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CONVOLUTIONS AND GREEN'S FUNCTIONS FOR TWO FAMILIES OF BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider families of two-point boundary value problems for fractional differential equations where the fractional derivative is assumed to be the Riemann-Liouville fractional derivative. The problems considered are such that appropriate differential operators commute and the problems can be constructed as nested boundary value problems for lower order fractional differential equations. Green's functions are then constructed as convolutions of lower order Green's functions. Comparison theorems are known for the Green's functions for the lower order problems and so, we obtain analogous comparison theorems for the two families of higher order equations considered here. We also pose a related open question for a family of Green's functions that do not apparently have convolution representations.

1. Introduction

Sign properties of Green's functions for boundary value problems and maximum principles for boundary value problems are very closely related [4, 6, 11]. In recent years, and possibly beginning with [2], many authors have obtained sign properties of a specific Green's function for a boundary value problem for a fractional differential equation in order to employ a fixed point theorem to obtain sufficient conditions for the existence of positive solutions.

In [7], the authors initiated the study of families of two-point boundary value problems for a fractional differential equation and obtained both sign properties and comparison theorems for the associated Green's functions. The results in [7] are motivated by the results for ordinary differential equations in, say, [5, 6, 11]. A primary application of this type of analysis is to the μ_0-positivity of operators defined on a cone in a Banach space. We refer the reader to the early development the theory of μ_0-positivity of operators defined on a cone in a Banach space in the definitive works of Krasnosel'skiĭ [15] and Krein and Rutman [16]; for applications to higher order ordinary differential equations we refer the reader to [14] or to [5]. That application is not pursued in this article.

The technique in [7] is naive; only one boundary condition is stacked at the right, and thus, Green's functions are constructed explicitly. Regardless, it serves as a beginning to carry the classical results for ordinary differential equations over

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to classes of fractional differential equations. For example, recently [8, 9, 10], the authors have employed comparison results to compare eigenvalues and characterize principal eigenvalues for boundary value problems for fractional differential equations. Such work has been extended to the Caputo fractional calculus [12] and the discrete fractional calculus [13].

In this article, we continue to consider families of two-point boundary value problems for linear fractional differential equations. The purpose of this work is to develop methods to allow for more than one boundary conditions stacked at the right. The primary tool is to construct Green's functions as convolutions of Green's functions for lower order boundary value problems. We are currently limited to consider problems for which the lower ordered fractional derivatives commute. Under the assumption that the lower order fractional derivatives commute, sign properties and comparison theorems will be established that are anticipated by the classical results for ordinary differential equations. We shall also consider a specific example in which the convolutions methods proposed here fail and numerical experiments indicate that anticipated comparison theorems are valid.

The paper is organized as follows: In section 2, we shall provide the basic definitions and properties of fractional calculus which are employed. We shall also state two comparison theorems proved in [7] for the sake of exposition. In section 3, we consider a specific problem and introduce the convolution method. In section 4, we develop the comparison theorems for two families of boundary value problems. We close in section 5 with an example to show that the expected comparison results may continue to be valid in the case when the appropriate lower order fractional derivatives do not commute.

2. Preliminaries

We recall the definitions of the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative. For the sake of exposition, the initial point throughout the article is $a = 0$.

Definition 2.1. Let $0 < \alpha$. For $t > 0$, the α-th Riemann-Liouville fractional integral of a function, u, is defined by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided the right-hand side exists. For $\alpha = 0$, define I_{0+}^{α} to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \le n$. The α-th Riemann-Liouville fractional derivative of the function $u : [0,\infty) \longrightarrow \mathbb{R}$, denoted

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{0+}^t (t-s)^{n-\alpha-1}u(s)ds = D^n I_0^{n-\alpha}u(t),$$

provided the right-hand side exists. Again for $\alpha = 0$, define D_{0+}^{α} to be the identity map.

We shall employ a standard notation, D_{0+}^{α}, to denote fractional derivatives for noninteger α and D^j to denote classical derivatives in the case, $\alpha = j$ is a nonnegative integer. We shall require only a few well known properties in fractional calculus which we state below. We refer the reader to the monograph of Diethelm [3] or the recent article [7] for basic definitions and properties.

$$I_{0+}^{\alpha_1}I_{0+}^{\alpha_2}u(t) = I_{0+}^{\alpha_1+\alpha_2}u(t) = I_{0+}^{\alpha_2}I_{0+}^{\alpha_1}u(t), \quad \text{if } \alpha_1, \alpha_2 > 0, \tag{2.1}$$

$$D_{0+}^{\alpha_1} I_{0+}^{\alpha_2} u(t) = I_{0+}^{\alpha_2-\alpha_1} u(t), \quad \text{if } 0 \le \alpha_1 \le \alpha_2, \tag{2.2}$$
$$D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t), \quad \text{if } 0 \le \alpha, \tag{2.3}$$
$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-n+(i-1)}, \quad \text{if } 0 \le \alpha, \tag{2.4}$$

We also require the power rules [3, 7],

$$I_{0+}^{\alpha_2} t^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_2+\alpha_1+1)} t^{\alpha_2+\alpha_1}, \quad \text{if } \alpha_1 > -1,\ \alpha_2 \ge 0, \tag{2.5}$$

and

$$D_{0+}^{\alpha_2} t^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1-\alpha_2)} t^{\alpha_1-\alpha_2}, \quad \text{if } \alpha_1 > -1,\ \alpha_2 \ge 0. \tag{2.6}$$

In (2.6), it is assumed that $\alpha_2 - \alpha_1$ is not a positive integer. If $\alpha_2 - \alpha_1$ is a positive integer, then the right hand side of (2.6) vanishes. To see this, one can appeal to the convention that $\frac{1}{\Gamma(\alpha_1+1-\alpha_2)} = 0$ if $\alpha_2 - \alpha_1$ is a positive integer, or one can perform the calculation on the left hand side and calculate

$$D^n t^{n-(\alpha_2-\alpha_1)} = 0.$$

In [7], the authors obtained comparison results for boundary value problems for lower order fractional differential equations. For the sake of self-containment, we summarize those results here.

Let $2 \le n$ denote an integer and let $n-1 < \bar{\alpha} \le n$. For each $0 < b$, $0 \le \beta \le n-1$, consider a boundary value problem (BVP) for the fractional differential equation of the form

$$D_{0+}^{\bar{\alpha}} u + h(t) = 0, \quad 0 < t < b, \tag{2.7}$$

with two-point boundary conditions of the form

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad D_{0+}^{\beta} u(b) = 0. \tag{2.8}$$

It is shown in [7] that the Green's function of (2.7)–(2.8) has the form

$$G(\bar{\alpha}, \beta, b; t, s) = \begin{cases} \frac{t^{\bar{\alpha}-1}(b-s)^{\bar{\alpha}-1-\beta}}{b^{\bar{\alpha}-1-\beta}\Gamma(\bar{\alpha})} - \frac{(t-s)^{\bar{\alpha}-1}}{\Gamma(\bar{\alpha})}, & 0 \le s \le t \le b, \\ \frac{t^{\bar{\alpha}-1}(b-s)^{\bar{\alpha}-1-\beta}}{b^{\bar{\alpha}-1-\beta}\Gamma(\bar{\alpha})}, & 0 \le t \le s \le b \end{cases} \tag{2.9}$$

and the solution of (2.7), (2.8) is $u(t) = \int_0^b G(\bar{\alpha}, \beta, b; t, s) h(s) ds$.

Theorem 2.2 ([7]). *If $0 \le \beta_1 < \beta_2 \le n-1$, then*

$$0 < G(\bar{\alpha}, \beta_1, b; t, s) < G(\bar{\alpha}, \beta_2, b; t, s), \quad (t, s) \in (0, b) \times (0, b). \tag{2.10}$$

If $n \ge 2$ the proof Theorem 2.2 is readily extended to obtain the following result.

Theorem 2.3 ([7]). *Let $j \in \{0, \dots, n-2\}$, $i \in \{0, \dots j\}$. If $0 \le \beta_1 < \beta_2 \le n-1$, then*

$$0 < \Big(\frac{\partial^i}{\partial t^i}\Big) G(\bar{\alpha}, \beta_1, b; t, s) < \Big(\frac{\partial^i}{\partial t^i}\Big) G(\bar{\alpha}, \beta_2, b; t, s), \quad (t, s) \in (0, b) \times (0, b). \tag{2.11}$$

Theorems 2.2 and 2.3 compare Green's functions as functions of β. The following theorem compares Green's functions as functions of b.

Theorem 2.4 ([7]). *Assume $0 < b_1 < b_2$. If $0 \le \beta < \bar\alpha - 1$, then*

$$0 < G(\bar\alpha, \beta, b_1; t, s) < G(\bar\alpha, \beta, b_2; t, s), \quad (t, s) \in (0, b_1) \times (0, b_1), \tag{2.12}$$

and if $\bar\alpha - 1 < \beta \le n - 1$, then

$$G(\bar\alpha, \beta, b_1; t, s) > G(\bar\alpha, \beta, b_2; t, s) > 0, \quad (t, s) \in (0, b_1) \times (0, b_1). \tag{2.13}$$

If $\beta = \bar\alpha - 1$, then $G(\bar\alpha, \bar\alpha - 1, b; t, s)$ is independent of b on $(0, b) \times (0, b)$.

3. Green's functions as convolutions

We begin with a specific family of two-point boundary value problems. Let $b > 0$. Assume $3 < \alpha \le 4$ and assume $0 \le \beta \le 1$. Consider a two-point BVP for a fractional differential equation of the form

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < b, \tag{3.1}$$

$$u(0) = 0, \quad D_{0+}^{\beta} u(b) = 0, \quad D_{0+}^{\alpha-2} u(0) = 0, \quad D_{0+}^{\alpha-2} u(b) = 0. \tag{3.2}$$

We shall construct the Green's function, $G(\alpha, \beta, b; t, s)$, with two strategies, one by direct computation and another by a convolution of lower order Green's functions. We shall also produce a calculation to verify the two constructions are equivalent.

To produce a direct computation, apply the operator, I_{0+}^{α} to (3.1), employ (2.4) and

$$u(t) + c_1 t^{\alpha-4} + c_2 t^{\alpha-3} + c_3 t^{\alpha-2} + c_4 t^{\alpha-1} + I_{0+}^{\alpha} h(t) = 0. \tag{3.3}$$

Since $u(0) = 0$, $c_1 = 0$ because of the singularity of $t^{\alpha-4}$ at $t = 0$, and $D_0^{\alpha-2} u(0) = 0$ implies $c_3 = 0$. Thus,

$$u(t) + c_2 t^{\alpha-3} + c_4 t^{\alpha-1} + I_{0+}^{\alpha} h(t) = 0.$$

Apply the boundary conditions, $D_{0+}^{\beta} u(b) = 0$ and $D_{0+}^{\alpha-2} u(b) = 0$ (and the power rule) to obtain the system of equations

$$\begin{gathered} c_2 b^{\alpha-3-\beta} \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-2-\beta)} + c_4 b^{\alpha-1-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} + I_{0+}^{\alpha-\beta} h(b) = 0, \\ c_4 \Gamma(\alpha) b + I_{0+}^{2} h(b) = 0. \end{gathered}$$

Thus, c_2 and c_4 are explicitly obtained and

$$\begin{gathered} c_2 = \frac{\Gamma(\alpha-2-\beta)}{b^{\alpha-3-\beta}\Gamma(\alpha-2)} \Big(\frac{b^{\alpha-2-\beta}}{\Gamma(\alpha-\beta)} I_{0+}^{2} h(b) - I_{0+}^{\alpha-\beta} h(b) \Big), \\ c_4 = -\frac{1}{\Gamma(\alpha) b} I_{0+}^{2} h(b), \end{gathered} \tag{3.4}$$

and (3.3) reduces to

$$u(t) + c_2 t^{\alpha-3} + c_4 t^{\alpha-1} + I_0^{\alpha} h(t) = 0,$$

where c_2 and c_4 are given in (3.4). Using (3.4) we define

$$g(\alpha, \beta, b; t, s) = \frac{t^{\alpha-1}(b-s)}{\Gamma(\alpha) b} + \frac{\big(-b^{\alpha-2-\beta}(b-s) + (b-s)^{\alpha-1-\beta}\big)\Gamma(\alpha-2-\beta) t^{\alpha-3}}{b^{\alpha-3-\beta}\Gamma(\alpha-2)\Gamma(\alpha-\beta)}.$$

Then the Green's function of BVP (3.1), (3.2), is

$$G(\alpha, \beta, b; t, s) = \begin{cases} g(\alpha, \beta, b; t, s), & 0 \le t < s \le b, \\ g(\alpha, \beta, b; t, s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le b, \end{cases} \tag{3.5}$$

and the solution u of (3.1), (3.2), has the form

$$u(t) = \int_0^1 G(\alpha, \beta, b; t, s)h(s)ds.$$

We now construct $G(\alpha, \beta, b; t, s)$ as a convolution of two Green's functions of lower order BVPs. Consider a change of variable, $v(t) = D_{0+}^{\alpha-2}u(t)$. Note that $3 < \alpha \leq 4$ implies $1 < \alpha - 2 \leq 2$ and

$$D^2 v(t) = D^2 D_{0+}^{\alpha-2}u(t) = D^2 D^2 I_{0+}^{4-\alpha}u(t) = D^4 I_{0+}^{4-\alpha}u(t) = D_{0+}^{\alpha}u(t).$$

Using only the boundary conditions $D_{0+}^{\alpha-2}u(0) = 0, D_{0+}^{\alpha-2}u(b) = 0$ from (3.2), it is the case that v satisfies a conjugate or Dirichlet boundary value problem for an ordinary differential equation,

$$v''(t) + h(t) = 0, \quad 0 < t < b, \tag{3.6}$$

$$v(0) = 0, \quad v(b) = 0. \tag{3.7}$$

Thus,

$$v(t) = \int_0^b G_{\rm conj}(b; t, s)h(s)ds, \quad 0 \leq t \leq b, \tag{3.8}$$

where $G_{\rm conj}(b; t, s)$ is well known and has the form

$$G_{\rm conj}(b; t, s) = \begin{cases} \frac{t(b-s)}{b}, & 0 \leq t < s \leq b, \\ \frac{s(b-t)}{b}, & 0 \leq s < t \leq b, \end{cases} \tag{3.9}$$

The function u satisfies a two-point BVP for a fractional differential equation of the form

$$D_{0+}^{\alpha-2}u(t) = v(t), \quad 0 < t < b, \tag{3.10}$$

$$u(0) = 0, \quad D_{0+}^{\beta}u(b) = 0. \tag{3.11}$$

The Green's function, $G(\alpha - 2, \beta, b; t, s)$, (with $\bar{\alpha} = \alpha - 2$) is given by (2.9) for (3.10), (3.11) and has the form

$$G(\alpha-2, \beta, b; t, s) = \begin{cases} \frac{t^{\alpha-3}(b-s)^{\alpha-3-\beta}}{b^{\alpha-3-\beta}\Gamma(\alpha-2)} - \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 \leq s \leq t \leq b, \\ \frac{t^{\alpha-3}(b-s)^{\alpha-3-\beta}}{b^{\alpha-3-\beta}\Gamma(\alpha-2)}, & 0 \leq t \leq s \leq b. \end{cases} \tag{3.12}$$

Thus,

$$u(t) = \int_0^b G(\alpha-2, \beta, b; t, s)(-v(s))ds, \quad 0 \leq t \leq b.$$

Since v has the form (3.8),

$$\begin{aligned} u(t) &= \int_0^b G(\alpha-2, \beta, b; t, s)\Big(-\int_0^b G_{\rm conj}(b; s, r)h(r)dr\Big)ds, \quad 0 \leq t \leq b, \\ &= \int_0^b \Big(-\int_0^b G(\alpha-2, \beta, b; t, s)G_{\rm conj}(b; s, r)ds\Big)h(r)dr \end{aligned}$$

and

$$u(t) = \int_0^b G(\alpha, \beta, b; t, s)h(s)ds, \quad 0 \leq t \leq b, \tag{3.13}$$

where $G(\alpha,\beta,b;t,s)$ is the Green's function for the original α fractional order BVP (3.1), (3.2); in particular,

$$G(\alpha,\beta,b;t,s) = -\int_0^b G(\alpha-2,\beta,b;t,r)G_{\rm conj}(b;r,s)dr, \quad (t,s)\in[0,b]\times[0,b]. \tag{3.14}$$

To gain confidence that the two constructions are valid, we show the equivalence of (3.5) and (3.14) in the case that $\beta=0$. To that end, recall a calculation that employs the special beta function. Make a change of variable, $t=\tau+s(x-\tau)$ to calculate

$$\begin{aligned}
&\int_\tau^x (x-t)^{m-1}(t-\tau)^{n-1}dt\\
&= (x-\tau)^{m+n-1}\int_0^1 (1-s)^{m-1}s^{n-1}ds = (x-\tau)^{m+n-1}B(m,n)
\end{aligned}$$

where $B(m,n)$ denotes the special beta function. Thus,

$$\int_\tau^x (x-t)^{m-1}(t-\tau)^{n-1}dt = (x-\tau)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Rewrite $G_{\rm conj}(b;t,s)$ in the form

$$G_{\rm conj}(b;t,s) = \begin{cases} \frac{t(b-s)}{b}, & 0\le t<s\le b,\\ \frac{t(b-s)}{b}-(t-s), & 0\le s<t\le b,\end{cases}$$

Then for $t<s$, write $-\int_0^b G(\alpha-2,0,b;t,r)G_{\rm conj}(b;r,s)dr$ as

$$\begin{aligned}
&-\int_0^b G(\alpha-2,0,b;t,r)G_{\rm conj}(b;r,s)dr\\
&= -\frac{t^{\alpha-3}}{b^{\alpha-3}\Gamma(\alpha-2)}\int_0^b (b-r)^{\alpha-3}rdr\frac{b-s}{b}\\
&\quad +\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-r)^{\alpha-3}rdr\frac{b-s}{b} + \frac{t^{\alpha-3}}{b^{\alpha-3}\Gamma(\alpha-2)}\int_s^b (b-r)^{\alpha-3}(r-s)dr\\
&= \frac{-1}{\Gamma(\alpha)}\Big(\Big(b(b-s)-\frac{(b-s)^{\alpha-1}}{b^{\alpha-3}}\Big)t^{\alpha-3}-\frac{(b-s)}{b}t^{\alpha-1}\Big),
\end{aligned}$$

and note that the three terms produced here match the three terms that are summed to produce $g(\alpha,0,b;t,s)$ for $t<s$.

Now we use the convolution representation (3.14) and extend comparison theorems for Green's functions in [7] (see Theorems 2.2 and 2.4) for boundary value problems with precisely one boundary condition specified at the right.

Theorem 3.1. *If $0\le\beta_1<\beta_2\le 1$, then*

$$0>G(\alpha,\beta_1,b;t,s)>G(\alpha,\beta_2,b;t,s), \quad (t,s)\in(0,b)\times(0,b). \tag{3.15}$$

Proof. Using (2.10), if $0\le\beta_1<\beta_2\le1$, then

$$0<G(\alpha-2,\beta_1,b;t,s)<G(\alpha-2,\beta_2,b;t,s), \quad (t,s)\in(0,b)\times(0,b).$$

Since

$$0<G_{\rm conj}(b;t,s), \quad (t,s)\in(0,b)\times(0,b),$$

the result follows immediately from the representation in (3.14). □

Theorem 3.2. *Assume* $0 < b_1 < b_2$. *If* $0 \le \beta < \alpha - 3$, *then*

$$0 > G(\alpha, \beta, b_1; t, s) > G(\alpha, \beta, b_2; t, s), \quad (t, s) \in (0, b_1) \times (0, b_1). \tag{3.16}$$

Proof. Applying (2.12), if $0 \le \beta < \alpha - 3$, then

$$0 < G(\alpha - 2, \beta, b_1; t, s) < G(\alpha - 2, \beta, b_2; t, s), \quad (t, s) \in (0, b_1) \times (0, b_1).$$

Since $0 < G_{\rm conj}(b; t, s)$ and $0 < \frac{\partial}{\partial b} G_{\rm conj}(b; t, s)$ for $(t, s) \in (0, b_1) \times (0, b_1)$ the comparison in Theorem (3.2) follows. □

4. Two families of boundary conditions

In this section, we shall employ the convolution construction and consider two families of boundary value problems for a higher order fractional differential equation. The first family we consider is motivated by the two-point Lidstone boundary value problem for ordinary differential equations [1, 17].

Assume $n \in \mathbb{N}$, $2n - 1 < \alpha \le 2n$, $0 \le \beta \le 1$ and consider the BVP

$$D^{\alpha}_{0+} u(t) + h(t) = 0, \quad 0 < t < b, \tag{4.1}$$

$$\begin{gathered} u(0) = 0, \quad D^{\beta}_{0+} u(b) = 0, \quad D^{\alpha - 2l}_{0+} u(0) = 0, \\ D^{\alpha - 2l}_{0+} u(b) = 0, \quad l = 1, \dots, n - 1. \end{gathered} \tag{4.2}$$

Denote by $G_n(\alpha, \beta, b; t, s)$ the Green's function for (4.1), (4.2). And so, if $3 < \alpha \le 4$, then

$$G_2(\alpha, \beta, b; t, s) = G(\alpha, \beta, b; t, s)$$

is given by (3.14).

Inductively, we construct functions $G_{n-k}(\alpha - 2k, \beta, b; t, s)$ by

$$G_{n-k}(\alpha - 2k, \beta, b; t, s) = -\int_0^b G_{n-(k+1)}(\alpha - 2(k+1), \beta, b; t, r) G_{\rm conj}(b; r, s) ds, \tag{4.3}$$

$n - k = 3, \dots, n - 1$. It then follows that the Green's function $G_n(\alpha, \beta, b; t, s)$ the Green's function for (4.1), (4.2) is of the form

$$G_n(\alpha, \beta, b; t, s) = -\int_0^b G_{n-1}(\alpha - 2, \beta, b; t, r) G_{\rm conj}(b; r, s) ds,$$

where $G_{n-1}(\alpha - 2, \beta, b; t, s)$ is the Green's function for the BVP

$$D^{\alpha-2}_{0+} u(t) + h(t) = 0, \quad 0 < t < b,$$

$$u(0) = 0, \quad D^{\beta}_{0+} u(b) = 0, \quad D^{\alpha - 2l}_{0+} u(0) = 0, \quad D^{\alpha - 2l}_{0+} u(b) = 0, \ l = 1, \dots, n - 2.$$

To see this, we make the change of variable $v(t) = D^{\alpha-2}_{0+} u(t)$. Then

$$D^2 v(t) = D^2 D^{\alpha-2}_{0+} u(t) = D^{\alpha}_{0+} u(t) = -h(t).$$

Since $v(0) = D^{\alpha-2}_{0+} u(0) = 0$ and $v(b) = D^{\alpha-2}_{0+} u(b) = 0$, v satisfies the Dirichlet BVP

$$\begin{gathered} v'' + h(t) = 0, \quad 0 < t < b, \\ v(0) = 0, \quad v(b) = 0. \end{gathered}$$

Also, u satisfies a lower order BVP

$$D^{\alpha-2}_{0+} u(t) = v(t), \quad 0 < t < b,$$

$$u(0) = 0, \quad D^{\beta}_{0+} u(b) = 0, \quad D^{\alpha - 2l}_{0+} u(0) = 0, \quad D^{\alpha - 2l}_{0+} u(b) = 0, \quad l = 2, \dots, k,$$

and by the induction hypothesis,

$$\begin{aligned} u(t) &= \int_0^b G_{n-1}(\alpha-2,\beta,b;t,s)(-v(s))ds \\ &= \int_0^b \Big(-\int_0^b G_{n-1}(\alpha-2,\beta,b;t,s)G_{\rm conj}(b;s,r)ds\Big)h(r)dr \\ &= \int_0^b G_n(\alpha,\beta,b;t,s)h(s)ds, \end{aligned}$$

where $G_n(\alpha,\beta,b;t,s) = -\int_0^b G_{n-1}(\alpha-2,\beta,b;t,r)G_{\rm conj}(b;r,s)ds$.

Since $0 < G_{\rm conj}(b;t,s)$ and $0 < \frac{\partial}{\partial b}G_{\rm conj}(b;t,s)$ for $(t,s) \in (0,b)\times(0,b)$ the following comparison results are immediate from Theorems 3.1 and 3.2 and the inductive construction in (4.3).

Theorem 4.1. *If $0 \le \beta_1 < \beta_2 \le 1$, then*

$$\begin{aligned} 0 &> (-1)^{n-k}\Big(\frac{d^{2k}}{dt^{2k}}\Big)G_n(\alpha,\beta_1,b;t,s) \\ &> (-1)^{n-k}\Big(\frac{d^{2k}}{dt^{2k}}\Big)G_n(\alpha,\beta_2,b;t,s), \quad (t,s)\in(0,b)\times(0,b), \end{aligned} \tag{4.4}$$

for $k = 0,\dots,n-1$.

Theorem 4.2. *Assume $0 < b_1 < b_2$ and assume $0 \le \beta < \alpha-(n-1)$. Then for $n\in\mathbb{N}$, $2n-1<\alpha\le 2n$,*

$$\begin{aligned} 0 &> (-1)^{n-k}\Big(\frac{d^{2k}}{dt^{2k}}\Big)G_n(\alpha,\beta,b_1;t,s) \\ &> (-1)^{n-k}\Big(\frac{d^{2k}}{dt^{2k}}\Big)G(\alpha,\beta,b_2;t,s), \quad (t,s)\in(0,b_1)\times(0,b_1), \end{aligned} \tag{4.5}$$

for $k = 0,\dots,n-1$.

For a second family we begin with a low order analogue of a right focal boundary value problem. Let $b > 0$. Assume $3 < \alpha \le 4$ and $0 \le \beta \le 1$. Consider a two-point BVP for a fractional differential equation of the form

$$D_{0+}^{\alpha}u(t) + h(t) = 0, \quad 0 < t < b, \tag{4.6}$$

$$u(0) = 0, \quad D_{0+}^{\beta}u(b) = 0, \quad D_{0+}^{\alpha-2}u(0) = 0, \quad D_{0+}^{\alpha-1}u(b) = 0. \tag{4.7}$$

Again, we set $v(t) = D_{0+}^{\alpha-2}u(t)$. Then v satisfies a right focal BVP for an ordinary differential equation,

$$\begin{gathered} v''(t) + h(t) = 0, \quad 0 < t < b, \\ v(0) = 0, \quad v'(b) = 0, \\ v(t) = \int_0^b G_{\rm foc}(b;t,s)h(s)ds, \quad 0 \le t \le b, \end{gathered}$$

where $G_{\rm foc}(b;t,s)$ is well known and has the form

$$G_{\rm foc}(b;t,s) = \begin{cases} t, & 0 \le t < s \le b, \\ s = t-(t-s), & 0 \le s < t \le b, \end{cases}$$

Then u, a solution of (4.6), (4.7), also satisfies the BVP,

$$D_{0+}^{\alpha-2}u(t) = v(t), \quad 0 < t < b,$$

along with boundary conditions $u(0) = 0, D^{\beta}_{0+}u(b) = 0$. Thus, a solution u of (4.6), (4.7) has the form

$$u(t) = \int_0^b \mathcal{G}(\alpha, \beta, b; t, s)h(s)ds, \quad 0 \le t \le b,$$

where $\mathcal{G}(\alpha, \beta, b; t, s)$ is the convolution

$$\mathcal{G}(\alpha, \beta, b; t, s) = -\int_0^b G(\alpha - 2, \beta, b; t, r)G_{\rm foc}(b; r, s)dr, \quad (t, s) \in [0, b] \times [0, b], \tag{4.8}$$

and $G(\alpha - 2, \beta, b; t, s)$ is given by (2.9) with $\bar{\alpha} = \alpha - 2$.

Since $G_{\rm foc}(b; t, s) > 0$ on $(0, b) \times (0, b)$ and $\frac{\partial}{\partial b} G_{\rm foc}(b; t, s) \equiv 0$ on $(0, b) \times (0, b)$, the following results follow from the known comparison results for $G(\alpha - 2, \beta, b; t, s)$.

Theorem 4.3. *If $0 \le \beta_1 < \beta_2 \le 1$, then*

$$0 > \mathcal{G}(\alpha, \beta_1, b; t, s) > \mathcal{G}(\alpha, \beta_2, b; t, s), \quad (t, s) \in (0, b) \times (0, b).$$

Theorem 4.4. *Assume $0 < b_1 < b_2$. If $0 \le \beta < \alpha - 3$, then*

$$0 > \mathcal{G}(\alpha, \beta, b_1; t, s) > \mathcal{G}(\alpha, \beta, b_2; t, s), \quad (t, s) \in (0, b_1) \times (0, b_1),$$

and if $\alpha - 3 < \beta \le 1$, then

$$0 > \mathcal{G}(\alpha, \beta, b_2; t, s) > \mathcal{G}(\alpha, \beta, b_1; t, s), \quad (t, s) \in (0, b_1) \times (0, b_1). \tag{4.9}$$

Note that Theorem 4.4 includes the case $\alpha - 3 < \beta \le 1$, and Theorem 3.2 does not. Since

$$\frac{\partial}{\partial b} G_{\rm foc}(b; t, s) \equiv 0 \quad \text{on } (0, b) \times (0, b),$$

Inequality (2.13) can be applied to obtain (4.9).

We also point out that a further type of comparison result is valid since

$$0 < G_{\rm conj}(b; t, s) < G_{\rm foc}(b; t, s) \quad \text{on } (0, b) \times (0, b).$$

Theorem 4.5. *If $0 \le \beta \le 1$, then*

$$0 > G(\alpha, \beta, b; t, s) > \mathcal{G}(\alpha, \beta, b; t, s), \quad (t, s) \in (0, b) \times (0, b),$$

where $G(\alpha, \beta, b; t, s)$ is given by (3.14).

The ideas produced here can be extended inductively. We do so but modify on the higher order boundary conditions. In (4.7), v satisfies right focal boundary conditions; for simplicity, to proceed inductively, v will satisfy initial conditions.

Let $b > 0$. Let $n \ge 3$ denote an integer and assume $n - 1 < \alpha \le n$. Let $k \in \{1, \dots, n - 1\}$ denote an integer and assume $0 \le \beta \le k$. Consider a two-point BVP for a fractional differential equation of the form

$$D^{\alpha}_{0+}u(t) + h(t) = 0, \quad 0 < t < b, \tag{4.10}$$

$$\begin{gathered} u^{(i)}(0) = 0, \quad i = 0, \dots, k - 1, \quad D^{\beta}_{0+}u(b) = 0, \\ D^{\alpha - j}_{0+}u(b) = 0, \quad j = 1, \dots n - k - 1. \end{gathered} \tag{4.11}$$

Denote the Green's function, if it exists, of (4.10), (4.11) as $\mathcal{G}(\alpha, \beta, k, b; t, s)$. And from what follows, $\mathcal{G}(\alpha, \beta, k, b; t, s)$ will exist as a convolution of Green's functions of lower order problems.

To treat this boundary value problem as a nested family of problems, we make the substitution $v = D_{0+}^{\alpha-(n-k-1)}u$. Then v satisfies the initial value problem,

$$v^{(n-k-1)}(t) + h(t) = 0, \quad 0 < t < b, \quad v^{(j)}(b) = 0, \quad j = 0, \dots, n-k-2,$$

and a solution u of (4.10), (4.11) satisfies the BVP

$$D_{0+}^{\alpha-(n-k-1)}u(t) = v(t), \quad 0 < t < b, \tag{4.12}$$

$$u^{(i)}(0) = 0, \quad i = 0, \dots, k-1, \quad D_{0+}^{\beta}u(b) = 0. \tag{4.13}$$

We write

$$v(t) = \int_b^t \frac{(t-s)^{n-k-2}}{\Gamma(n-k-1)}(-h(s))ds = \int_0^b G_{ivp}(b;t,s)h(s)ds,$$

where

$$G_{ivp}(b;t,s) = \begin{cases} 0, & 0 \le s \le t \le b, \\ \frac{(t-s)^{n-k-2}}{\Gamma(n-k-1)}, & 0 \le t \le s \le b. \end{cases}$$

The Green's function for the BVP (4.12), (4.13) has been constructed in [7] (see (2.9)) and has the form

$$\begin{aligned} &G(\alpha-(n-k-1),\beta,b;t,s) \\ &= \begin{cases} \frac{t^{\alpha-(n-k-2)}(b-s)^{\alpha-(n-k-2)-\beta}}{b^{\alpha-(n-k-2)-\beta}\Gamma(\alpha-(n-k-1))} - \frac{(t-s)^{\alpha-(n-k-2)}}{\Gamma(\alpha-(n-k-1))}, & 0 \le s \le t \le b, \\ \frac{t^{\alpha-(n-k-2)}(b-s)^{\alpha-(n-k-2)-\beta}}{b^{\alpha-(n-k-2)-\beta}\Gamma(\alpha-(n-k-1))}, & 0 \le t \le s \le b. \end{cases} \end{aligned} \tag{4.14}$$

Thus, $\mathcal{G}(\alpha,\beta,k,b;t,s)$ exists and can be written as the convolution

$$\mathcal{G}(\alpha,\beta,k,b;t,s) = -\int_0^b G(\alpha-(n-k-1),\beta,b;t,r)G_{ivp}(b;r,s)dr. \tag{4.15}$$

Theorem 4.6. *If $j \in \{0,\dots,k-1\}$, and if $j \le \beta_1 < \beta_2 \le k$, then, for $i = 0,\dots,j$,*

$$0 < (-1)^{n-k}\big(\frac{\partial^i}{\partial t^i}\big)\mathcal{G}(\alpha,\beta_1,k,b;t,s) < (-1)^{n-k}\big(\frac{\partial^i}{\partial t^i}\big)\mathcal{G}(\alpha,\beta_2,k,b;t,s), \tag{4.16}$$

for $(t,s) \in (0,b)\times(0,b)$.

Proof. Applying Theorem 2.3, if $j \in \{0,\dots,k-1\}$, and if $j \le \beta_1 < \beta_2 \le k$, then, for $i = 0,\dots,j$, and $(t,s) \in (0,b)\times(0,b)$, we have

$$0 < \big(\frac{\partial^i}{\partial t^i}\big)G(\alpha-(n-k-1),\beta_1,k,b;t,s) < \big(\frac{\partial^i}{\partial t^i}\big)G(\alpha-(n-k-1),\beta_2,k,b;t,s).$$

The parity of G_{ivp} is $(-1)^{n-k}$ and so, again, the result follows immediately from the representation in (4.15). $\square$

Theorem 2.4 implies the following comparisons.

Theorem 4.7. *Assume $0 < b_1 < b_2$. If $0 \le \beta < \alpha-(n-k)$, then*

$$0 > (-1)^{n-k}\mathcal{G}(\alpha,\beta,k,b_1;t,s) > (-1)^{n-k}\mathcal{G}(\alpha,\beta,k,b_2;t,s), \tag{4.17}$$

for $(t,s) \in (0,b_1)\times(0,b_1)$, and if $\alpha-(n-k) < \beta \le k$, then

$$0 > (-1)^{n-k}G(\alpha,\beta,k,b_2;t,s) > (-1)^{n-k}G(\alpha,\beta,k,b_1;t,s), \tag{4.18}$$

for $(t,s) \in (0,b_1)\times(0,b_1)$.

5. An open question

In the preceding, we have employed convolutions of Green's functions to obtain the expected comparison theorems for two point boundary value problems for higher order fractional equations. If the Green's function is not constructed as a convolution, it is open as to whether there exist families of two-point problems that maintain the validity of the expected comparison theorems. We introduce a specific family here to frame the open question.

Let $b > 0$. Assume $3 < \alpha \le 4$ and assume $\alpha - 2 < \gamma < \alpha - 1$. Consider a two-point BVP for a fractional differential equation of the form

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < b, \tag{5.1}$$

$$u(0) = 0, \quad u(b) = 0, \quad D_{0+}^{\alpha-2} u(0) = 0, \quad D_{0+}^{\gamma} u(b) = 0. \tag{5.2}$$

The technique to set $v = D_{0+}^{\alpha-2} u$ is now fruitless, since we do not know how to transform $D_{0+}^{\gamma} u$ to v. So apply (2.4) directly to (5.1), (5.2), and a solution u of (5.1), (5.2) has the form

$$u(t) + c_1 t^{\alpha-4} + c_2 t^{\alpha-3} + c_3 t^{\alpha-2} + c_4 t^{\alpha-1} + I_0^{\alpha} h(t) = 0.$$

Then apply the boundary conditions to obtain the system of equations

$$c_2 b^{\alpha-3} + c_4 b^{\alpha-1} + I_0^{\alpha} h(b) = 0,$$

$$c_2 \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-2-\gamma)} b^{\alpha-3-\gamma} + c_4 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} b^{\alpha-1-\gamma} + I_0^{\alpha-\gamma} h(b) = 0.$$

Thus, c_2 and c_4 are explicitly obtained and

$$c_2 = \frac{1}{\Delta}\Big(b^{\alpha-1} I_0^{\alpha-\gamma} h(b) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} b^{\alpha-1-\gamma} I_0^{\alpha} h(b)\Big),$$

$$c_4 = \frac{1}{\Delta}\Big(\frac{\Gamma(\alpha-2)}{\Gamma(\alpha-2-\gamma)} b^{\alpha-3-\gamma} I_0^{\alpha} h(b) - b^{\alpha-3} I_0^{\alpha-\gamma} h(b)\Big),$$

where

$$\Delta = \Big(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} - \frac{\Gamma(\alpha-2)}{\Gamma(\alpha-2-\gamma)}\Big) b^{2\alpha-\gamma-4}.$$

Let

$$\begin{aligned} g(\alpha,\gamma,b;t,s) &= \frac{-1}{\Delta}\left(b^{\alpha-1}(b-s)^{\alpha-\gamma-1} - b^{\alpha-1-\gamma}(b-s)^{\alpha-1}\right) \frac{t^{\alpha-3}}{\Gamma(\alpha-\gamma)} \\ &\quad + \frac{-1}{\Delta}\Big(\frac{\Gamma(\alpha-2)}{\Gamma(\alpha-2-\gamma)} b^{\alpha-3-\gamma} \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} - b^{\alpha-3} \frac{(b-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\Big) t^{\alpha-1}. \end{aligned}$$

Then the Green's function is

$$G(\alpha,\gamma,b;t,s) = \begin{cases} g(\alpha,\gamma,b;t,s), & 0 \le t < s \le b, \\ g(\alpha,\gamma,b;t,s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le b, \end{cases} \tag{5.3}$$

and the solution u has the form

$$u(t) = \int_0^1 G(\alpha,\gamma,b;t,s) h(s) ds.$$

Note that at $\gamma = \alpha - 2$,

$$G(\alpha,\gamma,b;t,s) = G(\alpha,0,b;t,s)$$

where $G(\alpha, 0, b; t, s)$ is given by (3.14) at $\beta = 0$ and at $\gamma = \alpha - 1$,

$$G(\alpha, \gamma, b; t, s) = \mathcal{G}(\alpha, 0, b; t, s)$$

where $\mathcal{G}(\alpha, 0, b; t, s)$ is given by (4.8) at $\beta = 0$.

A natural question to ask in the context of Theorem 4.5 is the following: Let $b > 0$. Assume $3 < \alpha \le 4$ and assume $\alpha - 2 < \gamma_1 < \gamma_2 < \alpha - 1$. Let $G(\alpha, \gamma_i, b; t, s)$, $i = 1, 2$ denote the Green's function for (5.1), (5.2) for $\gamma = \gamma_i$, $i = 1, 2$ respectively. Let $G(\alpha, 0, b; t, s)$ be given by (3.14) and $\mathcal{G}(\alpha, 0, b; t, s)$ be given by (4.8). Is

$$0 > G(\alpha, 0, b; t, s) > G(\alpha, \gamma_1, b; t, s) > G(\alpha, \gamma_2, b; t, s) > \mathcal{G}(\alpha, 0, b; t, s),$$

for $(t, s) \in (0, b) \times (0, b)$, valid?

Preliminary numerical experiments, not produced here, indicate that the answer to this particular question is yes.

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