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IDENTIFICATION OF AN UNKNOWN SOURCE TERM FOR A TIME FRACTIONAL FOURTH-ORDER PARABOLIC EQUATION

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Abstract. In this article, we considered two inverse source problems for fourth-order parabolic differential equation with fractional derivative in time. Determination of a space dependent source term from the data given at some time $t = T$ is considered in one problem while other addresses the recovery of a time dependent source term from the integral type over-determination condition. Existence and uniqueness of the solution of both inverse source problems are proved. The stability results for the inverse problems are presented.

1. Introduction

We are concerned with the fourth-order parabolic equation

$$D_{0_+}^{\alpha,\gamma}u(x,t) + u_{xxxx}(x,t) = F(x,t), \quad (x,t) \in \Omega := [0,1] \times (0,T], \tag{1.1}$$

with initial condition

$$I_{0_+}^{1-\gamma}u(x,t)|_{t=0} = \varphi(x), \quad x \in [0,1], \tag{1.2}$$

and nonlocal boundary conditions

$$u_x(0,t) = u_x(1,t), \quad u(0,t) = 0, \tag{1.3}$$

$$u_{xxx}(0,t) = u_{xxx}(1,t), \quad u_{xx}(1,t) = 0, \quad t \in (0,T], \tag{1.4}$$

where $D_{0_+}^{\alpha,\gamma}(\cdot)$ stands for the generalized left sided fractional derivative of order α and type γ in the time variable (also known as Hilfer fractional derivative), introduced by Hilfer [12] and is given by

$$D_{0_+}^{\alpha,\gamma}w(t) := \Big[I_{0_+}^{(\gamma-\alpha)}\frac{d}{dt}\Big(I_{0_+}^{(1-\gamma)}\Big)\Big]w(t), \quad 0 < \alpha \le \gamma < 1. \tag{1.5}$$

The left sided fractional integral is defined by

$$I_{0_+}^{\beta}w(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}w(\tau)d\tau, \quad t > 0, \quad \beta > 0, \tag{1.6}$$

where $w \in L^1_{loc}[0,T]$, $0 < t < T \le \infty$, is a locally integrable real-valued function and $\Gamma(\cdot)$ is the Euler gamma function. The fractional derivative in (1.1) interpolates the Riemann-Liouville fractional derivative and Caputo fractional derivative for $\gamma = \alpha$ and $\gamma = 1$, respectively. The Riemann-Liouville fractional derivative may has

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singularity at $t = 0$ and usually has initial conditions in terms of fractional integral whereas Caputo fractional derivative are used more frequently in the literature because with Caputo derivative the initial conditions are more natural [24]. Both Riemann-Liouville and Caputo fractional derivatives can be used in the modelling of anomalous diffusion and the fractional derivative $D_{0_+}^{\alpha,\gamma}(\cdot)$ has the properties of both of these fractional derivatives.

The nonlocal boundary conditions such as in (1.3)-(1.4) arise when we cannot measure data directly at the boundary. Such type of boundary conditions usually known as Samarskii-Ionkin boundary conditions which arise from particle diffusion in turbulent plasma and in heat propagation where the law of variation of total quantity of the heat is given [13]. For applications of more general nonlocal boundary conditions see [7, 36, 35].

The direct problem for (1.1)-(1.4) is the unique determination of $u(x,t)$ in $\bar{\Omega}$ such that $u(\cdot,t) \in C^4[0,1]$, $D_{0_+}^{\alpha,\gamma}u(x,\cdot) \in C(0,T]$, when the initial condition $\varphi(x)$ and the source term $F(x,t)$ are given and continuous. The direct problem with $\gamma = 1$ of homogenous equation (1.1), i.e., $F(x,t) = 0$ with initial condition $u(x,0) = au(x,1) + \phi(x)$ and boundary conditions (1.3)-(1.4) was considered by Berdyshev et al. in [3]. They proved existence and uniqueness of the regular solution of the direct problem. The main concern of this paper are the following inverse problems related to (1.1)-(1.4).

Inverse source problem I (ISP-I): For the first problem, we suppose the source term $F(x,t)$ depends only on the space variable, i.e., $F(x,t) = f(x)$. The inverse problem is to determine the source term $f(x)$ and $u(x,t)$ such that $u(x,t)$ satisfies the equation (1.1)-(1.4) from $u(x,T) = \psi(x)$. Indeed, we are looking for the map

$$\psi(x) \to \{f(x), u(x,t)\}, \quad t < T.$$

By a regular solution of the ISP-I we mean a pair of functions $\{u(x,t), f(x)\}$ such that $u(\cdot,t) \in C^4[0,1]$, $D_{0_+}^{\alpha,\gamma}u(x,\cdot) \in C(0,T]$ and $f(x) \in C[0,1]$.

Inverse source problem II (ISP-II): For the second problem, we consider the source term as $F(x,t) = a(t)f(x,t)$. We are interested in recovering the time dependent source term $a(t)$ and $u(x,t)$. The inverse source problems of determination of a time dependent source term was considered by many, for example see [37, 28, 41]. Physically, such type of source; that is, $a(t)f(x,t)$ arise in microwave heating process, in which the external energy is supplied to a target at a controlled level, represented by $a(t)$ and $f(x,t)$ is the local conversion rate of the microwave energy.

For problem (1.1)-(1.4) the ISP-II is not uniquely solvable an over-determination condition of integral type given by

$$\int_0^1 xu(x,t)dx = g(t), \quad t \in [0,T], \tag{1.7}$$

is considered, where $g(t) \in AC[0,T]$, the space of absolutely continuous functions. The integral type condition arise naturally as over-determination condition for recovering the time dependent source term, in chemical engineering [6], fluid flow in porous medium [8] and in some other applications see for example [32, 17]. A regular solution for the ISP-II is a pair of functions $\{u(x,t), a(t)\}$ such that $u(\cdot,t) \in C^4[0,1]$, $D_{0_+}^{\alpha,\gamma}u(x,\cdot) \in C(0,T]$ and $a(t) \in C[0,T]$.

The spectral problem for (1.1)-(1.4) is not self-adjoint and a bi-orthogonal system of functions is constructed from eigenfunctions of spectral and its adjoint problem. We proved that both inverse problems are well posed in the sense of Hadamard (see Section 3 and 4).

It is well known that the inverse problems for the parabolic equations are ill-posed apart from this the inverse problems considered here are not easy to handle due to the nonlocal boundary conditions (1.3)-(1.4) and the presence of generalized fractional derivative in time. The fourth order parabolic differential equations have been considered in applications to combustion theory [2], image smoothing and denoising [25, 10], incompressible elasticity problem, phase transition and surface tension problem [5], thin film theory, lubrication theory [1].

The calculus of arbitrary order integrals and derivative usually known as fractional calculus could be considered as old as integer order calculus. For the history of the subject the interested readers are referred to [26]. Fractional calculus got considerable attention in mathematics and other fields of science, because fractional integrals and derivatives were used in the modeling of many physical, chemical, biological process (see the monographs [27, 38]).

Let us dwell with some of the articles which considered the inverse problems related to time fractional parabolic equations. A stable algorithm using mollification techniques has been proposed by Murio [30] for the inverse problem of boundary function for time fractional diffusion equation from a given noisy temperature distribution.

Kirane et al [19] considered two dimensional inverse source problem for time fractional diffusion equation and prove the well posedness of the inverse source problem. Jin and Rundell [16] consider the problem of recovering a spatially varying potential for a one dimensional time fractional diffusion equation from the flux measurements at a particular time. Li et al [21] propose algorithms for simultaneous inversion of order of fractional derivative and a space dependent diffusion coefficient for a one dimensional time fractional diffusion equation. Li and Yamamoto [20] considered the recovery of orders of fractional derivatives for a multi term time fractional diffusion equation. The determination of orders of space and time fractional derivatives for space-time fractional diffusion equation was considered by Tatar et al [39]. Furati et al [9] proved existence and uniqueness results for the solution of the inverse source problem posed for the heat equation involving generalized fractional derivative given by (1.5). Direct and inverse problems for fourth order parabolic equation with fractional derivative in time was considered in [4]. For time fractional diffusion equation, determination of a time dependent source was considered in [15]. Liu et al [22] considered reconstruction of time dependent boundary sources for time fractional diffusion equation. The inverse problems of recovering the space dependent sources for time fractional diffusion equations were considered in [23], [40].

The rest of the paper is organized as follows: in Section 2, we recall some basic definitions needed in the sequel and provide the statements of our main results. Section 3, presents our results concerning the existence, uniqueness and continuous dependence of the solution of ISP-I. In Section 4 we give the solution of ISP-II. In the last section we provide some examples.

2. Preliminaries and statements of the main results

In this section, we provide some basic definitions, notations from fractional calculus (for more details see [34]) and statements of our main results.

The left sided Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$D^{\alpha}_{0_+} f(t) := \frac{d}{dt} I^{1-\alpha}_{0_+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{2.1}$$

The Riemann-Liouville fractional derivative of a constant is not equal to zero.

For $f \in AC[0,T]$ the left-hand sided Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$^{C}D^{\alpha}_{0_+} f(t) := I^{1-\alpha}_{0_+} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{2.2}$$

Notice that the generalized fractional derivative $D^{\alpha,\gamma}_{0_+}$ reduces to the Riemann-Liouville fractional derivative and Caputo fractional derivative for $\gamma = \alpha$ and $\gamma = 1$, respectively,

$$D^{\alpha,\alpha}_{0_+} w(t) := D^{\alpha}_{0_+} w(t), \quad D^{\alpha,1}_{0_+} w(t) := {}^{C}D^{\alpha}_{0_+} w(t),$$

where $D^{\alpha}_{0_+} w(t)$ and ${}^{C}D^{\alpha}_{0_+} w(t)$ are the left sided Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha < 1$ given by (2.1) and (2.2), respectively. The Laplace transform of the generalized fractional derivative (1.5) is given by [12],

$$\mathcal{L}\{D^{\alpha,\gamma}_{0_+} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - s^{\alpha-\gamma} I^{1-\gamma}_{0_+} f(t)\Big|_{t=0}, \quad 0 < \alpha \le \gamma < 1. \tag{2.3}$$

Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. A set of functions $\mathfrak{F}$ in $\mathcal{H}$ is called complete in the interval I if there exists no function f in $\mathcal{H}$, essentially different from zero, which is orthogonal to all the functions of the set $\mathfrak{F}$ in the interval I. Two sets S_1 and S_2 of functions of $\mathcal{H}$ form a bi-orthogonal system of functions if a one-to-one correspondence can be established between them such that the scalar product of two corresponding functions is equal to unity and the scalar product of two non-corresponding functions is equal to zero, i.e.,

$$\langle f_i, g_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

where $f_i \in S_1$, $g_i \in S_2$ and δ_{ij} is the Kronecker symbol. The bi-orthogonal system is complete in $\mathcal{H}$ if the sets S_1 and S_2 forming bi-orthogonal system are complete in $\mathcal{H}$.

The Mittag-Leffler function for any $z \in \mathbb{C}$ with parameter ξ is given by

$$E_{\xi}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + 1)} \quad \operatorname{Re} \xi > 0. \tag{2.4}$$

Notice that for $\xi = 1$, we have $E_1(z) = e^z$.

The Mittag-Leffler type function of two parameters $E_{\xi,\beta}(z)$ which is a generalization of (2.4) is defined by

$$E_{\xi,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + \beta)}, \quad z, \beta \in \mathbb{C}; \quad \operatorname{Re} \xi > 0. \tag{2.5}$$

The Mittag-Leffler type functions $E_\xi(-\mu t^\xi)$ and $t^{\beta-1}E_{\xi,\beta}(-\mu t^\xi)$ for $\mu>0$, $0<\xi\le\beta\le 1$ are *completely monotone* functions, i.e.,

$$(-1)^n[E_\xi(-\mu t^\xi)]^{(n)}\ge 0,\quad (-1)^n[t^{\beta-1}E_{\xi,\beta}(-\mu t^\xi)]^{(n)}\ge 0,\quad n\in\mathbb{N}\cup\{0\}. \tag{2.6}$$

The function $E_{\xi,\beta}$ is an entire function [33] and thus is bounded in any finite interval, that is

$$E_{\xi,\beta}(\mu t^\xi)\le M,\quad t\in[b,c],\ b\ge 0,$$

for some positive constant M and furthermore, we have

$$\int_0^t \tau^{\beta-1}E_{\xi,\beta}(\mu\tau^\xi)d\tau<\infty,\quad \text{for } t\in[b,c], \tag{2.7}$$

(see [33, page 9]). The Mittag-Leffler type function $t^{\beta-1}E_{\xi,\beta}(z)$ whose fractional integral is

$$I_{0_+}^{1-\gamma}[t^{\beta-1}E_{\xi,\beta}(\lambda t^\xi)]=t^{\beta-\gamma}E_{\xi,\beta-\gamma+1}(\lambda t^\xi),\quad 0\le\gamma\le 1,\ \xi,\ \beta>0,\ \lambda\in\mathbb{R}, \tag{2.8}$$

plays an important role in the forthcoming sections.

The Laplace transform of $t^{\beta-1}E_{\xi,\beta}(\lambda t^\xi)$ is

$$\mathcal{L}\{t^{\beta-1}E_{\xi,\beta}(\lambda t^\xi)\}=\frac{s^{\xi-\beta}}{(s^\xi-\lambda)},\quad \operatorname{Re}s>0,\ |\lambda s^{-\xi}|<1, \tag{2.9}$$

where $\xi,\beta,\lambda\in\mathbb{C}$, $\operatorname{Re}\xi>0$ and $\operatorname{Re}\beta>0$. Also from [31], we have

$$\lambda t^\xi|E_{\xi,\beta}(-\lambda t^\xi)|\le\mathcal{M},\quad 0<\xi<2,\ \beta\in\mathbb{C},\ t\ge 0,\ \lambda\ge 0, \tag{2.10}$$

for some constant $\mathcal{M}>0$.

For ISP-I we have the following results:

Theorem 2.1. *Suppose following conditions hold:*

(1) $\varphi(x)\in C^5[0,1]$ *be such that* $\varphi(0)=0$, $\varphi'(0)=\varphi'(1)$, $\varphi''(1)=0=\varphi^{iv}(0)$ *and* $\varphi'''(0)=\varphi'''(1)$.
(2) $\psi(x)\in C^5[0,1]$ *be such that* $\psi(0)=0$, $\psi'(0)=\psi'(1)$, $\psi''(1)=0=\psi^{iv}(0)$ *and* $\psi'''(0)=\psi'''(1)$.

Then, there exist a regular solution of the ISP-I.

Theorem 2.2. *A regular solution of the ISP-I (if it exists) is unique.*

Theorem 2.3. *The solution of the ISP-I, under the assumptions of Theorem 2.1, depends continuously on the given data.*

For second inverse problem (ISP-II), we have the following results:

Theorem 2.4. *Suppose the following conditions hold:*

(1) $\varphi(x)\in C^4[0,1]$ *be such that* $\varphi(0)=0$, $\varphi'(0)=\varphi'(1)$, $\varphi''(1)=0$ *and* $\varphi'''(0)=\varphi'''(1)$.
(2) $f(\cdot,t)\in C^4[0,1]$ *be such that* $f(0,t)=0$, $f_x(0,t)=f_x(1,t)$, $f_{xx}(1,t)=0$ *and* $f_{xxx}(0,t)=f_{xxx}(1,t)$. *Furthermore* $\int_0^1 xf(x,t)\,dx\neq 0$ *and*
$$0<\frac{1}{M^*}\le\Big|\int_0^1 xf(x,t)\,dx\Big|,\quad \textit{where } M^*>0.$$
(3) $g(t)\in AC[0,T]$ *and* $g(t)$ *satisfies the consistency condition* $\int_0^1 x\varphi(x)\,dx=I_{0_+}^{1-\gamma}g(t)|_{t=0}$. *Then, the ISP-II has a regular solution, furthermore the regular solution of the ISP-II is unique.*

Theorem 2.5. *A regular solution of the ISP-II (under the assumptions of Theorem 2.4) is unique.*

Theorem 2.6. *The solution of the ISP-II, under the assumptions of Theorem 2.4, depends continuously on the given data.*

3. Inverse Source Problem I

In this section, we present proofs of our main results. Before we proceed further let us construct a bi-orthogonal system of functions consisting of eigenfunctions of the spectral problem (1.1)–(1.4) and its adjoint problem.

3.1. **Construction of two Riesz basis for the space $L^2(0,1)$.** The spectral problem for the initial boundary value problem (1.1)–(1.4) given by

$$X^{iv}(x) = \lambda X(x), \qquad x \in (0,1), \tag{3.1}$$

$$X(0) = X''(1) = 0, \quad X'(0) = X'(1), \quad X'''(0) = X'''(1). \tag{3.2}$$

is non-self-adjoint and the adjoint problem of the spectral problem (3.1)–(3.2) is

$$Y^{iv}(x) = \lambda Y(x), \qquad x \in (0,1), \tag{3.3}$$

$$Y(0) = Y(1), \quad Y''(0) = Y''(1), \quad Y'(0) = Y'''(1) = 0. \tag{3.4}$$

The set of eigenfunctions for the boundary value problem (3.1)–(3.2), corresponding to eigenvalues $\lambda_0 = 0$ and $\lambda_n = (2\pi n)^4$, is

$$\{X_0(x) = 2x,\ X_{2n-1}(x) = 2\sin 2\pi nx,\ X_{2n}(x) = \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} + \cos 2\pi nx\}$$

for $n \in \mathbb{N}$ and is a complete set of functions in $L^2(0,1)$. Furthermore, this set forms a Riesz basis for the space $L^2(0,1)$ (see [3, Lemma 2, and Proposition 1]). The set of eigenfunctions is not orthogonal as

$$\int_0^1 X_0(x) X_{2n-1} dx \neq 0.$$

For the adjoint problem (3.3)–(3.4), the eigenfunctions corresponding to eigenvalues $\lambda_0 = 0$ and $\lambda_n = (2\pi n)^4$ are given by

$$\{Y_0(x) = 1,\ Y_{2n-1}(x) = \frac{e^{2\pi nx} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} + \sin 2\pi nx,\ Y_{2n}(x) = 2\cos 2\pi nx\}.$$

The set of functions form a bi-orthogonal system of functions under the following one-to-one correspondence

$$\begin{array}{ccc} \{\underbrace{X_0(x)}, & \underbrace{X_{2n-1}(x)}, & \underbrace{X_{2n}(x)}\}, \\ \downarrow & \downarrow & \downarrow \\ \{Y_0(x), & Y_{2n-1}(x), & Y_{2n}(x)\}, \end{array}$$

i.e., $\langle X_i, Y_j \rangle = \delta_{ij}$ for $i, j = 0, 2n-1, 2n$, for $n \in \mathbb{N}$, where

$$\langle g_1, g_2 \rangle := \int_0^1 g_1(x) g_2(x)\, dx.$$

We are in a position to present the proof of the Theorem 2.1.

Proof of Theorem 2.1. Expanding $u(x,t)$ and $f(x)$ using bi-orthogonal system of functions, we have

$$u(x,t) = u_0(t)X_0(x) + \sum_{n=1}^{\infty} u_{2n-1}(t)X_{2n-1}(x) + \sum_{n=1}^{\infty} u_{2n}(t)X_{2n}(x), \tag{3.5}$$

$$f(x) = f_0X_0(x) + \sum_{n=1}^{\infty} f_{2n-1}X_{2n-1}(x) + \sum_{n=1}^{\infty} f_{2n}X_{2n}(x), \tag{3.6}$$

where $u_0(t), u_{2n-1}(t),\ u_{2n}(t),\ f_0, f_{2n-1},$ and f_{2n} for $n \in \mathbb{N}$, are unknowns to be determined.

From the expansion of $u(x,t)$ given by (3.5) and using properties of the bi-orthogonal system of functions, we have

$$u_0(t) = \langle u(x,t), Y_0(x)\rangle, \quad u_{2n-1}(t) = \langle u(x,t), Y_{2n-1}(x)\rangle,$$
$$u_{2n}(t) = \langle u(x,t), Y_{2n}(x)\rangle.$$

Consider

$$u_{2n-1}(t) = \langle u(x,t), Y_{2n-1}(x)\rangle := \int_0^1 u(x,t)Y_{2n-1}\, dx\,.$$

Taking the fractional derivative under the integral and using (1.1) with $F(x,t) = f(x)$, we have

$$D_{0_+}^{\alpha,\gamma} u_{2n-1}(t) = -\int_0^1 u_{xxxx}Y_{2n-1}(x)\, dx + \int_0^1 f(x)Y_{2n-1}(x)\, dx.$$

Integrating by parts and using the boundary conditions (1.3)–(1.4), we obtain

$$D_{0_+}^{\alpha,\gamma} u_{2n-1}(t) + \lambda_n u_{2n-1}(t) = f_{2n-1}. \tag{3.7}$$

Similarly, we have the linear fractional differential equations

$$D_{0_+}^{\alpha,\gamma} u_0(t) = f_0, \tag{3.8}$$

$$D_{0_+}^{\alpha,\gamma} u_{2n}(t) + \lambda_n u_{2n}(t) = f_{2n}. \tag{3.9}$$

Taking Laplace transform of (3.7) and using formula (2.3), we obtain

$$\mathcal{L}\{u_{2n-1}(t)\} = I_{0_+}^{1-\gamma} u_{2n-1}(t)r|_{t=0}\Big(\frac{s^{\alpha-\gamma}}{s^\alpha + \lambda_n}\Big) + \frac{f_{2n-1}}{s(s^\alpha+\lambda_n)}.$$

The solution of (3.7) is obtained by applying inverse Laplace transform, formula (2.9) and $\mathcal{L}^{-1}(\mathcal{L}\{f_1(t)\}\mathcal{L}\{f_2(t)\}) = (f_1 * f_2)(t)$,

$$\begin{aligned} u_{2n-1}(t) &= I_{0_+}^{1-\gamma} u_{2n-1}(t)\Big|_{t=0} t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n t^\alpha) \\ &\quad + f_{2n-1}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau. \end{aligned} \tag{3.10}$$

Similarly, the solutions of (3.8) and (3.9) are given by

$$u_0(t) = I_{0_+}^{1-\gamma} u_0(t)\Big|_{t=0}\frac{t^{\gamma-1}}{\Gamma(\gamma)} + f_0\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{3.11}$$

$$u_{2n}(t) = I_{0_+}^{1-\gamma} u_{2n}(t)\Big|_{t=0} t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n t^\alpha) + f_{2n}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau, \tag{3.12}$$

respectively. By the initial condition (1.2), we have

$$I_{0_+}^{1-\gamma} u_0(t)r|_{t=0} = \varphi_0, \quad I_{0_+}^{1-\gamma} u_{2n-1}(t)r|_{t=0} = \varphi_{2n-1}, \quad I_{0_+}^{1-\gamma} u_{2n}(t)r|_{t=0} = \varphi_{2n},$$

where $\varphi_0, \varphi_{2n-1}$ and φ_{2n} are the coefficients of series expansion of $\varphi(x)$ when expanded using the bi-orthogonal system and are given by

$$\varphi_0 = \int_0^1 \varphi(x)Y_0(x)\,dx, \quad \varphi_{2n-1} = \int_0^1 \varphi(x)Y_{2n-1}(x)\,dx,$$
$$\varphi_{2n} = \int_0^1 \varphi(x)Y_{2n}(x)\,dx. \tag{3.13}$$

Alike, using the condition $u(x,T) = \psi(x)$, we have

$$u_0(T) = \psi_0, \quad u_{2n-1}(T) = \psi_{2n-1}, \quad u_{2n}(T) = \psi_{2n}, \tag{3.14}$$

where ψ_0, ψ_{2n-1} and ψ_{2n} are the coefficients of series expansion of the function $\psi(x)$ in terms of the bi-orthogonal system of functions. $\square$

Before we proceed further let us fix some notation

$$\mathcal{E}_n^{(1)}(t) := t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n t^\alpha), \quad \mathcal{E}_n^{(2)}(t) := \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)d\tau.$$

By using these notation and taking (3.10)–(3.12) into account we can write

$$u_0(t) = \varphi_0 \frac{t^{\gamma-1}}{\Gamma(\gamma)} + f_0 \frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_{2n-1}(t) = \varphi_{2n-1}\mathcal{E}_n^{(1)}(t) + f_{2n-1}\mathcal{E}_n^{(2)}(t),$$
$$u_{2n}(t) = \varphi_{2n}\mathcal{E}_n^{(1)}(t) + f_{2n}\ mathcalE_n^{(2)}(t).$$

Due to (3.14)–(3.1) the unknowns f_0, f_{2n-1}, f_{2n} are determined as

$$f_0 = \Big(\psi_0 - \frac{\varphi_0 T^{\gamma-1}}{\Gamma(\gamma)}\Big)\frac{\Gamma(1+\alpha)}{T^\alpha}, \tag{3.15}$$

$$f_{2n-1} = \frac{\psi_{2n-1} - \varphi_{2n-1}\mathcal{E}_n^{(1)}(T)}{\mathcal{E}_n^{(2)}(T)}, \tag{3.16}$$

$$f_{2n} = \frac{\psi_{2n} - \varphi_{2n}\mathcal{E}_n^{(1)}(T)}{\mathcal{E}_n^{(2)}(T)}. \tag{3.17}$$

The solution of the ISP-I is given by the series (3.5) and (3.6), where $u_0(t)$, $u_{2n-1}(t)$, $u_{2n}(t)$, f_0, f_{2n-1} and f_{2n} given by (3.1)–(3.17), respectively.

Before proceeding further, we recall [18, Lemma 5 on page 89].

Lemma 3.1. *Let $f \in L^2(0,1)$ and*

$$a_n = \int_0^1 f(x)e^{\mu n(x-1)}dx, \quad b_n = \int_0^1 f(x)e^{-\mu n x}dx,$$

where μ is any complex number such that $\operatorname{Re}\mu > 0$. Then the series

$$\sum_{n=1}^\infty |a_n|^2, \quad \sum_{n=1}^\infty |b_n|^2$$

are convergent.

Existence of the solution of the ISP-I: To show that the solution of the inverse problem represented by the series (3.5) and (3.6) is a regular solution we need to show that

- The series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$, and $D_{0+}^{\alpha,\gamma}u(x,t)$ represent continuous functions.
- The series corresponding to $f(x)$ is continuous on $[0,1]$.

Let

$$u(x,t) = \mathcal{W}_0 + \sum_{n=1}^{\infty} \mathcal{W}_{2n-1} + \sum_{n=1}^{\infty} \mathcal{W}_{2n}, \tag{3.18}$$

where $\mathcal{W}_0 = u_0(t)X_0(x)$, $\mathcal{W}_{2n-1} = u_{2n-1}(t)X_{2n-1}(x)$, $\mathcal{W}_{2n} = u_{2n}(t)X_{2n}(x)$, and $u_0(t), u_{2n-1}(t)$ and $u_{2n}(t)$ are given by (3.1)–(3.1).

We shall show that all the series involved in (3.18) represents a continuous function on $\Omega_\epsilon := [0,1] \times [\epsilon, T]$ for $\epsilon > 0$. By using (2.10) the bound for $\mathcal{E}_n^{(1)}(t)$ is obtained as

$$\mathcal{E}_n^{(1)}(t) \le \frac{C_1}{t^{1+\alpha-\gamma}\lambda_n}, \quad t \in [\epsilon, T], \tag{3.19}$$

and using (2.7), we can have

$$\mathcal{E}_n^{(2)}(t) \le C_2, \quad t \in [\epsilon, T],$$

where C_1 and C_2 are constants. For some fixed time (say) T, using above estimates together with (2.6)–(2.7), we can choose $\mathcal{M}_1$, and $\mathcal{M}_2$, independent of n, such that

$$|\mathcal{E}_n^{(1)}(T)| \le \mathcal{M}_1, \quad |\mathcal{E}_n^{(2)}(T)|^{-1} \le \mathcal{M}_2, \quad n \in \mathbb{N}.$$

From (3.13) and integration by parts, we have

$$|\varphi_{2n-1}| = \frac{1}{\lambda_n}\langle \varphi^{iv}(x), Y_{2n-1}(x)\rangle, \quad |\varphi_{2n}| = \frac{\sqrt{2}}{(2\pi n)}\langle \varphi'(x), \sqrt{2}\sin 2\pi n x\rangle,$$

using elementary inequality $ab \le 1/2(a^2+b^2)$ for all $a, b \in \mathbb{R}$, we obtain

$$|\varphi_{2n-1}| \le \frac{1}{2}\big(\frac{1}{\lambda_n^2} + \mathcal{I}_n^2\big), \quad |\varphi_{2n}| \le \frac{1}{\sqrt{2}}\big\{\frac{1}{(2\pi n)^2} + (\langle \varphi'(x), \sqrt{2}\sin 2\pi n x\rangle)^2\big\},$$

where $\mathcal{I}_n = \langle \varphi^{iv}(x), Y_{2n-1}(x)\rangle$. By Lemma 3.1 we conclude that the series $\sum_{n=1}^{\infty} \mathcal{I}_n^2$ converges absolutely. The sequence $\{\sqrt{2}\sin 2\pi n x\}_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(0,1)$, hence by Bessel's inequality, we have

$$\sum_{n=1}^{\infty} |\varphi_{2n}| \le \frac{1}{\sqrt{2}}\big\{\sum_{n=1}^{\infty} \frac{1}{(2\pi n)^2} + \|\varphi'(x)\|_{L^2(0,1)}^2\big\}.$$

Also, we have

$$|\varphi_0| = \langle \varphi(x), Y_0(x)\rangle \le 2\|\varphi(x)\|_{L^2(0,1)}.$$

Similarly, the estimates for ψ_0, ψ_{2n-1} and ψ_{2n} are obtained as

$$|\psi_0| \le 2\|\psi(x)\|_{L^2(0,1)}, \quad \sum_{n=1}^{\infty} |\psi_{2n-1}| \le \frac{1}{2}\Big(\frac{1}{\lambda_n^2} + \mathcal{J}_n^2\Big),$$

$$\sum_{n=1}^{\infty} |\psi_{2n}| \le \frac{1}{\sqrt{2}}\Big\{\sum_{n=1}^{\infty} \frac{1}{(2\pi n)^2} + \|\psi'(x)\|_{L^2(0,1)}^2\Big\},$$

where $\mathcal{J}_n = \langle \psi^{iv}(x), Y_{2n-1}(x)\rangle$. Consequently, from (3.15)–(3.17), we obtained the following estimates

$$T^{1+\alpha-\gamma}|f_0| \le 2C_3\left(\|\psi(x)\|_{L^2(0,1)} + \|\varphi(x)\|_{L^2(0,1)}\right), \tag{3.20}$$

$$\sum_{n=1}^{\infty}|f_{2n-1}| \le \frac{\mathcal{M}_2}{2}\Big\{\frac{1}{\lambda_n^2}+\mathcal{J}_n^2+\mathcal{M}_1\Big(\frac{1}{\lambda_n^2}+\mathcal{I}_n^2\Big)\Big\}, \tag{3.21}$$

$$\begin{aligned}\sum_{n=1}^{\infty}|f_{2n}| \le \frac{\mathcal{M}_2}{\sqrt{2}}\Big\{&\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}+\|\psi'(x)\|^2_{L^2(0,1)}\\ &+\mathcal{M}_1\Big(\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}+\|\varphi'(x)\|^2_{L^2(0,1)}\Big)\Big\},\end{aligned} \tag{3.22}$$

where

$$C_3=\max\Big\{\frac{\Gamma(1+\alpha)}{\Gamma(\gamma)}, t^{1-\gamma}\Gamma(1+\alpha), \frac{t^{\alpha}}{\Gamma(\gamma)}, \frac{t^{1+2\alpha-\gamma}}{\Gamma(1+\alpha)}\Big\},$$

for all $t\in[\epsilon,T]$. From estimates (3.20)–(3.22) the series expansion of $f(x)$ given by (3.6) represents a continuous function on Ω_ϵ.

Using (3.20)–(3.22) and $|X_n(x)|\le 2$ for $n\in\mathbb{N}\cup\{0\}$, we have the following estimates for the series involved in (3.18),

$$t^{1+\alpha-\gamma}|\mathcal{W}_0|\le 4C_3\{\|\varphi(x)\|_{L^2(0,1)}+C_3(\|\psi(x)\|_{L^2(0,1)}+\|\varphi(x)\|_{L^2(0,1)})\},$$

$$t^{1+\alpha-\gamma}\sum_{n=1}^{\infty}|\mathcal{W}_{2n-1}|\le 2\Big[\frac{C_1C_4}{\lambda_n}+\frac{t^{1+\alpha-\gamma}C_2\mathcal{M}_2}{2}\Big\{\frac{1}{\lambda_n^2}+\mathcal{J}_n^2+\mathcal{M}_1\Big(\frac{1}{\lambda_n^2}+\mathcal{I}_n^2\Big)\Big\}\Big],$$

$$\begin{aligned}t^{1+\alpha-\gamma}\sum_{n=1}^{\infty}|\mathcal{W}_{2n}|\le 2\Big[\frac{C_1C_5}{\lambda_n}+\frac{t^{1+\alpha-\gamma}C_2\mathcal{M}_2}{\sqrt{2}}\Big\{&\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}+\|\psi'(x)\|^2_{L^2(0,1)}\\ &+\mathcal{M}_1(\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}+\|\varphi'(x)\|^2_{L^2(0,1)})\Big\}\Big],\end{aligned}$$

where C_4 and C_5 are positive constants such that

$$\sum_{n=1}^{\infty}|\varphi_{2n-1}|\le C_4, \quad \text{and} \quad \sum_{n=1}^{\infty}|\varphi_{2n}|\le C_5.$$

Thus all the series in (3.18) are bounded above by uniformly convergent numerical series. Consequently, by Weierstrass M-test the series expansion of $u(x,t)$ given by (3.18) is uniformly convergent in Ω_ϵ.

Notice that

$$X_0^{iv}(x)=0, \quad X_{2n-1}^{iv}(x)=\lambda_n X_{2n-1}(x), \quad X_{2n}^{iv}(x)=\lambda_n X_{2n}(x).$$

Let us show that the series representation of $u_{xxxx}(x,t)$ obtained from (3.18) is uniformly convergent series.

Integration by parts leads us to the following estimates

$$|\varphi_{2n-1}|=\frac{1}{(2\pi n)^5}\langle\varphi^v(x), \quad \frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\rangle=\frac{\mathcal{I}_n^*}{(2\pi n)^5}, \tag{3.23}$$

$$|\varphi_{2n}|=\frac{1}{(2\pi n)^5}\langle\varphi^v(x), 2\sin(2\pi nx)\rangle\le\frac{\sqrt{2}}{(2\pi n)^5}\|\varphi^v(x)\|_{L^2(0,1)}, \tag{3.24}$$

$$|\psi_{2n-1}|=\frac{1}{(2\pi n)^5}\langle\psi^v(x), \frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\rangle=\frac{\mathcal{J}_n^*}{(2\pi n)^5}, \tag{3.25}$$

$$|\psi_{2n}|=\frac{1}{(2\pi n)^5}\langle\psi^v(x), 2\sin(2\pi nx)\rangle\le\frac{\sqrt{2}}{(2\pi n)^5}\|\psi^v(x)\|_{L^2(0,1)}, \tag{3.26}$$

where $\mathcal{I}_n^* = \langle \varphi^v(x), (e^{2\pi nx} - e^{2\pi n(1-x)})/(e^{2\pi n} - 1) - \cos 2\pi nx\rangle$ and

$$\mathcal{J}_n^* = \langle \psi^v(x), (e^{2\pi nx} - e^{2\pi n(1-x)})/(e^{2\pi n} - 1) - \cos 2\pi nx\rangle.$$

Using (3.23)–(3.26) in (3.15)–(3.17) the estimates for f_{2n-1} and f_{2n}, are

$$|f_{2n-1}| \le \mathcal{M}_2\{\frac{1}{(2\pi n)^5}\mathcal{J}_n^* + \frac{\mathcal{M}_1}{\lambda_n}\mathcal{I}_n^*\}, \tag{3.27}$$

$$|f_{2n}| \le \mathcal{M}_2\{\frac{2}{(2\pi n)^5}\|\psi^v(x)\|_{L^2(0,1)} + \frac{2\mathcal{M}_1}{(2\pi n)^5}\|\varphi^v(x)\|_{L^2(0,1)}\}. \tag{3.28}$$

From (3.23)–(3.28) we have

$$t^{1+\alpha-\gamma}\sum_{n=1}^{\infty}|\frac{\partial^4\mathcal{W}_{2n-1}}{\partial x^4}| \le \sum_{n=1}^{\infty}2\lambda_n\Big\{\frac{C_1\mathcal{I}_n^*}{\lambda_n(2\pi n)^5} + t^{1+\alpha-\gamma}\mathcal{M}_2C_2(\frac{\mathcal{J}_n^*}{(2\pi n)^5} + \frac{\mathcal{M}_1\mathcal{I}_n^*}{(2\pi n)^5})\Big\}, \tag{3.29}$$

$$\begin{aligned} t^{1+\alpha-\gamma}\sum_{n=1}^{\infty}|\frac{\partial^4\mathcal{W}_{2n}}{\partial x^4}| &\le \sum_{n=1}^{\infty}2\lambda_n\Big\{\frac{C_1\|\varphi^v(x)\|_{L^2(0,1)}}{\lambda_n(2\pi n)^5} + t^{1+\alpha-\gamma}\mathcal{M}_2C_2 \\ &\quad\times\Big(\frac{\|\varphi^v(x)\|_{L^2(0,1)}}{(2\pi n)^5} + \frac{\mathcal{M}_1\|\psi^v(x)\|_{L^2(0,1)}}{(2\pi n)^5}\Big)\Big\}. \end{aligned} \tag{3.30}$$

By using the inequality $2ab \le (a^2+b^2)$ and Lemma 3.1 the series involved in (3.29)–(3.30) are uniformly convergent. Moreover by the assumptions on $\varphi(x)$ and $\psi(x)$ it can be concluded that the series expansion of $u_{xxxx}(x,t)$ is bounded above by convergent series and represents a continuous function.

Next we show that the series corresponding to fractional derivative $D_{0_+}^{\alpha,\gamma}u(x,t)$ is uniformly convergent, i.e.,

$$D_{0_+}^{\alpha,\gamma}\sum_{n=1}^{\infty}\mathcal{W}_{2n-1}(t), \quad D_{0_+}^{\alpha,\gamma}\sum_{n=1}^{\infty}\mathcal{W}_{2n}(t),$$

are uniformly convergent. From (3.7)–(3.9), we have

$$D_{0_+}^{\alpha,\gamma}\mathcal{W}_0 = f_0X_0(x), \tag{3.31}$$

$$\sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}\mathcal{W}_{2n-1} = \sum_{n=1}^{\infty}[\lambda_nu_{2n-1}(t) + f_{2n-1}]\,X_{2n-1}(x), \tag{3.32}$$

$$\sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}\mathcal{W}_{2n} = \sum_{n=1}^{\infty}[\lambda_nu_{2n}(t) + f_{2n}]\,X_{2n}(x). \tag{3.33}$$

Using estimates (3.23)–(3.28) and Weierstrass M-test, the series $\sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}\mathcal{W}_{2n-1}$ and $\sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}\mathcal{W}_{2n}$ are uniformly convergent on Ω_ϵ.

At this stage let us recall the [34, Lemma 15.2, page 278].

Lemma 3.2. *Let the fractional derivative $D_{0_+}^{\alpha,\gamma}f_n$ exists for all $n\in\mathbb{N}$ and the series $\sum_{n=1}^{\infty}f_n$ and $\sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}f_n$ are uniformly convergent on every subinterval $[\epsilon,b]$ for $\epsilon>0$ then*

$$D_{0_+}^{\alpha,\gamma}\Big(\sum_{n=1}^{\infty}f_n(x)\Big) = \sum_{n=1}^{\infty}D_{0_+}^{\alpha,\gamma}f_n(x), \quad 0<\alpha\le\gamma<1,\ 0<x<b.$$

By the estimates (3.23)–(3.28) and Lemmas 3.2 and 3.3 it can be deduced that the series involved in $D_{0_+}^{\alpha,\gamma}u(x,t)$ are bounded above by uniformly convergent numerical series and hence by Weierstrass M-test $D_{0_+}^{\alpha,\gamma}u(x,t)$ is uniformly convergent.

Proof of Theorem 2.2. (Uniqueness of the solution of the ISP-I)
Suppose $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ are two solution sets of the ISP-I, then $\bar{u}(x,t) = u_1(x,t) - u_2(x,t)$ and $\bar{f}(x) = f_1(x) - f_2(x)$ satisfy

$$D_{0_+}^{\alpha,\gamma}\bar{u}(x,t) + \bar{u}_{xxxx}(x,t) = \bar{f}(x), \quad (x,t) \in \Omega, \tag{3.34}$$

$$I_{0_+}^{1-\gamma}\bar{u}(x,t)\Big|_{t=0} = 0, \quad \bar{u}(x,T) = 0, \quad x \in [0,1], \tag{3.35}$$

$$\bar{u}_x(0,t) = \bar{u}_x(1,t), \quad \bar{u}(0,t) = 0, \quad t \in [0,T], \tag{3.36}$$

$$\bar{u}_{xxx}(0,t) = \bar{u}_{xxx}(1,t), \quad \bar{u}_{xx}(1,t) = 0, \quad t \in [0,T], \tag{3.37}$$

Following the strategy in [29], we consider the functions

$$\begin{aligned} \bar{u}_0(t) &= \int_0^1 \bar{u}(x,t)Y_0(x)dx, \\ \bar{u}_{2n-1}(t) &= \int_0^1 \bar{u}(x,t)Y_{2n-1}(x)dx, \\ \bar{u}_{2n}(t) &= \int_0^1 \bar{u}(x,t)Y_{2n}(x)dx, \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} \bar{f}_0 &= \int_0^1 \bar{f}(x)Y_0(x)dx, \\ \bar{f}_{2n-1} &= \int_0^1 \bar{f}(x)Y_{2n-1}(x)dx, \\ \bar{f}_{2n} &= \int_0^1 \bar{f}(x)Y_{2n}(x)dx. \end{aligned} \tag{3.39}$$

Applying the time fractional derivative $D_{0_+}^{\alpha,\gamma}(\cdot)$ to both sides of each equation in (3.38), we obtain

$$\begin{aligned} D_{0_+}^{\alpha,\gamma}\bar{u}_0(t) &= \int_0^1 D_{0_+}^{\alpha,\gamma}\bar{u}(x,t)Y_0(x)dx, \\ D_{0_+}^{\alpha,\gamma}\bar{u}_{2n-1}(t) &= \int_0^1 D_{0_+}^{\alpha,\gamma}\bar{u}(x,t)Y_{2n-1}(x)dx, \\ D_{0_+}^{\alpha,\gamma}\bar{u}_{2n}(t) &= \int_0^1 D_{0_+}^{\alpha,\gamma}\bar{u}(x,t)Y_{2n}(x)dx. \end{aligned} \tag{3.40}$$

Let us take the third equation in (3.40). Using (3.34) together with the conditions (3.36)-(3.37), we obtain the fractional differential equation

$$D_{0_+}^{\alpha,\gamma}\bar{u}_{2n}(t) + \lambda_n\bar{u}_{2n}(t) = \bar{f}_{2n}. \tag{3.41}$$

By using Laplace transform technique the solution of (3.41) is

$$\bar{u}_{2n}(t) = I_{0_+}^{1-\gamma}\bar{u}_{2n}(t)\big|_{t=0}t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n t^\alpha) + \bar{f}_{2n}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau. \tag{3.42}$$

Since

$$\bar{u}_{2n}(t) = \int_0^1 \bar{u}(x,t)Y_{2n}(x)dx \;\Rightarrow\; I_{0_+}^{1-\gamma}\bar{u}_{2n}(t)\Big|_{t=0} = \int_0^1 I_{0_+}^{1-\gamma}\bar{u}(x,t)\Big|_{t=0} Y_{2n}(x)dx$$

and by using the initial condition from (3.35) the solution (3.42) takes the form

$$\bar{u}_{2n}(t) = \bar{f}_{2n}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau. \tag{3.43}$$

By using the final temperature condition from (3.35), we obtain $\bar{f}_{2n} = 0$ and consequently $\bar{u}_{2n}(t) = 0$ for all $t \in [0,T]$.

Similarly, we can shaow that for all $t \in [0,T]$,

$$\bar{u}_0(t) = 0, \quad \bar{u}_{2n-1}(t) = 0, \quad \bar{f}_0 = 0, \quad \bar{f}_{2n-1} = 0. \tag{3.44}$$

The uniqueness of the regular solution of the ISP-I follows from the completeness of the set $\{Y_0(x),\ Y_{2n-1}(x),\ Y_{2n}(x)\}$, $n \in \mathbb{N}$ (see [3, Lemma 2]).

It remains to show that $u(x,t)$ given by (3.18) agrees with the initial and final data. We have

$$\begin{gathered}
I_{0_+}^{1-\gamma}\mathcal{W}_0 = \big\{\varphi_0 + \frac{t^{1+\alpha-\gamma}}{\Gamma(2+\alpha-\gamma)}f_0\big\}X_0(x),\\
I_{0_+}^{1-\gamma}\mathcal{W}_{2n-1}\big\{E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_{2n-1} + t^{1+\alpha-\gamma}E_{\alpha,2+\alpha-\gamma}(-\lambda_n t^\alpha)f_{2n-1}\big\}X_{2n-1}(x),\\
I_{0_+}^{1-\gamma}\mathcal{W}_{2n} = \big\{E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_{2n} + t^{1+\alpha-\gamma}E_{\alpha,2+\alpha-\gamma}(-\lambda_n t^\alpha)f_{2n}\big\}X_{2n}(x).
\end{gathered}$$

The term by term fractional integral of (3.18) converges to $I_{0_+}^{1-\gamma}u(x,t)$ and it is uniformly convergent on $[\epsilon,T]$. For $t=0$ we have,

$$\begin{gathered}
I_{0_+}^{1-\gamma}\mathcal{W}_0|_{t=0} = \varphi_0 X_0(x), \quad I_{0_+}^{1-\gamma}\mathcal{W}_{2n-1}\big|_{t=0} = \varphi_{2n-1}X_{2n-1}(x),\\
I_{0_+}^{1-\gamma}\mathcal{W}_{2n}\big|_{t=0} = \varphi_{2n}X_{2n}.
\end{gathered}$$

Therefore,

$$I_{0_+}^{1-\gamma}u(x,t)\Big|_{t=0} = \varphi_0X_0 + \sum_{n=1}^\infty \varphi_{2n-1}X_{2n-1} + \sum_{n=1}^\infty \varphi_{2n}X_{2n},$$

which is the series expansion of $\varphi(x)$, when expanded using bi-orthogonal system.

Similarly, we can show that for $u(x,t)$ given by (3.18) the over-determination is also satisfied, that is, $u(x,T) = \psi(x)$. □

Before providing the proof of our stability result, i.e., Theorem 2.3 let us mention the following result from [14].

Lemma 3.3. *For any function $f \in L^2(0,1)$ the inequality*

$$r_1\|f\|^2_{L^2(0,1)} \le \sum_{n=0}^\infty f_n^2 \le R_1\|f\|^2_{L^2(0,1)}, \tag{3.45}$$

is valid, where r_1 and R_1 are constants and f_n are coefficients of the bi-orthogonal expansion of the function f in any Riesz basis $\{\mathcal{R}_n(x)\}$ given by

$$f_n = \langle f, \mathcal{W}_n\rangle, \quad n \in \mathbb{N}\cup\{0\},$$

where $\{\mathcal{W}_n(x)\}$ is corresponding bi-orthogonal set of Riesz basis $\{\mathcal{R}_n(x)\}$.

Proof of Theorem 2.3. Let $\{u(x,t), f(x)\}, \{\tilde{u}(x,t), \tilde{f}(x)\}$ be two solution sets of the ISP-I corresponding to the data $\{\varphi, \psi\}, \{\tilde{\varphi}, \tilde{\psi}\}$ respectively. By Lemma 3.3, we have

$$\|f-\tilde{f}\|_{L^2(0,1)}{}^2 \le \frac{1}{r_1}\sum_{n=0}^{\infty}(f_n-\tilde{f}_n)^2.$$

Consider

$$\begin{aligned}(f_0-\tilde{f}_0)^2 &= \Big(\frac{\Gamma(1+\alpha)}{T^\alpha}\Big)^2\Big[\Big(\psi_0-\frac{T^{\gamma-1}}{\Gamma(\gamma)}\varphi_0\Big)-\Big(\tilde{\psi}_0-\frac{T^{\gamma-1}}{\Gamma(\gamma)}\tilde{\varphi}_0\Big)\Big]^2\\ &\le 2C_3^2\Big[(\psi_0-\tilde{\psi}_0)^2+C_3^2(\varphi_0-\tilde{\varphi}_0)^2\Big],\end{aligned} \tag{3.46}$$

where we have used $(a\pm b)^2\le 2a^2+2b^2$. Similarly, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}(f_{2n-1}-\tilde{f}_{2n-1})^2\\ &\le \sum_{n=1}^{\infty}2(\mathcal{M}_2)^2\Big[\Big(\psi_{2n-1}-\tilde{\psi}_{2n-1}\Big)^2+(\mathcal{M}_1)^2(\varphi_{2n-1}-\tilde{\varphi}_{2n-1})^2\Big],\end{aligned} \tag{3.47}$$

$$\sum_{n=1}^{\infty}(f_{2n}-\tilde{f}_{2n})^2 \le \sum_{n=1}^{\infty}2(\mathcal{M}_2)^2\Big[\Big(\psi_{2n}-\tilde{\psi}_{2n}\Big)^2+(\mathcal{M}_1)^2(\varphi_{2n}-\tilde{\varphi}_{2n})^2\Big]. \tag{3.48}$$

Setting

$$N=\max\left\{2C_3^2,\, 2C_3^4,\, 2(\mathcal{M}_1)^2(\mathcal{M}_2)^2,\, 2(\mathcal{M}_2)^2\right\},$$

and using the estimates (3.46)-(3.48) we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(f_n-\tilde{f}_n)^2\\ &\le 3N\Big[(\varphi_0-\tilde{\varphi}_0)^2+\sum_{n=1}^{\infty}(\varphi_{2n-1}-\tilde{\varphi}_{2n-1})^2+\sum_{n=1}^{\infty}(\varphi_{2n}-\tilde{\varphi}_{2n})^2\\ &\quad+(\psi_0-\tilde{\psi}_0)^2+\sum_{n=1}^{\infty}(\psi_{2n-1}-\tilde{\psi}_{2n-1})^2+\sum_{n=1}^{\infty}(\psi_n-\tilde{\psi}_{2n})^2\Big]\\ &\le 3NR_1\Big(\|\varphi-\tilde{\varphi}\|_{L^2(0,1)}^2+\|\psi-\tilde{\psi}\|_{L^2(0,1)}^2\Big).\end{aligned} \tag{3.49}$$

By Lemma 3.3, we have

$$\|f-\tilde{f}\|_{L^2(0,1)}^2 \le \frac{1}{r_1}\sum_{n=0}^{\infty}(f_n-\tilde{f}_n)^2 \le \frac{3NR_1}{r_1}\Big(\|\varphi-\tilde{\varphi}\|_{L^2(0,1)}^2+\|\psi-\tilde{\psi}\|_{L^2(0,1)}^2\Big),$$

$$\|f-\tilde{f}\|_{L^2(0,1)} \le \sqrt{\frac{3NR_1}{r_1}}\Big(\|\varphi-\tilde{\varphi}\|_{L^2(0,1)}+\|\psi-\tilde{\psi}\|_{L^2(0,1)}\Big).$$

Similarly we can obtain a stability result for $u(x,t)$. □

4. Inverse source problem II

In this section, we shall deal with ISP-II for (1.1)–(1.4), with $F(x,t)=a(t)f(x,t)$, where $f(x,t)$ is known and a pair of functions $\{u(x,t), a(t)\}$ is to be determined.

Proof of Theorem 2.4. To determine the solution of ISP-II, i.e., the pair of functions $\{u(x,t),a(t)\}$, we expand $u(x,t)$ and $f(x,t)$ using bi-orthogonal system functions

$$u(x,t)=\sum_{n=1}^{\infty}u_0(t)X_0(x)+\sum_{n=1}^{\infty}u_{2n-1}(t)X_{2n-1}(x)+\sum_{n=1}^{\infty}u_{2n}(t)X_{2n}(x), \tag{4.1}$$

$$f(x,t)=\sum_{n=1}^{\infty}f_0(t)X_0(x)+\sum_{n=1}^{\infty}f_{2n-1}(t)X_{2n-1}(x)+\sum_{n=1}^{\infty}f_{2n}(t)X_{2n}(x), \tag{4.2}$$

where $u_0(t), u_{2n-1}(t)$ and $u_{2n}(t)$ are to be determined, $f_0(t), f_{2n-1}(t)$ and $f_{2n}(t)$ are coefficients of $f(x,t)$, when expanded by using bi-orthogonal system. The following linear fractional differential equations are obtained

$$D_{0_+}^{\alpha,\gamma}u_0(t)=a(t)f_0(t), \tag{4.3}$$

$$D_{0_+}^{\alpha,\gamma}u_{2n-1}(t)=-\lambda_n u_{2n-1}(t)+a(t)f_{2n-1}(t), \tag{4.4}$$

$$D_{0_+}^{\alpha,\gamma}u_{2n}(t)=-\lambda_n u_{2n}(t)+a(t)f_{2n}(t),\quad n\in\mathbb{N}. \tag{4.5}$$

The solutions of the fractional differential equations (4.3)–(4.5) are

$$u_0(t)=\varphi_0\frac{t^{\gamma-1}}{\Gamma(\gamma)}+a(t)f_0(t)*\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{4.6}$$

$$u_{2n-1}(t)=\varphi_{2n-1}\mathcal{E}_n^{(1)}(t)+a(t)f_{2n-1}(t)*\mathcal{E}_n^{(3)}(t), \tag{4.7}$$

$$u_{2n}(t)=\varphi_{2n}\mathcal{E}_n^{(1)}(t)+a(t)f_{2n}(t)*\mathcal{E}_n^{(3)}(t), \tag{4.8}$$

where * is the integral convolution operator and

$$\mathcal{E}_n^{(3)}(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha}).$$

Taking the generalized fractional derivative $D_{0_+}^{\alpha,\gamma}$, under the integral sign of the over-determination condition (1.7) and using (1.1) along with $F(x,t)=a(t)f(x,t)$, we obtain

$$a(t)=\Big(\int_0^1 xf(x,t)dx\Big)^{-1}\Big(D_{0_+}^{\alpha,\gamma}g(t)+\int_0^1 xu_{xxxx}(x,t)dx\Big). \tag{4.9}$$

From the conditions of Theorem 2.4, we have $\int_0^1 xf(x,t)dx\neq 0$ and is given by

$$\begin{aligned}&\int_0^1 xf(x,t)dx\\&=\frac{2}{3}f_0(t)-\sum_{n=1}^{\infty}\frac{1}{\pi n}f_{2n-1}(t)+\sum_{n=1}^{\infty}\Big(\frac{-1}{2\pi^2n^2}+\frac{1+e^{2\pi n}}{2\pi n(e^{2\pi n}-1)}\Big)f_{2n}(t),\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\int_0^1 xu_{xxxx}\,dx=\sum_{n=1}^{\infty}\lambda_n\Big\{&-\frac{1}{\pi n}\Big(\mathcal{E}_n^{(1)}(t)\varphi_{2n-1}(t)+a(t)f_{2n-1}(t)*\mathcal{E}_n^{(3)}(t)\Big)\\&+\Big(\frac{-1}{2\pi^2n^2}+\frac{1+e^{2\pi n}}{2\pi n(e^{2\pi n}-1)}\Big)\\&\times\Big(\mathcal{E}_n^{(1)}(t)\varphi_{2n}(t)+a(t)f_{2n}(t)*\mathcal{E}_n^{(3)}(t)\Big)\Big\}.\end{aligned} \tag{4.11}$$

By (4.10)–(4.11), we have the following linear Volterra type integral equation of second kind

$$a(t)=\Big(\int_0^1 xf(x,t)dx\Big)^{-1}\Big(D_{0_+}^{\alpha,\gamma}g(t)+\mathcal{T}(t)+\int_0^t K(t,\,\tau)a(\tau)\,d\tau\Big), \tag{4.12}$$

where

$$\begin{aligned}\mathcal{T}(t)=\sum_{n=1}^{\infty}\ \lambda_n\Big\{&-\frac{1}{\pi n}\mathcal{E}_n^{(1)}(t)\varphi_{2n-1}(t)\\&+\Big(\frac{-1}{2\pi^2n^2}+\frac{1+e^{2\pi n}}{2\pi n(e^{2\pi n}-1)}\Big)\mathcal{E}_n^{(1)}(t)\varphi_{2n}(t)\Big\},\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}K(t,\tau)=\sum_{n=1}^{\infty}\ \lambda_n\Big\{&-\frac{1}{\pi n}\Big(f_{2n-1}(\tau)\mathcal{E}_n^{(3)}(t-\tau)\Big)\\&+\Big(\frac{-1}{2\pi^2n^2}+\frac{1+e^{2\pi n}}{2\pi n(e^{2\pi n}-1)}\Big)\Big(f_{2n}(\tau)\mathcal{E}_n^{(3)}(t-\tau)\Big)\Big\}.\end{aligned} \tag{4.14}$$

□

Let us consider the space of continuous functions $C[0,T]$, equipped with the Chebyshev norm

$$\|f\|_{C[0,T]}:=\max_{0\le t\le T}|f(t)|.$$

Define an operator $\mathcal{B}(a(t)):=a(t)$, where the operator $\mathcal{B}$ is

$$\mathcal{B}(a(t))=\Big(\int_0^1 xf(x,t)dx\Big)^{-1}\Big(D_{0_+}^{\alpha,\gamma}g(t)+\mathcal{T}(t)+\int_0^t K(t,\,\tau)a(\tau)\,d\tau\Big). \tag{4.15}$$

To show that the mapping $\mathcal{B}\colon C[0,T]\to C[0,T]$ is a contraction map. First of all, we shall show that $a(t)\in C[0,T]$ implies that $\mathcal{B}(a(t))\in C[0,T]$.

By using (2.10) there exists a constant C_6 such that

$$t\mathcal{E}_n^{(3)}(t)\le\frac{C_6}{\lambda_n}\quad t\in[\epsilon,T]. \tag{4.16}$$

Using (3.13), integration by parts and Bessel's inequality, we obtained the inequalities

$$|\varphi_{2n-1}|\le\frac{1}{\lambda_n}\mathcal{I}_n,\quad\text{and}\quad|\varphi_{2n}|\le\frac{\sqrt{2}}{\lambda_n}\|\varphi^{iv}(x)\|_{L^2(0,1)}.$$

Similarly we obtain

$$|f_{2n-1}|\le\frac{1}{\lambda_n}\mathcal{H}_n,\quad\text{and}\quad|f_{2n}|\le\frac{\sqrt{2}}{\lambda_n}\|f^{iv}(x)\|_{L^2(0,1)},$$

where $\mathcal{H}_n=\langle f^{iv},Y_{2n-1}\rangle$. From estimates (3.19), (4.16) and using above relations we have

$$t^{1+\alpha-\gamma}|\mathcal{T}(t)|\le\sum_{n=1}^{\infty}C_1\Big\{\frac{\mathcal{I}_n}{\pi n\lambda_n}+\Big(\frac{1}{\pi^2n^2}+\frac{1}{\pi n}\Big)\frac{\sqrt{2}}{\lambda_n}\|\varphi^{iv}(x)\|_{L^2(0,1)}\Big\},$$

$$t|K(t,\tau)|\le\sum_{n=1}^{\infty}C_6\Big\{\frac{\mathcal{H}_n}{\pi n\lambda_n}+\Big(\frac{1}{\pi^2n^2}+\frac{1}{\pi n}\Big)\frac{\sqrt{2}}{\lambda_n}\|f^{iv}(x)\|_{L^2(0,1)}\Big\}.$$

Hence, the series (4.13) and (4.14) are uniformly convergent by Weierstrass M-test. The uniform convergence of the series (4.14) allow us to write

$$\|K(t,\tau)\|_{C[0,T]} \le K_1, \quad t \in (0,T],$$

where K_1 is a constant, consequently $\mathcal{B}(a(t)) \in C[0,\,T]$.

Without loss of generality we set T such that $T < 1/K_1M^*$.

Let us show that the mapping $\mathcal{B} : C[0,\,T] \to C[0,\,T]$ is contraction, for this we take

$$|\mathcal{B}(a) - \mathcal{B}(c)| \le M^* \int_0^t |a(\tau) - c(\tau)||K(t,\,\tau)|d\tau \le TK_1M^* \max_{0\le t\le T} |a(\tau) - c(\tau)|,$$
$$\|\mathcal{B}(a) - \mathcal{B}(c)\|_{C[0,T]} \le TK_1M^*\|a - c\|_{C[0,T]}, \tag{4.17}$$

thus, the mapping $\mathcal{B}(\cdot)$ is a contraction which assures the unique determination of $a \in C[0,T]$ by Banach fixed point theorem.

The solution $u(x,t)$ is formally given by the series (4.1); the uniform convergence of the series involved in $u(x,t)$,$u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,y)$ and $D_{0_+}^{\alpha,\gamma}u(x,t)$ directly follows from the estimates obtained in the previous section.

Proof of Theorem 2.5. (Uniqueness of the solution of the ISP-II) We have already proved uniqueness of the source term $a(t)$ in Theorem 2.4, it remains to prove uniqueness of $u(x,t)$.

Let $u(x,t)$ and $v(x,t)$ be two solutions, and let $\bar{u}(x,t) = u(x,t) - v(x,t)$. Then $\bar{u}(x,t)$ satisfy the equation

$$D_{0_+}^{\alpha,\gamma}\bar{u}(x,t) = \bar{u}_{xxxx}(x,t), \quad (x,t) \in \Omega, \tag{4.18}$$

with initial condition

$$I_{0_+}^{1-\gamma}\bar{u}(x,t)|_{t=0} = 0, \quad x \in [0,1], \tag{4.19}$$

and nonlocal boundary conditions

$$\bar{u}_x(0,t) = \bar{u}_x(1,t), \quad \bar{u}(0,t) = 0 \quad t \in [0,T], \tag{4.20}$$

$$\bar{u}_{xxx}(0,t) = 0 = \bar{u}_{xxx}(1,t) \quad \bar{u}_{xx}(1,t) = 0, \quad t \in [0,T]. \tag{4.21}$$

Consider the functions

$$\bar{u}_0(t) = \int_0^1 \bar{u}(x,t)Y_0(x)dx,$$
$$\bar{u}_{2n-1}(t) = \int_0^1 \bar{u}(x,t)Y_{2n-1}(x)dx,$$
$$\bar{u}_{2n}(t) = \int_0^1 \bar{u}(x,t)Y_{2n}(x)dx.$$

Following the same steps as in the proof of Theorem 2.2, we can show that

$$\bar{u}_0(t) = 0, \quad \bar{u}_{2n-1}(t) = 0, \quad \bar{u}_{2n}(t) = 0, \quad t \in [0,T].$$

Consequently, the uniqueness of the solution follows from the completeness of the set of function $\{Y_0(x), Y_{2n-1}(x), Y_{2n}(x)\}$, $n \in \mathbb{N}$. □

The proof of Theorem 2.6, the stability result is similar to the proof of Theorem 2.3. Theefore, we omit it.

5. Examples

In this section, we provide some examples for ISP-I and ISP-II.

Example 5.1. Consider the ISP-I with initial and final temperatures
$$\varphi(x) = \sin 2\pi x, \quad \psi(x) = (1 + T^\alpha)\sin 2\pi x.$$
The coefficients of the series expansions of $\phi(x)$ and $\psi(x)$ using bi-orthogonal system of functions are
$$\varphi_0 = 0, \quad \varphi_{2n} = 0, \quad \varphi_{2n-1} = \begin{cases} 1/2, & n = 1, \\ 0, & n \neq 1, \end{cases}$$
and
$$\psi_0 = 0, \quad \psi_{2n} = 0, \quad \psi_{2n-1} = \begin{cases} (1 + T^\alpha)/2, & n = 1, \\ 0, & n \neq 1. \end{cases}$$
Using (3.15)–(3.17), we have
$$f_0 = 0, \quad f_{2n} = 0, \quad f_{2n-1} = \begin{cases} \frac{(1+T^\alpha) - \mathcal{E}_1^{(1)}(T)}{2\mathcal{E}_1^{(2)}(T)}, & n = 1, \\ 0, & n \neq 1. \end{cases}$$
Substituting the series coefficient of $f(x)$ in (3.1)–(3.1) we obtain
$$u_0 = 0, \quad u_{2n} = 0,$$
$$u_{2n-1} = \begin{cases} \frac{\mathcal{E}_1^{(1)}(t)}{2} + \frac{(1+T^\alpha) - \mathcal{E}_1^{(1)}(T)}{2\mathcal{E}_1^{(2)}(T)} \mathcal{E}_1^{(2)}(t), & n = 1, \\ 0, & n \neq 1. \end{cases}$$
Hence the solution of ISP-I is
$$f(x) = \Big(\frac{(1 + T^\alpha) - \mathcal{E}_1^{(1)}(T)}{\mathcal{E}_1^{(2)}(T)} \Big) \sin(2\pi x),$$
$$u(x,t) = \Big(\mathcal{E}_1^{(1)}(t) + \frac{(1 + T^\alpha) - \mathcal{E}_1^{(1)}(T)}{\mathcal{E}_1^{(2)}(T)} \mathcal{E}_1^{(2)}(t) \Big) \sin(2\pi x).$$

Example 5.2. Consider the ISP-II with given the data
$$\varphi(x) = 0, \quad g(t) = \frac{2}{3}\Big(\frac{t^{\gamma+1}}{\Gamma(\gamma)} + \frac{t^{\alpha+2}}{\Gamma(\alpha+1)} \Big),$$
$$f(x,t) = 2\Big(\frac{\Gamma(\gamma+2)}{\Gamma(\gamma)\Gamma(\gamma-\alpha+2)} t^{\gamma-\alpha} + \frac{\Gamma(\alpha+3)}{\Gamma(3)\Gamma(\alpha+1)} t \Big) x.$$
By using the bi-orthogonal system, the coefficients of the series expansion are
$$\varphi_0 = 0, \quad \varphi_{2n} = 0, \quad \varphi_{2n-1} = 0,$$
and
$$f_0 = \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)\Gamma(\gamma-\alpha+2)} t^{\gamma-\alpha} + \frac{\Gamma(\alpha+3)}{\Gamma(3)\Gamma(\alpha+1)} t, \quad f_{2n} = 0, \quad f_{2n-1} = 0.$$
The solution is
$$u(x,t) = 2\Big(\frac{t^{\gamma+1}}{\Gamma(\gamma)} + \frac{t^{\alpha+2}}{\Gamma(\alpha+1)} \Big) x,$$
which satisfies the initial condition (1.2) and the over-determination condition (1.7). By using the value of $u(x,t)$ in (4.9), we obtained the source term as $a(t) = t$.

Hence $\{u(x,t), a(t)\}$ forms the solution set for the ISP-II.

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