated as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, as is known, the coincidence degree theory of Mawhin [19] has played an important role in dealing with the existence of solutions for these problems. For more recent results, we refer the reader to [3, 5, 6, 6, 8, 9, 14, 20, 22, 24, 25] and the references therein.

Moreover boundary value problems on the half line arise in many applications in physics such that in modeling the unsteady flow of a gas through semi-infinite porous media, in plasma physics, in determining the electrical potential in an isolated neutral atom, or in combustion theory. For an extensive literature of results as regards boundary value problems on unbounded domains, we refer the reader to the monograph by Agarwal and O'Regan [1].

Recently, there have been many works concerning the existence of solutions for the boundary value problems on the half-line. For instance see [2, 4, 10, 11, 12, 13, 15, 16, 17, 18, 21, 23] and the references therein. By the way, much of work on the existence of solutions for the boundary value problems on unbounded domains involves second or third-order differential equations.

However, for the resonance case, there is no work done for the higher-order boundary value problems with integral boundary conditions on the half-line, such as BVP (11.1)-(11.2).

The remaining part of this paper is organized as follows. We present in Section 2 some notations and basic results involved in the reformulation of the problem. In Section 3, we give the main theorem and some lemmas, then we will show that the proof of the main theorem is an immediate consequence of these lemmas and the coincidence degree of Mawhin.

12. Preliminaries

For the convenience of the readers, we recall some notation and two theorems which will be used later.

Let X, Y be two real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a linear operator which is Fredholm map of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Then $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus ImQ$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if the map $QN(\overline{\Omega})$ is bounded and $K_P(I - QN) : \overline{\Omega} \to X$ is compact.

Theorem 12.1 ([19]). *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1. *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$.*
2. *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$.*
3. *$\deg(QN|_{\ker L}, \Omega \cap \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Since the Arzelá-Ascoli theorem fails in the noncompact interval case, we use the following result in order to show that $K_P(I - QN) : \overline{\Omega} \to X$ is compact.

Theorem 12.2 ([1]). *Let $F \subset X$. Then F is relatively compact if the following conditions hold:*

1. *F is bounded in X.*
2. *The functions belonging to F are equi-continuous on any compact interval of $[0, \infty)$.*
3. *The functions from F are equi-convergent at $+\infty$.*

Let $AC[0,+\infty)$ denote the space of locally absolutely continuous functions on the interval $[0,+\infty)$. Let

$$X = \big\{x \in C^{n-1}[0,+\infty) : x^{(n-1)} \in AC_{\rm loc}[0,+\infty),\ \lim_{t\to\infty} e^{-t}|x(t)| \text{ exists}\big\}$$

endowed with the norm $\|x\| = \sup_{t\in[0,+\infty)}\ e^{-t}|x(t)|$. Let $Y = L^1[0,+\infty)$ with norm $\|y\|_1 = \int_0^{+\infty} |y(t)|dt$.

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by $Lx = x^{(n)}$, where

$$\operatorname{dom} L = \big\{x \in X : x^{(i)}(0) = 0,\ i = \overline{0, n-2}, x^{(n-1)}(\infty) = \frac{n!}{\xi^n}\int_0^\xi x(t)dt\big\}.$$

Let $N : X \to Y$ be the operator $Nx = f(t, x(t))$, $t \in [0,+\infty)$, then the BVP (11.1)–(11.2) can be written as $Lx = Nx$.

13. Main results

We can now state our results on the existence of a solution for (11.1)-(11.2).

Theorem 13.1. *Assume that the following conditions are satisfied:*

(H1) *There exists functions $\alpha, \beta \in L^1[0,\infty)$, such that for all $x \in \mathbb{R}$ and $t \in [0,\infty)$,*
$$|f(t, x)| \le e^{-t}\alpha(t)|x| + \beta(t). \tag{13.1}$$

(H2) *There exists a constant $M > 0$, such that for $x \in \operatorname{dom} L$, if $|x^{(n-1)}(t)| > M$, for all $t \in [0,\infty)$, then*
$$\int_0^\infty f(s, x(s))ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, x(s))ds \neq 0. \tag{13.2}$$

(H3) *There exists a constant $M^* > 0$, such that for any $x(t) = c_0t^{n-1} \in \ker L$ with $|c_0| > M^*/(n-1)!$, either*
$$c_0\Big[\int_0^\infty f(s, c_0s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0s^{n-1})ds\Big] < 0, \tag{13.3}$$
or
$$c_0\Big[\int_0^\infty f(s, c_0s^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n f(s, c_0s^{n-1})ds\Big] > 0. \tag{13.4}$$

Then (11.1)-(11.2)*, has at least one solution in $C[0,\infty)$, provided*
$$1 - 2M_n\|\alpha\|_1 > 0, \tag{13.5}$$
where $M_n = \sup_{t\in[0,\infty)} e^{-t}t^{n-1} = (\frac{n-1}{e})^{n-1}$.

To prove Theorem 13.1, we need to prove some Lemmas.

Lemma 13.2. *The operator $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero. Furthermore, the linear projector operator $Q : Y \to Y$ can be defined by*

$$Qy(t) = ae^{-t}\Big[\int_0^\infty y(s)ds - \frac{1}{\xi^n}\int_0^\xi (\xi - s)^n y(s)ds\Big],$$

where

$$1/a = 1 - \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\xi^k}$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *can be written as*

$$K_p y(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds, y \in \operatorname{Im} L.$$

Furthermore

$$\|K_p y\| \le \frac{M_n}{(n-1)!} \|y\|_1, \quad \text{for every } y \in \operatorname{Im} L. \tag{13.6}$$

Proof. It is clear that

$$\ker L = \{x \in \operatorname{dom} L : x = ct^{n-1},\ c \in \mathbb{R},\ t \in [0,\infty)\}.$$

Now we show that

$$\operatorname{Im} L = \Big\{ y \in Y : \int_0^\infty y(s) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n y(s) ds = 0 \Big\}. \tag{13.7}$$

The problem

$$x^{(n)}(t) = y(t) \tag{13.8}$$

has a solution $x(t)$ that satisfies the conditions $x^{(i)}(0) = 0$, for $i = 0, 1, \dots, n-2$, and $x^{(n-1)}(\infty) = \frac{n!}{\xi^n} \int_0^\xi x(t) dt$ if and only if

$$\int_0^\infty y(s) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n y(s) ds = 0. \tag{13.9}$$

In fact from (13.8) and the boundary conditions (11.2) we have

$$\begin{aligned} x(t) &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1} \\ &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds + ct^{n-1}. \end{aligned}$$

From $x^{(n-1)}(\infty) = \frac{n!}{\xi^n} \int_0^\xi \ x(t) dt$, we obtain

$$\int_0^\infty y(s) ds = \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n y(s) ds.$$

On the other hand, if (13.9) holds, setting

$$x(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds + ct^{n-1}$$

where c is an arbitrary constant, then $x(t)$ is a solution of (13.8). Hence (13.7) holds. Setting

$$Ry = \int_0^\infty y(s) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n y(s) ds,$$

define $Qy(t) = ae^{-t} Ry$, it is clear that $\dim \operatorname{Im} Q = 1$. We have

$$\begin{aligned} Q^2 y = Q(Qy) &= ae^{-t} (a.Ry) (\int_0^\infty e^{-s} ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n e^{-s} ds) \\ &= ae^{-t} Ry = Qy, \end{aligned}$$

that implies the operator Q is a projector. Furthermore, $\operatorname{Im} L = \ker Q$.

Let $y = (y - Qy) + Qy$, where $y - Qy \in \ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $\ker Q = \operatorname{Im} L$ and $Q^2 y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then, we have $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus $\dim \ker L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 1$, this means that L is a Fredholm operator of index zero. Now we define a projector P from X to X by setting

$$Px(t) = \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1}.$$

Then the generalized inverse $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L can be written as

$$K_p y = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds.$$

Obviously, $\operatorname{Im} P = \ker L$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $X = \ker P + \ker L$. By simple calculation, we obtain that $\ker L \cap \ker P = \{0\}$. Hence $X = \ker L \oplus \ker P$.

From the definitions of P and K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = (K_p y(t))^{(n)} = y(t),$$

and for $x \in \operatorname{dom} L \cap \ker P$, we know that

$$\begin{aligned}
(K_p L)x(t) = (K_p)x^{(n)}(t) &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} x^{(n)}(s) ds \\
&= x(t) - [x(0) + x'(0)t + \dots\dots . \frac{x^{(n-2)}(0)}{(n-2)!} t^{n-2} + \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1}].
\end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \ker P$, $x^{(i)}(0) = 0$, for $i = 0, 1, \dots, n-2$, and $Px = 0$, thus

$$(K_p L)x(t) = x(t).$$

This shows that $K_p = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. From the definition of K_p, we have

$$\begin{aligned}
\|K_p y\| = \sup_{t \in [0,\infty)} e^{-t} |K_p y| &\leq \sup_{t \in [0,\infty)} \frac{e^{-t}}{(n-1)!} \int_0^t (t-s)^{n-1} |y(s)| ds \\
&< \frac{M_n}{(n-1)!} \int_0^\infty |y(s)| ds = \frac{M_n}{(n-1)!} \|y\|_1.
\end{aligned}$$

This completes the proof. □

Lemma 13.3. *Let $\Omega_1 = \{x \in \operatorname{dom} L \backslash \ker L : Lx = \lambda Nx$ for some $\lambda \in [0,1]\}$. Then Ω_1 is bounded.*

Proof. Suppose that $x \in \Omega_1$, and $Lx = \lambda Nx$. Thus $\lambda \neq 0$ and $QNx = 0$, so that

$$\int_0^\infty f(s, x(s)) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n f(s, x(s)) ds = 0.$$

Thus, by condition (H2), there exists $t_0 \in \mathbb{R}_+$, such that $|x^{(n-1)}(t_0)| \leq M$. It follows from the absolute continuity of $x^{(n-1)}$ that

$$|x^{(n-1)}(0)| = \big|x^{(n-1)}(t_0) - \int_0^{t_0} x^{(n)}(s) ds\big|,$$

then, we have

$$|x^{(n-1)}(0)| \leq M + \int_0^\infty |Lx(s)| ds \leq M + \int_0^\infty |Nx(s)| ds = M + \|Nx\|_1. \tag{13.10}$$

Again for $x \in \Omega_1$ and $x \in \operatorname{dom} L \setminus \ker L$, we have $(I-P)x \in \operatorname{dom} L \cap \ker P$ and $LPx = 0$; thus from Lemma 13.2,

$$\begin{aligned} \|(I-P)x\| &= \|K_p L(I-P)x\| \\ &\le \frac{M_n}{(n-1)!}\|L(I-P)x\|_1 \\ &= \frac{M_n}{(n-1)!}\|Lx\|_1 \le \frac{M_n}{(n-1)!}\|Nx\|_1. \end{aligned} \tag{13.11}$$

So

$$\|x\| \le \|Px\| + \|(I-P)x\| = M_n|x^{(n-1)}(0)| + \frac{M_n}{(n-1)!}\|Nx\|_1, \tag{13.12}$$

again from (13.10) and (13.11), (13.12) becomes

$$\|x\| \le M_n M + M_n\|Nx\|_1 + \frac{M_n}{(n-1)!}\|Nx\|_1 \le M_n M + 2M_n\|Nx\|_1. \tag{13.13}$$

On the other hand by (13.1) we have

$$\|Nx\|_1 = \int_0^\infty |f(s,x(s))|ds \le \|x\|\|\alpha\|_1 + \|\beta\|_1. \tag{13.14}$$

Therefore, (13.13) and (13.14), it yield

$$\|x\| \le M_n M + 2M_n\|x\|\|\alpha\|_1 + 2M_n\|\beta\|_1;$$

since $1 - 2M_n\|\alpha\|_1 > 0$, we obtain

$$\|x\| \le \frac{M_n M}{1-2M_n\|\alpha\|_1} + \frac{2M_n\|\beta\|_1}{1-2M_n\|\alpha\|_1}.$$

So Ω_1 is bounded. □

Lemma 13.4. *The set $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ is bounded.*

Proof. Let $x \in \Omega_2$, then $x \in \ker L$ implies $x(t) = ct^{n-1}$, $c \in \mathbb{R}$, and $QNx = 0$; therefore

$$\int_0^\infty f(s,cs^{n-1})ds - \frac{1}{\xi^n}\int_0^\xi (\xi-s)^n f(s,cs^{n-1})ds = 0.$$

From condition (H_2), there exists $t_1 \in \mathbb{R}_+$, such as $|x^{(n-1)}(t_1)| \le M$. We have

$$(n-1)!|c| \le M$$

so $|c| \le \frac{M}{(n-1)!}$. On the other hand

$$\|x\| = \sup_{t\in[0\infty)} e^{-t}|x(t)| = |c|\sup_{t\in[0\infty)} e^{-t}t^{n-1} = |c|M_n,$$

i.e. $\|x\| \le \frac{M_n M}{(n-1)!} < \infty$, so Ω_2 is bounded. □

Lemma 13.5. *Suppose that the first part of Condition* (H3) *holds. Let*

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(ct^{n-1}) = ct^{n-1}$, for all $c \in \mathbb{R}$ $t \ge 0$. Then Ω_3 is bounded.

Proof. In fact $x_0 \in \Omega_3$, means that $x_0 \in \ker L$ i.e. $x_0(t) = c_0 t^{n-1}$ and $\lambda J x_0 = (1-\lambda) Q N x_0$. Then we obtain

$$\lambda c_0 t^{n-1} = (1-\lambda) a e^{-t} \Big(\int_0^\infty f(s, c_0 s^{n-1}) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1}) ds \Big).$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (13.3) one has

$$\lambda c_0^2 t^{n-1} = (1-\lambda) a e^{-t} c_0 (\int_0^\infty f(s, c_0 s^{n-1}) ds - \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n f(s, c_0 s^{n-1}) ds) < 0,$$

which contradicts the fact that $\lambda c_0^2 \geq 0$. So $|c_0| \leq M^*$, moreover

$$\|x_0\| = \sup e^{-t} |c_0| t^{n-1} = |c_0| M_n \leq M^* M_n.$$

Therefore Ω_3 is bounded. $\square$

Lemma 13.6. *Suppose that the second part of Condition* (H3) *holds. Let*

$$\Omega_3 = \{ x \in \ker L : \lambda J x + (1-\lambda) Q N x = 0,\ \lambda \in [0,1] \}$$

where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(ct^{n-1}) = ct^{n-1}$, for all $c \in \mathbb{R}$, $t \geq 0$. Then Ω_3 is bounded here J as in Lemma 13.5. Similar to the above argument, we can verify that Ω_3 is bounded.

Lemma 13.7. *Suppose that Ω is an open bounded subset of X such that* $\operatorname{dom}(L) \cap \overline{\Omega} \neq \emptyset$*. Then N is L-compact on $\overline{\Omega}$.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Without loss of generality, we may assume that $\Omega = B(0, r)$, then for any $x \in \overline{\Omega}$, $\|x\| \leq r$. For $x \in \overline{\Omega}$, and by condition (13.1), we obtain

$$\begin{aligned}
e^{-t} |QNx| &\leq a e^{-2t} \Big[\int_0^\infty |f(s, x(s))| ds + \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n |f(s, x(s))| ds \Big] \\
&\leq a e^{-2t} \Big[\int_0^\infty e^{-s} \alpha(s) |x(s)| + \beta(s) ds \\
&\quad + \frac{1}{\xi^n} \int_0^\xi (\xi - s)^n (e^{-s} \alpha(s) |x(s)| + \beta(s)) ds \Big] \\
&\leq a e^{-2t} \Big[r \int_0^\infty \alpha(s) ds + \int_0^\infty \beta(s) ds + r \int_0^\xi \alpha(s) ds + \int_0^\xi \beta(s) ds \Big] \\
&\leq a e^{-2t} [2r \|\alpha\|_1 + 2 \|\beta\|_1] \\
&\leq 2a [r \|\alpha\|_1 + \|\beta\|_1];
\end{aligned}$$

thus,

$$\|QNx\|_1 \leq 2a [r \|\alpha\|_1 + \|\beta\|_1], \tag{13.15}$$

which implies that $QN(\overline{\Omega})$ is bounded. Next, we show that $K_P(I - Q)N(\overline{\Omega})$ is compact. For $x \in \overline{\Omega}$, by (13.1) we have

$$\|Nx\|_1 = \int_0^\infty |f s, x(s)| ds \leq [r \|\alpha\|_1 + \|\beta\|_1]; \tag{13.16}$$

on the other hand, from the definition of K_P and together with (13.6), (13.15) and (13.16) one gets

$$\begin{aligned}
\|K_P(I - Q)N\| &\leq M_n \|(I - Q)N\|_1 \leq M_n [\|Nx\|_1 + \|QNx\|_1] \\
&\leq M_n [r(1 + 2a) \|\alpha\|_1 + (1 + 2a) \|\beta\|_1].
\end{aligned}$$

It follows that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

Let us prove that T is equicontinuous. For any $x\in\overline{\Omega}$ and any $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $T\in[0,\infty)$, we have

$$\begin{aligned}
&|e^{-t_1}K_P(I-Q)Nx(t_1)-e^{-t_2}K_P(I-Q)Nx(t_2)|\\
&=\frac{1}{(n-1)!}\Big|\int_0^{t_1}e^{-t_1}(t_1-s)^{n-1}(I-Q)Nx(s)ds\\
&\quad-\int_0^{t_2}e^{-t_2}(t_2-s)^{n-1}(I-Q)Nx(s)ds\Big|\\
&\le\frac{1}{(n-1)!}[\int_0^{t_1}e^{-t_2}(t_2-s)^{n-1}-e^{-t_1}(t_1-s)^{n-1}|(I-Q)Nx(s)|ds\\
&\quad+\int_{t_1}^{t_2}e^{-t_2}(t_2-s)^{n-1}|(I-Q)Nx(s)|ds]\\
&\le\frac{1}{(n-1)!}\Big[\int_0^{t_1}(e^{-(t_2-s)}(t_2-s)^{n-1}-e^{-(t_1-s)}(t_1-s)^{n-1})\\
&\quad\times\ e^{-s}|(I-Q)Nx(s)|ds\\
&\quad+\int_{t_1}^{t_2}e^{-(t_2-s)}(t_2-s)^{n-1}e^{-s}|(I-Q)Nx(s)|ds\Big]\\
&\le\frac{1}{(n-1)!}[M_n'(t_2-t_1)\int_0^{t_1}e^{-s}|(I-Q)Nx(s)|ds\\
&\quad+e^{-t_2}(t_2-t_1)^{n-1}\int_{t_1}^{t_2}|(I-Q)Nx(s)|ds\Big]\to0,\quad\text{as } t_1\to t_2.
\end{aligned}$$

So $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous on every compact subset of $[0,\infty)$. In addition, we claim that $K_P(I-Q)N(\overline{\Omega})$ is equiconvergent at infinity. In fact,

$$\begin{aligned}
&|e^{-t}K_p(I-Q)Nx(t)|\\
&\le\frac{1}{(n-1)!}\int_0^t e^{-(t-s)}(t-s)^{n-1}e^{-s}|(I-Q)Nx(s)|ds\\
&\le\frac{M_n}{(n-1)!}\int_0^t|(I-Q)Nx(s)|ds\le\frac{M_n}{(n-1)!}\|(I-Q)Nx\|_1\\
&\le\frac{M_n}{(n-1)!}[\|Nx\|_1+\|QNx\|_1]<\infty;
\end{aligned}$$

thus, $\lim_{t\to\infty}|e^{-t}K_p(I-Q)Nx(t)|<\infty$. Which means that $K_P(I-Q)N(\overline{\Omega})$ is equiconvergent □

Now we are able to give the proof of Theorem 13.1, which is an immediate consequence of Theorem 12.1 and the above lemmas.

Proof of Theorem 13.1. We shall prove that all conditions of Theorem 12.1 are satisfied. Set Ω to be an open bounded subset of X such that $\cup_{i=1}^3\overline{\Omega}_i\subset\Omega$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω we have

(i) $Lx\neq\lambda Nx$ pour tout $(x,\lambda)\in[(\operatorname{dom}L\setminus\ker L)\cap\partial\Omega]\times(0,1)$;
(ii) $Nx\notin\operatorname{Im}L$ pour tout $x\in\ker L\cap\partial\Omega$.

At last we prove that condition (iii) of Theorem 12.1 is satisfied. To this end, let

$$H(x, \lambda) = \pm\lambda Jx + (1 - \lambda)QNx$$

By the definition of Ω we know that $\overline{\Omega}_3 \subset \Omega$, thus $H(x, \lambda) \neq 0$ for every $x \in \ker L \cap \partial\Omega$. Then, by the homotopy property of degree, we obtain

$$\begin{aligned} \deg(QN|_{\ker L}, \Omega \cap \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \cap \ker L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \cap \ker L, 0) \\ &= \deg(\pm J, \Omega \cap \cap \ker L, 0) \neq 0. \end{aligned}$$

So, the third assumption of Theorem 12.1 is fulfilled and $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e. (11.1)-(11.2) has at least one solution in X. The prove is complete. □

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