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STURM-LIOUVILLE BVPS WITH CARATHEODORY NONLINEARITIES

ABDELHAMID BENMEZAÏ, WASSILA ESSERHANE, JOHNNY HENDERSON

Abstract. In this article we study the existence and multiplicity of solutions for several classes of Sturm-Liouville boundary value problems having Caratheodory nonlinearities. Many results existing in the literature for such boundary value problems in the continuous framework will find in this work their extensions to the Caratheodory setting.

1. Introduction

Sturm-Liouville boundary value problems (BVP for short) have been the subject of hundreds of articles during the previous five decades, where existence and multiplicity of solutions have been investigated. Often, these works are considered in the continuous framework. For this reason, we are concerned here with existence and multiplicity of solutions for Sturm-Liouville BVPs posed in the Caratheodory framework given by,

$$\begin{gathered}
\pounds u = f(t,u,\mu) \quad \text{in } (\xi,\eta) \text{ a.e.},\\
au(\xi) + bpu'(\xi) = 0,\\
cu(\eta) + dpu'(\eta) = 0,
\end{gathered} \tag{1.1}$$

where $-\infty \le \xi < \eta \le +\infty$, $\pounds u = -(pu')' + qu$ for $u \in \operatorname{dom}(\pounds)$, $1/p, q \in L^1(\xi,\eta)$, $p > 0$ in (ξ,η) a.e., $(a^2+b^2)(c^2+d^2) \neq 0$ and $f : (\xi,\eta)\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function, that is,

(i) $f(t,\cdot,\cdot)$ is continuous for a.e. $t \in (\xi,\eta)$,
(ii) $f(\cdot,u,\mu)$ is measurable for all $u,\mu \in \mathbb{R}$.

In what follows, we let $m : (\xi,\eta) \to [0,+\infty)$ be in $L^1(\xi,\eta)$ such that m is positive on a subset of positive measure, $\alpha,\beta \in L^1(\xi,\eta)$ and $g : (\xi,\eta)\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function. Our first contribution in this work concerns the linear version of (1.1), namely the case where $f(t,u,\mu) = \mu m(t)u$ and (1.1) takes the form

$$\begin{gathered}
\pounds u = \mu m u \quad \text{in } (\xi,\eta) \text{ a.e.},\\
au(\xi) + bpu'(\xi) = 0,\\
cu(\eta) + dpu'(\eta) = 0.
\end{gathered} \tag{1.2}$$

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So far we know, the best result existing in the literature (see [41, Theorem 4.9.1]) states that (1.1) admits an increasing sequence of simple eigenvalues $(\mu_k)_{k\geq 1}$such that $\lim_{k\to\infty}\mu_k = +\infty$ and if ϕ_k is the eigenfunction associated with μ_k and z_k is its number of zeros, then $z_{k+1} = z_k + 1$. Moreover, if $m > 0$ in (ξ, η) a.e., then $z_1 = 0$. We obtain in this work (see Corollary 3.14) that although $m(t) > 0$ a.e. $t \in (\xi', \eta') \subsetneq (\xi, \eta)$, we have always $z_1 = 0$.

In fact Corollary 3.14 is a consequence of Theorem 3.10 which is the second contribution in this work. This result concerns the case where $f(t, u, \mu) = \mu m(t)u+\alpha(t)u^+ - \beta(t)u^-$, and the BVP (1.1) takes the form

$$\begin{gathered}\pounds u = \mu m u + \alpha u^+ - \beta u^- \quad \text{in } (\xi, \eta) \text{ a.e.},\\ au(\xi) + bpu'(\xi) = 0,\\ cu(\eta) + dpu'(\eta) = 0.\end{gathered} \tag{1.3}$$

Note that such a nonlinearity f is positively 1-homogeneous and it is linear on $[0, +\infty)$ and on $(-\infty, 0]$. For this reason, the BVP (1.3) is said to be half-linear and if (μ, u) is a nontrivial solution, we say that μ is a half-eigenvalue of BVP (1.3). Clearly, if $\alpha = \beta = 0$ then BVP (1.3) coincides with the linear eigenvalue BVP (1.2) and this exhibits that the concept of half-eigenvalue generalizes that of eigenvalue. Such types of BVPs have been considered for the first time in [6], where the author introduced the concept of half-eigenvalue. He proved in the case where $-\infty < \xi < \eta < +\infty$, $p \in C^1[\xi, \eta]$, $q, m, \alpha, \beta \in C[\xi, \eta]$ and $m > 0$ in $[\xi, \eta]$, that BVP (1.3) admits two increasing sequences of simple half-eigenvalues $(\mu_k^+)_{k\geq 1}$ and $(\mu_k^-)_{k\geq 1}$. Theorem 3.10 states that the Berestycki's result holds for our more general case. In [9], Binding and Rynne studied existence of half-eigenvalues and their properties for the periodic version of BVP ((1.3). The importance of the concept of half-eigenvalue in the theory of Sturm-Liouville BVPs appears clearly in all existence and multiplicity results (see [9, Theorems 5.1, 5.3, 5.4]).

Our third contribution consists in Theorem 4.3 of Section 4, where is examined the perturbed version of the BVP (1.3),

$$\begin{gathered}\pounds u = \mu m u + u g(t, u) \quad \text{in } (\xi, \eta) \text{ a.e.},\\ au(\xi) + bpu'(\xi) = 0,\\ cu(\eta) + dpu'(\eta) = 0,\end{gathered} \tag{1.4}$$

where $g(t, 0) = 0$, $\lim_{u\to+\infty} g(t, u) = \alpha(t)$, $\lim_{u\to-\infty} g(t, u) = \beta(t)$ a.e. $t \in (\xi, \eta)$. Theorem 4.3 concerns the bifurcation diagram of the BVP (1.4). It describes the asymptotic behavior of the two components ζ_k^+ and ζ_k^- bifurcating from the k^{th}-eigenvalue μ_k of the BVP (1.2). More precisely, it states that each one of the components ζ_k^+ and ζ_k^- rejoins respectively the points (μ_k^+, ∞) and (μ_k^-, ∞) where $(\mu_k^+)_{k\geq 1}$ and $(\mu_k^-)_{k\geq 1}$ are the two sequences of half-eigenvalues of BVP (1.3). Note that if either $\mu_k^\kappa < 1 < \mu_k$, or $\mu_k < 1 < \mu_k^\kappa$ with $\kappa = +$ or $-$, then the BVP

$$\begin{gathered}\pounds u = u\widetilde{g}(t, u) \quad \text{in } (\xi, \eta) \text{ a.e.},\\ au(\xi) + bpu'(\xi) = 0,\\ cu(\eta) + dpu'(\eta) = 0,\end{gathered} \tag{1.5}$$

where $\widetilde{g}(t, u) = m(t) + g(t, u)$, admits a nontrivial solution. Thus, in Section 5, we present situations where this is the case and our contribution consists in Theorem 5.1 and its corollary (Corollary 5.2). In fact, Theorem 5.1 is composed of four assertions and each assertion presents a situation where (1.5) admits nodal solutions. The first two assertions generalize and improve many results existing in the literature and so far we know, the last two ones presents new existence results.

In the last section, we consider the case where $f(t,u,\mu) = g(t,u) - \mu\phi + h$, $\phi, h \in L^1(\xi,\eta)$, and the BVP (1.1) takes the form

$$\begin{gathered} \pounds u = g(t,u) - \mu\phi + h \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ au(\xi) + bpu'(\xi) = 0, \\ cu(\eta) + dpu'(\eta) = 0, \end{gathered} \tag{1.6}$$

Such a class of nonlinearities is known in the literature by jumping nonlinearities, and the particular case of BVP (1.6) having such a nonlinearity

$$\begin{gathered} -u'' = \psi(u) - \mu\sin(t) - h \quad \text{in } (0,\pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.7}$$

where $h \in C[0,\pi]$ and $\int_\xi^\eta h(t)\sin(t)dt = 0$, has been widely investigated in the literature. Denote by $(\lambda_k)_{k\geq 1}$ the sequence of eigenvalues of the BVP

$$\begin{gathered} -u'' = \lambda u \quad \text{in } (0,\pi), \\ u(0) = u(\pi) = 0, \end{gathered}$$

and note that $\sin(t)$ is the eigenfunction associated with the first eigenvalue λ_1. Suppose that $\psi \in C^1(\mathbb{R})$ and set $a_\pm = \lim_{u\to\pm\infty}\psi'(u)$, the first existence result for BVP (1.7) was obtained by Hammerstein in [20], where he proved that if $a_-, a_+ < \lambda_1$ then BVP (1.7) admits at least one solution. Moreover, if $\psi'(u) < \lambda_1$ for all $u \in \mathbb{R}$, then the solution is unique. Dolph extended Hammerstein's result in [18], to the case where $\lambda_k < a_-, a_+ < \lambda_{k+1}$ for some integer $k \geq 1$ and he proved that the solution is unique whenever $\lambda_k < \psi'(u) < \lambda_{k+1}$. The nonlinearity ψ under the hypothesis $a_-, a_+ < \lambda_1$ or $\mu_k < a_-, a_+ < \lambda_{k+1}$ is said to be without jump since there is no eigenvalue in the interval $I = (\min(a_-,a_+), \max(a_-,a_+))$.

The case where I contains exactly one eigenvalue, has been considered for the first time in [2], under the assumptions that $\psi \in C^2(\mathbb{R})$ is convex and $0 < a_- < \lambda_1 < a_+ < \lambda_2$, in which case the authors proved by means of a generalized version of the global inversion theorem to operators having singularities, existence of a manifold Γ in $C[0,\pi]$ such that $C[0,\pi] \setminus \Gamma$ consists of two components Γ_0 and Γ_2, and (1.7) has no solution if $\tilde{h} = \mu\sin(t) + h \in \Gamma_0$, exactly two solutions if $\tilde{h} \in \Gamma_2$, and a unique solution if $\tilde{h} \in \Gamma$. In [32], the authors relaxed the condition $0 < a_- < \lambda_1 < a_+ < \mu_2$ to that $-\infty < a_- < \lambda_1 < a_+ < \mu_2$, and in [8] the authors proved existence of $\bar{\mu}$ such that $\Gamma = \{\tilde{h} = \mu\sin(t) + h : \mu = \bar{\mu}\}$, $\Gamma_0 = \{\tilde{h} = \mu\sin(t) + h : \mu < \bar{\mu}\}$ and $\Gamma_2 = \{\tilde{h} = \mu\sin(t) + h : \mu > \bar{\mu}\}$. Many other extensions of the Ambrosetti-Prodi result are obtained in [1, 3, 11, 17, 22, 38]. The case where I contains more than one eigenvalue is considered in [10, 12, 15, 21, 24, 25, 26, 27, 36, 35, 37, 39]. The best result obtained for the minorant of the number of solutions to BVP (1.7) in the above cited references is: if $\lambda_{j-1} < a_- < \lambda_j < \cdots < \lambda_i < a_+ < \lambda_{i+1}$ for some integers $i, j \geq 1$ with $i \geq 2(j-1)$, then the BVP (1.7) admits $2(i-(j-1))$ nontrivial solutions for μ large.

In this section, we assume that g and $\frac{\partial g}{\partial u}$ are Caratheodory functions and the nonlinearity g has the linear behavior at $\pm\infty$, $\lim_{u\to+\infty} g(t,u)/u = \alpha(t)$, and

$\lim_{u\to-\infty} g(t,u)/u = \beta(t)$ a.e. $t \in (\xi,\eta)$. Our first contribution consists in Theorem 6.1 and its corollary (Corollary 6.3). This theorem provide an existence and uniqueness result of a solution to (1.6) for all $\mu \in \mathbb{R}$ and $\phi, h \in L^1(\xi,\eta)$, and Corollary 6.3 consider the case where the nonlinearity g is a separated variables function and shows that Theorem 6.1 is an extension of Hammerstein's and Dolph's results to the case of Sturm-Liouville BVPs posed in the Caratheodory frame-work. Theorem 6.1 is proved by means of degree theory and eigenvalue properties. The second contribution in this section consists in Theorem 6.7 and its corollary (Corollary 6.11). Theorem 6.7 provides a multiplicity result for BVP (1.6) and Corollary 6.11 consider the case where the nonlinearity g is a separated variables function and shows that Theorem 6.7 recuperates the minorant of the number of solutions to (1.7) obtained in [10, 12, 15, 21, 24, 25, 26, 27, 36, 35, 37, 39] for our general case of Sturm-Liouville BVPs posed in the Caratheodory frame-work.

In the last part of the last section, we present a result (Theorem 6.14) which states that the Ambrosetti-Prodi situation holds for the particular case of BVP (1.7) where the nonlinearity g is a separated variables function; Namely we consider the BVP

$$\begin{gathered} \pounds u = m(t)g_1(u) - \mu\phi + h \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ au(\xi) + bpu'(\xi) = 0, \\ cu(\eta) + dpu'(\eta) = 0, \end{gathered} \tag{1.8}$$

where $g_1 \in C^2(\mathbb{R},\mathbb{R})$ and $\lim_{u\to\pm\infty} g_1'(u) = g_\pm$. We prove by means of a shooting method that if $g_1'' > 0$ and $g_- < \mu_1 < g_+ < \mu_2$ where μ_1 and μ_2 are respectively and the second eigenvalues of (1.2), then there exists μ_* such that (1.8) admits

(a) no solution if $\mu < \mu_*$,
(b) a unique solution if $\theta = \mu_*$, and
(c) exactly two solutions if $\theta > \mu_*$.

The main tool used in this article to obtain multiplicity results, is the global bifurcation theory established by Rabinowitz in [34] on which Dancer gives more precision in [16]. This theory remains a very powerful tool to prove existence and multiplicity results for BVP (1.1), see for example [4, 5, 13, 14, 19, 28, 29, 30, 31, 40].

All the above contributions are presented in Sections 3-6 and Section 2 is devoted to some preliminary results. All these results are not original and we can find in the literature similar utterances, for example the case where $\tau \in \mathbb{R}$ of Theorem 2.2 can be easily found in the literature, although its extension to the case $\tau = \pm\infty$ is easy to prove, we haven't find in the literature a result providing this situation. Also, we met the spirit of Lemmas 2.8 and 2.9 in [6] but these two results are not clearly stated in the above cited wok. For this reason and for sake of completeness, some results in Section Preleminaries are stated and proved in the manner which agree with the spirit of this work. We end this introduction with the following useful lemma:

Lemma 1.1 ([23, Corollary 4.7]). *Let $p \in [1,\infty)$, $f \in L^p(\Omega)$ and (f_n) be a sequence in $L^p(\Omega)$ where Ω is a measurable set in $\mathbb{R}^N$. If $f_n \to f$ a.e. in Ω and $\lim \|f_n\|_p = \|f\|_p$, then $\lim \|f - f_n\|_p = 0$.*

2. Preliminaries

2.1. **Notation.**

$$\begin{gathered}
\Delta_1=\{(\xi,\eta):-\infty\leq\xi<\eta\leq+\infty\}=\overline{\mathbb{R}}\times\overline{\mathbb{R}},\\
\Delta_2=\{(\xi,\eta,p):\rho_1=(\xi,\eta)\in\Delta_1 \text{ and } 1/p\in K_\rho^+\},\\
\Delta_3=\{(\xi,\eta,p,q):\rho_1=(\xi,\eta)\in\Delta_1,\ ;(\xi,\eta,p)\in\Delta_2 \text{ and } q\in L^1_{\rho_1}\},\\
\Delta_4=\{(\xi,\eta,p,0,a,b,c,d):(\xi,\eta,p)\in\Delta_2 \text{ and } (a^2+b^2)(c^2+d^2)\neq 0\},\\
\Delta=\{(\xi,\eta,p,q,a,b,c,d):(\xi,\eta,p,q)\in\Delta_3 \text{ and } (\xi,\eta,p,0,a,b,c,d)\in\Delta_4\}\,.
\end{gathered}$$

For $\rho_1=(\xi,\eta)\in\Delta_1$, we define

$$\begin{gathered}
L^1_{\rho_1}=\big\{m:(\xi,\eta)\to\mathbb{R} \text{ measurable } \int_\xi^\eta|m(s)|ds<\infty\big\},\\
K_{\rho_1}=\{m\in L^1_{\rho_1}:\ m\geq 0 \text{ a.e. in } (\xi,\eta)\},\\
K^*_{\rho_1}=\{m\in K_{\rho_1}:\ m \text{ is positive in a subset of positive measure}\},\\
K^+_{\rho_1}=\{m\in K_{\rho_1}:m>0 \text{ a.e. in } (\xi,\eta)\},\\
C_{\rho_1}=\Big\{u:(\xi,\eta)\to\mathbb{R}:u \text{ is continuous and}\\
\lim_{t\to\xi}u(t),\ \lim_{t\to\eta}u(t) \text{ exist and are finite}\Big\},\\
AC_{\rho_1}=\{u\in C_{\rho_1}:u'\in L^1_{\rho_1}\}.
\end{gathered}$$

For $\rho_2=(\xi,\eta,p)\in\Delta_2$, we define the linear spaces

$$W_{\rho_2}=\{u\in AC_{\rho_1}:u^{[p]}\in C_{\rho_1}\},\quad \tilde{W}_{\rho_2}=\{u\in W_{\rho_2}:u^{[p]}\in AC_{\rho_1}\},$$

where $\rho_1=(\xi,\eta)$ and $u^{[p]}=pu'$ is the quasi-derivative of u. These two spaces, respectively, with the norms

$$\|u\|_1=\sup_{t\in(\xi,\eta)}|u(t)|+\sup_{t\in(\xi,\eta)}|u^{[p]}(t)|,\quad \|u\|_2=\|u\|_1+\int_\xi^\eta|u^{[p]}(t)|dt$$

become Banach spaces.

For the sake of simplicity, we write for $u\in W_{\rho_2}$, $u(+\infty)$, $u^{[p]}(+\infty)$ instead of $\lim_{t\to+\infty}u(t)$, $\lim_{t\to+\infty}u^{[p]}(t)$ when $\eta=+\infty$ and $u(-\infty)$, $u^{[p]}(-\infty)$ instead of $\lim_{t\to-\infty}u(t)$, $\lim_{t\to-\infty}u^{[p]}(t)$ when $\xi=-\infty$. Let $u\in W_{\rho_2}$ and t_0 be such that $\xi\leq t_0\leq\eta$. If $u(t_0)=0$ and $u^{[p]}(t_0)\neq 0$, then t_0 is said to be a simple zero of u.

Throughout this paper, for $\rho_3=(\xi,\eta,p,q)\in\Delta_3$, $\pounds_{\rho_3}$ is the differential operator defined for $u\in\widetilde{W}_{\rho_2}$ where $\rho_2=(\xi,\eta,p)$ by

$$\pounds_{\rho_3}u(x)=-(u^{[p]})'(x)+q(x)u(x).$$

For $\rho_4=(\xi,\eta,p,0,a,b,c,d)\in\Delta_4$, $B^l_{\rho_4}$, $B^r_{\rho_4}$ are the operators given, for $u\in\tilde{W}_{\rho_2}$ where $\rho_2=(\xi,\eta,p)$, by

$$B^l_{\rho_4}u=au(\xi)+bu^{[p]}(\xi),\quad B^r_{\rho_4}u=cu(\eta)+du^{[p]}(\eta),$$

and E_{ρ_4} is the subspace of W_{ρ_2} defined by

$$E_{\rho_4}=\{u\in W_{\rho_2}:B^l_{\rho_4}u=B^r_{\rho_4}u=0\}.$$

For integers $k\geq 1$, $S^{k,+}_{\rho_4}$ denotes the set of functions $u\in E_{\rho_4}$ having exactly $(k-1)$ zeros in (ξ,η), all are simple and u is positive in a right neighbourhood of ξ. It is well known that $S^{k,+}_{\rho_4}$, $S^{k,-}_{\rho_4} = -S^{k,+}_{\rho_4}$ and $S^{k}_{\rho_4} = S^{k,+}_{\rho_4} \cup S^{k,-}_{\rho_4}$ are open sets in E_{ρ_4} and if $u \in \partial S^{k,\kappa}_{\rho_4}$, $(\kappa = +,-)$, then there exists $\tau \in (\xi,\eta)$ such that $u(\tau) = u^{[p]}(\tau) = 0$. For $u \in S^{k}_{\rho_4}$, $(z_j)_{j=0}^{j=k}$ with $\xi = z_0 < z_1 < \dots < z_k = \eta$ and $u(z_j) = 0$ for $j = 1,\dots,k-1$, is said to be the sequence of zeros of u.

For $\rho_1 \in \Delta_1$ and $\kappa = +$ or $-$, let $I^\kappa : C_{\rho_1} \to C_{\rho_1}$ be defined by $I^\kappa u(x) = \max(\kappa u(x), 0)$.

For all $u \in E$, we have

$$u = I^+u - I^-u, \quad |u| = I^+u + I^-u.$$

This implies that, for all $u, v \in E$,

$$\begin{aligned} |I^+u - I^+v| &\le \frac{|u-v|}{2} + \frac{||u|-|v||}{2} \le |u-v|,\\ |I^-u - I^-v| &\le \frac{|u-v|}{2} + \frac{||u|-|v||}{2} \le |u-v|, \end{aligned} \tag{2.1}$$

and the operators I^+, I^- are continuous.

Remark 2.1. Throughout this paper, when there is no confusion, we write for $\rho = (\xi,\eta,p,q,a,b,c,d) \in \Delta$, L^1_ρ, K_ρ, K^*_ρ, K^+_ρ, C_ρ, AC_ρ, W_ρ, $\tilde{W}_\rho$, E_ρ, $S^{k,+}_\rho$, $S^{k,-}_\rho$, S^k_ρ, $\pounds_\rho$, B^l_ρ, B^r_ρ instead of $L^1_{\rho_1}$, K_{ρ_1}, $K^*_{\rho_1}$, $K^+_{\rho_1}$, C_{ρ_1}, AC_{ρ_1}, W_{ρ_2}, $\tilde{W}_{\rho_2}$, $\pounds_{\rho_3}$, $B^l_{\rho_4}$, $B^r_{\rho_4}$, E_{ρ_4}, $S^{k,+}_{\rho_4}$, $S^{k,-}_{\rho_4}$, $S^k_{\rho_4}$, where for $i \in \{1,2,3,4\}$, ρ_i is the projection of ρ onto Δ_i.

2.2. **Initial value problem.** In this subsection we let $\rho_3 = (\xi,\eta,p,q) \in \Delta_3$, $\rho_1 = (\xi,\eta)$, $\rho_2 = (\xi,\eta,p)$, $\gamma,\delta \in \mathbb{R}$ and τ is such that $\xi \le \tau \le \eta$. Consider the initial value problem (IVP for short);

$$\begin{gathered} \pounds_{\rho_3} u = f(t,u),\\ u(\tau) = \gamma,\\ u^{[p]}(\tau) = \delta, \end{gathered} \tag{2.2}$$

where $f : (\xi,\eta) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function; that is,

(1) $f(\cdot,u)$ is measurable for all $u \in \mathbb{R}$,
(2) $f(t,\cdot)$ is continuous for a.e. $t \in (\xi,\eta)$.

Suppose that

$$f(\cdot,0) \in L^1_{\rho_1}. \tag{2.3}$$

By a solution to (2.2), we mean a function $\phi \in \tilde{W}_{\rho_2}$ such that $\pounds_{\rho_3}\phi = f(t,\phi)$ and $\phi(\tau) = \gamma$, $\phi^{[p]}(\tau) = \delta$.

Theorem 2.2. *Assume that Hypothesis* (2.3) *holds and there exists $\psi \in L^1_{\rho_1}$ such that for all $x, y \in \mathbb{R}$ and a.e. $t \in (\xi,\eta)$,*

$$|f(t,x) - f(t,y)| \le \psi(t)|x-y|.$$

Then (2.2) *admits a unique solution.*

Proof. Clearly, u is a solution to (2.2) if and only if $(u, u^{[p]})$ is a solution to the first-order IVP

$$\begin{gathered} U' = F(t,U)\\ U(\tau) = (\gamma,\delta) \end{gathered} \tag{2.4}$$

where for $U = (u,v)$ and $t \in (\xi,\eta)$, $F(t,U) = \big(\frac{v}{p(t)}, q(t)u - f(t,u)\big)$.

Let $\kappa > 1$ and $X = C_{\rho_1} \times C_{\rho_1}$ be equipped with the norm,

$$\|(u,v)\|_\kappa = \sup_{t\in(\xi,\eta)} \Big(\exp(-\kappa\Big|\int_\tau^t \omega(r)dr\Big|)(|u(t)| + |v(t)|)\Big)$$

where $\omega = |q| + \psi + \frac{1}{p}$. Note that the norm $\|\cdot\|_\kappa$ is equivalent to the norm $\|\cdot\|_\infty$ defined for $(u,v)\in X$ by $\|(u,v)\|_\infty = \sup_{t\in(\xi,\eta)} |u(t)| + \sup_{t\in(\xi,\eta)} |v(t)|$.

At this stage, we have that $U = (u,v) \in X$ is a solution to (2.4) if and only if $U(t) = TU(t)$ where $TU(t) = (\gamma,\delta) + \int_\tau^t F(s,U(s))ds$. Since

$$\begin{aligned}|F(s,U(s))| &\le |F(s,U(s)) - F(s,0)| + |F(s,0)|\\ &\le \frac{1}{p(s)}|v(s)| + (|q(s)| + \psi(s))|u(s)| + |f(s,0)|\end{aligned}$$

the operator $T : X \to X$ is well defined. Therefore, it suffices to prove that T is a contraction.

To this aim let $U_1 = (u_1,v_1), U_2 = (u_2,v_2) \in X$, we have

$$\begin{aligned}&|F(s,U_1(s)) - F(s,U_2(s))|\\ &\le \frac{|v_1(s) - v_2(s)|}{p(s)} + (|q(s)| + \psi(s))|u_1(s) - u_2(s)|\\ &\le \omega(s)|U_1(s) - U_2(s)|\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}S(t) &= \exp(-\kappa|\int_\tau^t \omega(r)dr|)|TU_1(t) - TU_2(t)|\\ &= |\int_\tau^t e^{-\kappa|\int_s^t \omega(r)dr|}(F(s,U_1(s)) - F(s,U_2(s)))e^{-\kappa|\int_\tau^s \omega(r)dr|}ds|.\end{aligned} \tag{2.6}$$

Hence, we obtain from (2.5) and (2.6) that if $t > \tau$, then

$$\begin{aligned}S(t) &\le \int_\tau^t e^{-\kappa\int_s^t \omega(r)dr}|F(s,U_1(s)) - F(s,U_2(s))|e^{-\kappa\int_\tau^s \omega(r)dr}ds\\ &\le \int_\tau^t e^{-\kappa\int_s^t \omega(r)dr}\omega(s)|U_1(s) - U_2(s)|e^{-\kappa\int_\tau^s \omega(r)dr}ds\\ &\le (\int_\tau^t e^{-\kappa\int_s^t \omega(r)dr}\omega(s)ds)\|U_1 - U_2\|_\kappa\\ &\le \frac{1}{\kappa}\|U_1 - U_2\|_\kappa\end{aligned}$$

and if $t < \tau$, then

$$\begin{aligned}S(t) &\le \int_t^\tau e^{-\kappa\int_t^s \omega(r)dr}|F(s,U_1(s)) - F(s,U_2(s))|e^{-\kappa\int_s^\tau \omega(r)dr}ds\\ &\le \int_t^\tau e^{-\kappa\int_t^s \omega(r)dr}\omega(s)|U_1(s) - U_2(s)|e^{-\kappa\int_s^\tau \omega(r)dr}ds\\ &\le \Big(\int_t^\tau \omega(s)e^{-\kappa\int_t^s \omega(r)dr}ds\Big)\|U_1 - U_2\|_\kappa\\ &\le \frac{1}{\kappa}\|U_1 - U_2\|_\kappa.\end{aligned}$$

The above estimates on $S(t)$ lead to $\|TU_1 - TU_2\|_\kappa \le \frac{1}{\kappa}\|U_1 - U_2\|_\kappa$ and (2.2) admits a unique solution, thus completing the proof. ☐

The following corollary is obtained from Theorem 2.2 and is an extension of [41, Theorem 2.2.1] to the case where τ can be infinite.

Corollary 2.3. *For all $\rho_3 = (\xi, \eta, p, q) \in \Delta_3$, $\gamma, \delta \in \mathbb{R}$ and $\xi \le \tau \le \eta$ and $f \in L^1_{\rho_1}$ with $\rho_1 = (\xi, \eta)$, the IVP*

$$\begin{gathered} \pounds_{\rho_3} u = f, \\ u(\tau) = \gamma, \\ u^{[p]}(\tau) = \delta, \end{gathered}$$

admits a unique solution.

Now consider the IVP

$$\begin{gathered} \pounds_{\rho_3} u = u g(t, u), \\ u(\tau) = 0, \\ u^{[p]}(\tau) = 0, \end{gathered} \tag{2.7}$$

where $g : (\xi, \eta) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

Corollary 2.4. *Assume that*

$$|g(t, u)| \le \psi(t) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } t \in (\xi, \eta)$$

for some $\psi \in L^1_{\rho_1}$. Then the trivial function is the unique solution for (2.7).

Proof. Indeed, if (λ, u) is a solution to (2.7) then u is a solution of the IVP

$$\begin{gathered} -(pv')' + (q + q_u)v = 0, \\ v(\tau) = 0, \\ v^{[p]}(\tau) = 0, \end{gathered}$$

where $q_u(t) = -g(t, u(t))$. Since the hypothesis in Corollary 2.4 guarantees that $q_u \in L^1_{\rho_1}$, we have from Corollary 2.3 that u is the unique solution of (2.7). ☐

2.3. **Comparison results.**

Definition 2.5. *Let $\rho_2 = (\xi, \eta, p) \in \Delta_2$ and $u, v \in W_{\rho_2}$. The function $Wr(u, v) = uv^{[p]} - u^{[p]}v$ is called the Wronksian of u, v.*

It is easy to prove the following lemma.

Lemma 2.6. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and $u, v \in W_\rho$. We have*

- [i] *If $B^l_\rho u = B^l_\rho v = 0$, then $Wr(u, v)(\xi) = 0$;*
- (ii) *If $B^r_\rho u = B^r_\rho v = 0$, then $Wr(u, v)(\eta) = 0$;*
- (iii) *If $Wr(u, v)(t_0) \neq 0$ for some $t_0 \in (\xi, \eta)$ and $\pounds_\rho u = \pounds_\rho v = 0$, then $\{u, v\}$ form a basis of the space of solutions to the differential equation $\pounds_\rho w = 0$.*

The proof of the following lemma is similar to that of [6, Lemma 2], so it is omitted.

Lemma 2.7. *Let j and k be two integers such that $j \geq k \geq 2$. Suppose that there exist two families of real numbers*

$$\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \eta,$$
$$\eta_0 = \xi < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \eta.$$

Then, if $\xi_1 < \eta_1$, there exist two integers m and n having the same parity, $1 \leq m \leq k-1$ and $1 \leq n \leq j-1$ such that

$$\xi_m < \eta_n \leq \eta_{n+1} \leq \xi_{m+1}.$$

Lemma 2.8. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and let for $i = 1, 2$, $\phi_i \in S_\rho^{k_i,\kappa}$ having a sequence of zeros $(z_j^i)_{j=0}^{j=k_i}$. If for some integers m, n with $m \leq k_1-1$ and $n \leq k_2-1$ we have $\phi_1\phi_2 > 0$ and $z_m^1 \leq z_n^2 < z_{n+1}^2 \leq z_{m+1}^1$, then $\int_{z_n^2}^{z_{n+1}^2} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \geq 0$. Moreover, $\int_{z_n^2}^{z_{n+1}^2} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 = 0$ if and only if $z_m^1 = z_n^2 < z_{n+1}^2 = z_{m+1}^1$.*

Proof. Without loss of generality, suppose that $\phi_1, \phi_2 > 0$ in (z_n^2, z_{n+1}^2) and let Wr be the Wronksian of ϕ_1 and ϕ_2. Set $I = \int_{z_n^2}^{z_{n+1}^2} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1$ and note that $I = Wr(z_n^2) - Wr(z_{n+1}^2)$.

We distinguish four cases:

(i) $\xi = z_n^2 < z_{n+1}^2 = \eta$: In this case we have $I = Wr(\xi) - Wr(\eta) = 0$.

(ii) $\xi = z_n^2 < z_{n+1}^2 < \eta$: In this case we have $Wr(\xi) = 0$, $\phi_1(z_{n+1}^2) \geq 0$, $\phi_2(z_{n+1}^2) = 0$, $\phi_2^{[p]}(z_{n+1}^2) < 0$, leading to
$$I = -Wr(z_{n+1}^2) = -\phi_1(z_{n+1}^2)\phi_2^{[p]}(z_{n+1}^2) \geq 0.$$
Clearly, if $I = 0$ then $\phi_1(z_{n+1}^2) = 0$ and $z_{m+1}^1 = z_{n+1}^2$.

(iii) $\xi < z_n^2 < z_{n+1}^2 = \eta$: In this case we have $Wr(\eta) = 0$, $\phi_1(z_n^2) \geq 0$, $\phi_2(z_n^2) = 0$, $\phi_2^{[p]}(z_n^2) > 0$, leading to $I = Wr(z_n^2) = \phi_1(z_n^2)\phi_2^{[p]}(z_n^2) \geq 0$. Clearly, if $I = 0$ then $\phi_1(z_n^2) = 0$, proving that $z_m^1 = z_n^2$.

(iv) $\xi < z_n^2 < z_{n+1}^2 < \eta$: In this case we have $\phi_1(z_n^2) \geq 0$, $\phi_1(z_{n+1}^2) \geq 0$, $\phi_2(z_n^2) = 0$, $\phi_2(z_{n+1}^2) = 0$, $\phi_2^{[p]}(z_n^2) > 0$, $\phi_2^{[p]}(z_{n+1}^2) < 0$ (see Figure 1), leading to $I = \phi_1(z_n^2)\phi_2^{[p]}(z_n^2) - \phi_1(z_{n+1}^2)\phi_2^{[p]}(z_{n+1}^2) \geq 0$. Clearly, if $I = 0$ then $\phi_1(z_n^2) = \phi_1(z_{n+1}^2) = 0$, proving that $z_m^1 = z_n^2$ and $z_{m+1}^1 = z_{n+1}^2$. ∎

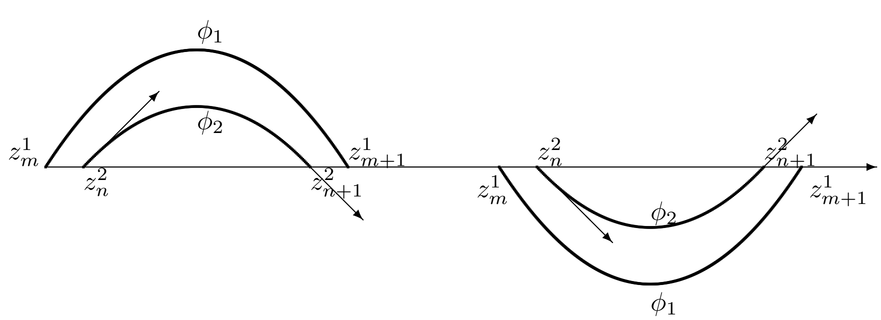

Figure 1. Bumps

Lemma 2.9. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and let ϕ_1, ϕ_2 be respectively two functions in $S_\rho^{k,\kappa} \cap \tilde{W}_\rho$. Then, there exist two intervals (ξ_1, η_1) and (ξ_2, η_2) such that $\phi_1\phi_2 > 0$ in (ξ_1, η_1) and in (ξ_2, η_2). Moreover,*

$$\int_{\xi_1}^{\eta_1} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \geq 0, \quad \int_{\xi_2}^{\eta_2} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \leq 0.$$

Proof. Without loss of generality, suppose that $\kappa = +$ and let for $i = 1, 2$, $(z_j^i)_{j=0}^{j=k}$ the sequence of zeros of ϕ_i. Since the case $k = 1$ is obvious, we suppose that $k \geq 2$. We distinguish two cases

(i) $z_1^1 = z_1^2$: In this case let $\theta = \inf(z_2^1, z_2^2)$. From Lemma 2.8, we have

$$\int_{\xi}^{z_1^1} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 = 0, \quad \int_{z_1^1}^{\theta} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \begin{cases} \geq 0 & \text{if } \theta = z_2^2 \\ \leq 0 & \text{if } \theta = z_2^1. \end{cases}$$

Thus, if $\theta = z_2^1$, we take $(\xi_1, \eta_1) = (\xi, z_1^1)$, $(\xi_2, \eta_2) = (z_1^1, z_2^1)$ and if $\theta = z_2^2$, we take $(\xi_1, \eta_1) = (\xi, z_1^1)$, $(\xi_2, \eta_2) = (z_2^1, z_2^2)$.

(ii) $z_1^2 < z_1^1$, (the case $z_1^1 < z_1^2$ is checked similarly): In this case Lemma 2.7 guarantees existence of two integers $m, n \geq 1$ having the same parity such that $z_m^2 < z_n^1 < z_{n+1}^1 \leq z_{m+1}^2$. Thus, we take $(\xi_1, \eta_1) = (\xi, z_1^2)$ and $(\xi_2, \eta_2) = (z_n^1, z_{n+1}^1)$ and we have from Lemma 2.8,

$$\int_{\xi_1}^{\eta_1} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \geq 0, \quad \int_{\xi_2}^{\eta_2} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \leq 0.$$

This completes the proof. □

Lemma 2.10 ([6]). *Let $\rho \in \Delta$ and let $w_1,\ w_2$ be two functions in $\tilde{W}_\rho$ and assume that w_2 does not vanish identically and $\pounds_\rho w_1 = m_1 w_1$ and $\pounds_\rho w_2 = m_2 w_2$ where $m_1,\ m_2 \in L_\rho^1$ are such that $(m_1 - m_2) \in K_\rho^*$. Suppose that either*

(1) *$w_2(\xi) = w_2(\eta) = 0$, or*
(2) *for $i = 1, 2$ $B_\rho^l w_i = 0$ and $w_2(\eta) = 0$, or*
(3) *for $i = 1, 2$ $B_\rho^r w_i = 0$ and $w_2(\xi) = 0$,or*
(4) *for $i = 1, 2$ $B_\rho^l w_i = 0$ and $B_\rho^r w_i = 0$.*

Then there exists $\tau \in (\xi, \eta)$ such that $w_1(\tau) = 0$.

2.4. **Green's function.** For $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ let Φ_ρ and Ψ_ρ be respectively the solutions obtained from Theorem 2.3 to the equations

$$\begin{aligned} \pounds_\rho u &= 0 & \pounds_\rho u &= 0 \\ u(\xi) &= b, & u(\eta) &= d, \\ u^{[p]}(\xi) &= -a, & u^{[p]}(\eta) &= -c, \end{aligned}$$

and $Wr_\rho = Wr(\Phi_\rho, \Psi_\rho)$. Note that because $W'r_\rho = 0$, we have $Wr_\rho(t) = Wr(\Phi_\rho, \Psi_\rho)(\xi)$ for all $t \in (\xi, \eta)$.

Theorem 2.11. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and assume that the trivial function 0 is the unique solution to the BVP*

$$\begin{gathered} \pounds_\rho u = 0 \quad \text{a.e. in } (\xi, \eta), \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{2.8}$$

Then, there exists a unique function $G_\rho : (\xi, \eta) \times (\xi, \eta) \to \mathbb{R}$ such that

(1) *G_ρ is uniformly continuous, bounded and symmetric.*
(2) *For $s_0 \in (\xi,\eta)$ fixed, the function $H_0(t) = G_\rho(t,s_0)$ satisfies the differential equation (2.8) in each of intervals (ξ,s_0) and (s_0,η) and the boundary conditions in (2.8).*
(3) *For $s_0 \in (\xi,\eta)$ fixed, $G_\rho^{[p]}(s_0^+,s_0)$, $G_\rho^{[p]}(s_0^-,s_0)$ exist and we have*
$$G_\rho^{[p]}(s_0^+,s_0) - G_\rho^{[p]}(s_0^-,s_0) = 1.$$
(4) *Moreover, for all $f \in L^1_\rho$, $u \in \tilde{W}_\rho$ is a solution to*
$$\pounds_\rho u = f \ \text{a.e. in } (\xi,\eta),$$
$$B^l_\rho u = B^r_\rho u = 0,$$
if and only if $u(t) = \int_\xi^\eta G_\rho(t,s)f(s)ds = L_\rho f(t)$.
(5) *The operator $L_\rho : L^1_\rho \to C_\rho$ is compact.*

Proof. The function
$$G_\rho(t,s) = \frac{1}{Wr_\rho}\begin{cases} \Phi_\rho(s)\Psi_\rho(t) & \text{if } s \le t \\ \Phi_\rho(t)\Psi_\rho(s) & \text{if } t \le s \end{cases}$$
is what we are seeking, where $Wr_\rho = Wr(\Phi_\rho,\Psi_\rho) = Wr(\Phi_\rho,\Psi_\rho)(\xi)$.

Since $q, 1/p \in L^1_\rho$, from [41, Theorem 2.3.1] we have that the functions, Φ_ρ, Ψ_ρ, $\Phi_\rho^{[p]}$, $\Psi_\rho^{[p]}$ are bounded by a constant $M > 0$. Therefore, for $t_1, t_2 \in (\xi,\eta)$ we have
$$|\Phi_\rho(t_2) - \Phi_\rho(t_1)| \le M\Big|\int_{t_1}^{t_2}\frac{ds}{p(s)}\Big|, \quad |\Psi_\rho(t_2) - \Psi_\rho(t_1)| \le M|\int_{t_1}^{t_2}\frac{ds}{p(s)}|,$$
proving that Φ_ρ, Ψ_ρ are uniformly continuous. Then G_ρ is uniformly continuous on $(\xi,\eta)\times(\xi,\eta)$. Clearly, the function G_ρ satisfies Properties 1, 2, 3, and Property 4 is proved by the method of variation of constants.

At the end, note that $L_\rho = i_\rho \circ \widetilde{L}_\rho$, where $\widetilde{L}_\rho : L^1_\rho \to W_\rho$ with $\widetilde{L}_\rho u = L_\rho u$ for all $u \in L^1_\rho$, is continuous and i_ρ is the continuous embedding of W_ρ in C_ρ. Because the estimate
$$|u(t_2) - u(t_1)| \le \Big|\int_{t_1}^{t_2}\frac{ds}{p(s)}\Big|\|u\|_1$$
holds for all $u \in W_\rho$ and t_1, t_2 with $\xi \le t_1 < t_2 \le \eta$, the embedding i_ρ is compact, and then L_ρ is compact. □

Lemma 2.12. *Assume that $Wr_\rho \ne 0$, for some $\rho = (\xi,\eta,p,q,a,b,c,d) \in \Delta$, and let for $\theta \in (\xi,\eta)$, $\rho_l(\theta) = (\xi,\theta,p,q,a,b,1,0)$ and $\rho_r(\theta) = (\theta,\eta,p,q,a,b,1,0)$.*
(i) *If $\Phi_\rho(\theta) \ne 0$ for all $\theta \in (\xi,\eta)$, then for all $\theta \in (\xi,\eta)$, $G_{\rho_l(\theta)}$ exists and we have $G_{\rho_l(\theta)}(t,s) = G_\rho(t,s) - (\Psi_\rho(\theta)/Wr_\rho\Phi_\rho(\theta))\Phi_\rho(t)\Phi_\rho(s)$.*
(ii) *If $\Psi_\rho(\theta) \ne 0$ for all $\theta \in (\xi,\eta)$, then for all $\theta \in (\xi,\eta)$, $G_{\rho_r(\theta)}$ exists and we have $G_{\rho_r(\theta)}(t,s) = G_\rho(t,s) - (\Phi_\rho(\theta)/Wr_\rho\Psi_\rho(\theta))\Psi_\rho(t)\Psi_\rho(s)$.*

Proof. We need to prove that $\Phi_\rho(\theta) \ne 0$ for all $\theta \in (\xi,\eta)$.

(i) Let $\Phi_{\rho_l(\theta)}(t) = \Phi_\rho(t)$ and $\Psi_{\rho_l(\theta)}(t) = -(\Psi_\rho(\theta)/\Phi_\rho(\theta))\Phi_\rho(t) + \Psi_\rho(t)$. Then Φ_θ, Ψ_θ are respectively the unique solutions to
$$\begin{aligned} \pounds_{\rho_l(\theta)}u &= 0, & \pounds_{\rho_l(\theta)}u &= 0,\\ u(\xi) &= b, & u(\theta) &= 0,\\ u^{[p]}(\xi) &= -a, & u^{[p]}(\theta) &= Wr_\rho/\Phi_\rho(\theta), \end{aligned}$$

and for all $\theta \in (\xi, \eta)$, we have $Wr_{\rho_l(\theta)} = Wr_\rho \neq 0$ and

$$\begin{aligned} G_{\rho_l(\theta)}(t,s) &= \frac{1}{Wr_{\rho_l(\theta)}} \times \begin{cases} \Phi_{\rho_l(\theta)}(s)\Psi_{\rho_l(\theta)}(t) & \text{if } s \le t \\ \Phi_{\rho_l(\theta)}(t)\Psi_{\rho_l(\theta)}(s) & \text{if } t \le s \end{cases} \\ &= G_\rho(t,s) - (\Psi_\rho(\theta)/Wr_\rho\Phi_\rho(\theta))\Phi_\rho(t)\Phi_\rho(s). \end{aligned}$$

(ii) Let $\Phi_{\rho_r(\theta)}$ and $\Psi_{\rho_r(\theta)}$ be defined by $\Phi_{\rho_l(\theta)}(t) = \Phi_\rho(t) - (\Psi_\rho(\theta)/\Phi_\rho(\theta))\Phi_\rho(t)$ and $\Psi_{\rho_r(\theta)}(t) = \Psi_\rho(t)$. Then, $\Phi_{\rho_r(\theta)}, \Psi_{\rho_r(\theta)}$ are respectively the unique solutions of

$$\begin{aligned} \pounds_{\rho_r(\theta)}u &= 0, & \pounds_{\rho_r(\theta)}u &= 0, \\ u(\theta) &= 0, & u(\eta) &= d, \\ u^{[p]}(\theta) &= Wr_\rho/\Psi_\rho(\theta), & u^{[p]}(\eta) &= -c, \end{aligned}$$

and we have for all $\theta \in (\xi, \eta)$, $Wr_{\rho_r(\theta)} = Wr_\rho \neq 0$ and

$$\begin{aligned} G_{\rho_r(\theta)}(t,s) &= \frac{1}{Wr_{\rho_r(\theta)}} \times \begin{cases} \Phi_{\rho_r(\theta)}(s)\Psi_{\rho_r(\theta)}(t) & \text{if } s \le t \\ \Phi_{\rho_r(\theta)}(t)\Psi_{\rho_r(\theta)}(s) & \text{if } t \le s \end{cases} \\ &= G_\rho(t,s) - (\Phi_\rho(\theta)/Wr_\rho\Psi_\rho(\theta))\Psi_\rho(t)\Psi_\rho(s). \end{aligned}$$

□

2.5. **Linear eigenvalue problem.** For $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and $m \in K_\rho^*$, consider the eigenvalue problem

$$\begin{gathered} \pounds_\rho u = \mu m u \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r = 0. \end{gathered} \tag{2.9}$$

Theorem 2.13 ([41, Theorem 4.9.1]). *For $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and $m \in K_\rho^+$, BVP (2.9) admits an increasing sequences of eigenvalues $(\mu_k(\rho, m))_{k \ge 1}$ such that*

1. *$\lim \mu_k(\rho, m) = +\infty$,*
2. *$\mu_k(\rho, m)$ is simple,*
3. *If ϕ_k is an eigenvalue associated with $\mu_k(\rho, m)$, then $\phi_k \in S_\rho^k$.*

In what follows, we present some important properties of eigenvalues needed for the proofs of the main results of this paper.

Lemma 2.14. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m_1, m_2 \in K_\rho^+$ and assume that $m_1 \le m_2$ a.e. in (ξ, η) and $m_1 < m_2$ in a subset of positive measure. If for some integer $k \ge 1$, either $\mu_k(\rho, m_1) \ge 0$ or $\mu_k(\rho, m_2) \ge 0$, then $\mu_k(\rho, m_1) > \mu_k(\rho, m_2) \ge 0$.*

Proof. For $i = 1, 2$, set $\mu_i = \mu_k(\rho, m_i)$ and let ϕ_i be the eigenfunction associated with μ_i having a sequence of zeros $(z_j^i)_{j=0}^{j=k}$. First, we claim that there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$. Indeed, assume that $\phi_1(z_j^2) = 0$ for all $j \in \{1, \dots, k-1\}$ and $\mu_1 < \mu_2$ and note that there exists $j_1 \in \{1, \dots, k-1\}$ such that $\operatorname{meas}(\{m_2 > m_1\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$ and $\phi_1\phi_2 > 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$. Applying Lemma 2.10, we get that there exists $\tau \in (z_{j_1}^2, z_{j_1+1}^2)$ such that $\phi_1(\tau) = 0$ and this contradicts $\phi_1 \in S_\rho^{k,\kappa}$.

Now, let $k_1 = \max\{l \le k : z_j^1 = z_j^2 \text{ for all } j \le l\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$. Then we distinguish two cases.

(i) $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case we have

$$\begin{aligned} 0 &< \int_{\eta_0}^{\eta_1} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = \int_{\eta_0}^{\eta_1} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\ &= \int_{\eta_0}^{\eta_1} (\mu_1 - \mu_2) m_1 \phi_1\phi_2 + \int_{\eta_0}^{\eta_1} \mu_2 (m_1 - m_2)\phi_1\phi_2 \\ &= \int_{\eta_0}^{\eta_1} \mu_1 (m_1 - m_2)\phi_1\phi_2 + \int_{\eta_0}^{\eta_1} (\mu_1 - \mu_2) m_2 \phi_1\phi_2 \end{aligned}$$

and this proves that in both the cases $\mu_1 \geq 0$ or $\mu_2 \geq 0$, we have $\mu_1 > \mu_2$.

(ii) $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case Lemma 2.7 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z^1_{k_1+m} < \eta_n = z^2_{k_1+n} < \eta_{n+1} = z^2_{k_1+n+1} \leq \xi_{m+1} = z^1_{k_1+m+1}.$$

As above, we have

$$\begin{aligned} 0 &< \int_{\eta_n}^{\eta_{n+1}} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = \int_{\eta_n}^{\eta_{n+1}} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\ &= \int_{\eta_n}^{\eta_{n+1}} (\mu_1 - \mu_2) m_1 \phi_1\phi_2 + \int_{\eta_n}^{\eta_{n+1}} \mu_2 (m_1 - m_2)\phi_1\phi_2 \\ &= \int_{\eta_n}^{\eta_{n+1}} \mu_1 (m_1 - m_2)\phi_1\phi_2 + \int_{\eta_n}^{\eta_{n+1}} (\mu_1 - \mu_2) m_2 \phi_1\phi_2 \end{aligned}$$

and this proves that in both the cases $\mu_1 \geq 0$ or $\mu_2 \geq 0$, we have $\mu_1 > \mu_2$. This completes the proof. □

Lemma 2.15. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K^+_\rho$ and $\gamma, \delta \in \mathbb{R}$ with $\xi < \gamma < \delta < \eta$. Then for all integers $k \geq 1$, $\mu_k(\rho, m) \leq \mu_k(\overline{\rho}, m)$ where $\overline{\rho} = (\gamma, \delta, p, q, 1, 0, 1, 0)$.*

Proof. Fix $k \geq 1$ and set $\mu_1 = \mu_k(\rho, m)$ and $\mu_2 = \mu_k(\overline{\rho}, m)$. For $i = 1, 2$, let ϕ_i be an eigenfunction associated with μ_i, having a sequence of zeros $(z^i_j)^{j=k}_{j=0}$, and without loss of generality, suppose that $\phi_1\phi_2 > 0$ in a right neighborhood of γ. We distinguish two cases.

(i) $\phi_1 > 0$ in (γ, δ): In this case we have

$$\begin{aligned} 0 &\leq -\phi_1(\delta)\phi_2^{[p]}(\delta) + \phi_1(\gamma)\phi_2^{[p]}(\gamma) = \int_\gamma^\delta \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \\ &= (\mu_2 - \mu_1) \int_\gamma^\delta m\phi_1\phi_2 \end{aligned}$$

leading to $\mu_2 \geq \mu_1$.

(ii) $\phi_1(t_0) = 0$ for some $t_0 \in (\gamma, \delta)$: In this case consider the family $(\xi_j)^{j=k_0}_{j=0}$ defined by $\xi_0 = \gamma$, $\xi_{k_0} = \delta$ and $\phi_1(\xi_j) = 0$ for $j \in \{1, \dots, k_0 - 1\}$ and note that $k_0 \leq k$. Thus, from Lemma 2.7 there exist two integers m, n having the same parity, such that $\xi_m < z^2_n < z^2_{n+1} \leq \xi_{m+1}$. Therefore, we have $\phi_1, \phi_2 > 0$ in (z^2_n, z^2_{n+1}) and

$$\begin{aligned} 0 &\leq -\phi_1(z^2_{n+1})\phi_2^{[p]}(z^2_{n+1}) + \phi_1(z^2_n)\phi_2^{[p]}(z^2_n) \\ &= \int_{z^2_n}^{z^2_{n+1}} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 \end{aligned}$$

$$= (\mu_2 - \mu_1) \int_{z_n^2}^{z_{n+1}^2} m\phi_1\phi_2$$

leading to $\mu_2 \geq \mu_1$. This completes the proof. □

Lemma 2.16. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and $m \in K_\rho^+$ and set for all $\theta \in (\xi, \eta)$, $\rho_r(\theta) = (\theta, \eta, p, q, 1, 0, c, d)$ (resp. $\rho_l(\theta) = (\xi, \theta, p, q, a, b, 1, 0)$). Then, the mapping $\theta \to \mu_1(\rho_r(\theta), m)$ is continuous increasing on (ξ, η) (resp. $\theta \to \mu_1(\rho_l(\theta), m)$ is continuous decreasing on (ξ, η)), and we have $\lim_{\theta\to\eta} \mu_1(\rho_r(\theta), m) = +\infty$ (resp. $\lim_{\theta\to\xi} \mu_1(\rho_l(\theta), m) = +\infty$).*

Proof. The continuity of the mapping $\theta \to \mu_1(\rho_r(\theta), m)$ follows from [41, Theorem 4.4.1]. Let θ_1, θ_2 be such that $\xi \leq \theta_1 < \theta_2 < \eta$ and let for $i = 1, 2$, ϕ_i be the eigenvector corresponding to the eigenvalue $\mu_i = \mu_1(\rho_r(\theta_i), m)$. Taking into consideration $\phi_2(\theta_2) = 0$ and $Wr(\phi_1, \phi_2)(\eta) = 0$, from simple computations,

$$(\mu_2 - \mu_1) \int_{\theta_2}^{\eta} m\phi_1\phi_2 = \int_{\theta_2}^{\eta} \phi_1 \pounds_{\rho_r(\theta_2)}\phi_2 - \phi_2 \pounds_{\rho_r(\theta_1)}\phi_1 = \phi_1(\theta_2)\phi_2^{[p]}(\theta_2) > 0,$$

thus proving that $\mu_2 > \mu_1$.

Now, we understand from Theorem 2.13 that there exists $\overline{\mu} > 0$ such that $\mu_*(\rho) = \mu_1(\rho, m) + \overline{\mu} > 0$ and this, together with $\theta \to \mu_1(\rho_r(\theta), m)$ is increasing, leads to

$$\mu_*(\theta) = \mu_1(\rho_r(\theta), m) + \overline{\mu} = \mu_1(\widetilde{\rho_r}(\theta), m) \geq \mu_*(\rho) = \mu_1(\widetilde{\rho}, m) > 0$$

where $\widetilde{\rho_r}(\theta) = (\theta, \eta, p, q + \overline{\mu}m, 1, 0, c, d)$ and $\widetilde{\rho} = (\xi, \eta, p, q + \overline{\mu}m, 1, 0, c, d)$.

To prove $\lim_{\theta\to\eta} \mu_1(\rho_r(\theta), m) = +\infty$, we need to prove the existence of a positive constant $M(d)$ such that $\sup_{t\in(\theta,\eta)}(\Psi_{\widetilde{\rho}}(t)/\Psi_{\widetilde{\rho}}(\theta)) \leq M(d)$. Note that $\Psi_{\widetilde{\rho}}(t) \neq 0$ for all $t \in (\xi, \eta)$; indeed, if $\Psi_{\widetilde{\rho}}(t_0) = 0$ for some $t_0 \in (\xi, \eta)$, then there exists an integer $k_0 \geq 1$ such that $\Psi_{\widetilde{\rho}}$ will be an eigenfunction associated with $\mu_{k_0}(\widetilde{\rho_r}(t_0), m) = 0$ and yields the contradiction

$$0 = \mu_{k_0}(\widetilde{\rho_r}(t_0), m) \geq \mu_1(\widetilde{\rho_r}(t_0), m) = \mu_*(t_0) > 0.$$

Without loss of generality, suppose that $\Psi_{\widetilde{\rho}} > 0$ in (ξ, η) and note then that $d \geq 0$. We distinguish two cases:

(i) $d > 0$: In this case we have $\inf_{t\in(\xi,\eta)} \Psi_{\widetilde{\rho}}(t) > 0$ and

$$\sup_{t\in(\theta,\eta)} (\Psi_{\widetilde{\rho}}(t)/\Psi_{\widetilde{\rho}}(t)) \leq \|\Psi_{\widetilde{\rho}}\| / \inf_{t\in(\xi,\eta)} \Psi_{\widetilde{\rho}}(t) = M(d).$$

(ii) $d = 0$: In this case we have $c > 0$ and there exists $\delta > 0$ such that $\Psi_{\widetilde{\rho}}^{[p]}(t) < 0$ for all $t \in (\delta, \eta)$. We have then $\sup_{t\in(\theta,\eta)}(\Psi_{\widetilde{\rho}}(t)/\Psi_{\widetilde{\rho}}(t)) = 1$ if $\theta \in (\delta, \eta)$ and $\sup_{t\in(\theta,\eta)}(\Psi_{\widetilde{\rho}}(t)/\Psi_{\widetilde{\rho}}(t)) \leq \|\Psi_{\widetilde{\rho}}\| / \inf_{t\in(\xi,\delta)} \Psi_{\widetilde{\rho}}(t)$. Thus,

$$\sup_{t\in(\theta,\eta)} (\Psi_{\widetilde{\rho}}(t)/\Psi_{\widetilde{\rho}}(t)) \leq M(d) = \sup(1, \|\Psi_{\widetilde{\rho}}\| / \inf_{t\in(\xi,\delta)} \Psi_{\widetilde{\rho}}(t)).$$

Since $\mu_*(\theta) > 0$, $G_{\widetilde{\rho}_r(\theta)}$ exists and we have for all $\theta \in (\xi, \eta)$ and all $t \in (\theta, \eta)$

$$\begin{aligned} |G_{\widetilde{\rho}_r(\theta)}(t, s)| &= |G_{\widetilde{\rho}}(t, s) - (\Phi_{\widetilde{\rho}}(\theta)/Wr_{\widetilde{\rho}}\Psi_{\widetilde{\rho}}(\theta))\Psi_{\widetilde{\rho}}(t)\Psi_{\widetilde{\rho}}(s)| \\ &\leq \|G_{\widetilde{\rho}}\|_\infty + Wr_{\widetilde{\rho}}^{-1} M(d)\|\Phi_{\widetilde{\rho}}\|\|\Psi_{\widetilde{\rho}}\|. \end{aligned}$$

Therefore,

$$0 < 1/\mu_*(\theta) \leq \sup_{t\in(\theta,\eta)} \int_\theta^\eta |G_{\widetilde{\rho}(\theta)}(t, s)| m(s) ds$$

$$\leq (\|G_{\widetilde{\rho}}\|_\infty + Wr_{\widetilde{\rho}}^{-1}M(d)\|\Phi_{\widetilde{\rho}}\|\|\Psi_{\widetilde{\rho}}\|)\int_\theta^\eta m(s)ds \to 0 \quad \text{as } \theta \to \eta,$$

thus proving that $\lim_{\theta\to\eta}\mu_k(\rho(\theta), m) = +\infty$. This completes the proof. □

3. On the half-eigenvalue problem

For $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$, and $\alpha, \beta \in L_\rho^1$, consider the BVP

$$\begin{gathered} \pounds_\rho u = \lambda m u + \alpha u^+ - \beta u^- \text{ in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{3.1}$$

Definition 3.1. We say that λ_0 is a half-eigenvalue of (3.1) if there exists a non-trivial solution (λ_0, u_0) of (3.1). In this situation, $\{(\lambda_0, tu_0),\ t > 0\}$ is a half-line of nontrivial solutions of (3.1) and λ_0 is said to be simple if all solutions (λ_0, u) of (3.1), with $uu_0 > 0$ in a right neighborhood of ξ, are on this half-line. There may exist another half-line of solutions $\{(\lambda_0, tv_0),\ t > 0\}$, but then we say that λ_0 is simple, if $u_0 v_0 < 0$ in a right neighborhood of ξ and all solutions (λ_0, v) of (3.1) lie on these two half lines.

Berestycki [6] proved that if $-\infty < \xi < \eta < +\infty$, $p \in C^1([\xi, \eta])$, $q, m, \alpha, \beta \in C([\xi, \eta])$ and m is positive, then (3.1) admits two increasing sequences of half-eigenvalues. So, the main goal of this section is to prove that the Beresticki's result holds for the case $1/p, q, m, \alpha, \beta \in L_\rho^1$. We begin with the following list of lemmas.

Lemma 3.2. *If (λ, ϕ) is a non trivial solution of (3.1), then $\phi \in S_\rho^{k,\kappa}$, for some integer $k \geq 1$ and $\kappa = +, -$.*

Proof. We have to prove that ϕ has a finite number of zeros and all are simple. Clearly if for some $\tau, \xi \leq \tau \leq \eta, \phi(\tau) = \phi^{[p]}(\tau) = 0$, we obtain from Corollary 2.4 that $\phi = 0$ and this contradicts the lemma's hypothesis.

Now, suppose that ϕ has an infinite sequence of zeros (t_n) in (ξ, η) converging to $\hat{t}$. Then we have $\phi(\hat{t}) = \lim_{n\to+\infty}\phi(t_n) = 0$. We claim that $\phi^{[p]}(\hat{t}) = 0$; indeed, if for instance $\phi^{[p]}(\hat{t}) > 0$ then there exists $\delta_0 > 0$ such that $\phi^{[p]}(t) > 0$ for all $t \in [\hat{t} - \delta_0, \hat{t} + \delta_0]$, and we get

$$\phi(t) = \int_{\hat{t}}^t \Big(\frac{1}{p(s)}\Big)\phi^{[p]}(s)ds \begin{cases} > 0 & \text{if } t \in (\hat{t}, \hat{t} + \delta_0) \\ < 0 & \text{if } t \in (\hat{t} - \delta_0, \hat{t}) \end{cases}$$

contradicting $\lim t_n = \hat{t}$. Again, we obtain from Corollary 2.4 that $\phi = 0$, contradicting the Lemma's hypothesis. Thus, we have proved that ϕ has a finite number of zeros and that all are simple. In other words, $\phi \in S_\rho^{k,\kappa}$ for some integer $k \geq 1$ and $\kappa = +, -$. The proof is complete. □

Lemma 3.3. *If λ is a half-eigenvalue of (3.1), then λ is simple.*

Proof. Let λ be a half-eigenvalue and ϕ_1, ϕ_2 be two eigenfunctions associated with λ such that $\phi_1, \phi_2 > 0$ in a right neighborhood of ξ. Therefore, $\phi_1, \phi_2 \in S_\rho^{k,+}$ for some integer $k \geq 1$, and denote for $i = 1, 2$, $(z_j^i)_{j=0}^{j=k}$ the sequence of zeros of ϕ_i. We have that $z_j^1 = z_j^2$ for all $j = 0, \dots, k$. By induction, clearly $z_0^1 = z_0^2 = \xi$ and if $z_j^1 = z_j^2$ then $z_{j+1}^1 = z_{j+1}^2$. Indeed, if for example $z_{j+1}^1 < z_{j+1}^2$, From Lemma 2.8

we have the contradiction

$$0 < \int_{z_j^1}^{z_{j+1}^1} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = 0.$$

Because of the positive homogeneity of (3.1), we have that $\psi_1 = -\phi_2^{[p]}(z_1^1)\phi_1$ and $\psi_2 = -\phi_1^{[p]}(z_1^1)\phi_2$ are eigenfunctions associated with λ satisfying

$$\psi_1(z_1^1) = \psi_2(z_1^1) = 0 \quad \text{and} \quad \psi_1^{[p]}(\xi) = \psi_2^{[p]}(\xi) = -\phi_2^{[p]}(z_1^1)\phi_1^{[p]}(z_1^1).$$

Therefore, $\psi = \psi_1 - \psi_2$ satisfies

$$\begin{gathered}\pounds_\rho \psi = \lambda m\psi + \alpha\psi^+ - \beta\psi^- \quad \text{in } (\xi,\eta) \text{ a.e.},\\ \psi(\xi) = \psi^{[p]}(\xi) = 0,\end{gathered}$$

and from Corollary 2.4 we have $\psi_1 = \psi_2$. This shows that the half-eigenvalue λ is simple and completes the proof. □

Lemma 3.4. *For all $\rho \in \Delta$, $m \in K_\rho^*$, $\alpha, \beta \in L_\rho^1$, $k \geq 1$ and $\kappa = +, -$, BVP (3.1) admits at most one half-eigenvalue having an eigenfunction in $S_\rho^{k,\kappa}$.*

Proof. Let $(\lambda_1, \phi_1), (\lambda_2, \phi_2) \in \mathbb{R} \times (S_\rho^{k,\kappa} \cap \tilde{W}_\rho)$ be two solutions of (3.1) such that $\lambda_1 \neq \lambda_2$ and $\phi_1, \phi_2 \in S_\rho^{k,\kappa}$ for some integer $k \geq 1$ and $\kappa = +, -$, and denote for $i = 1, 2$ $(z_j^i)_{j=0}^{j=k}$ the sequence of zeros of ϕ_i. First, we claim that there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$; indeed, assume that $\phi_1(z_j^2) = 0$ for all $j \in \{1, \dots, k-1\}$ and $\lambda_1 < \lambda_2$ and note that there exists $j_1 \in \{1, \dots, k-1\}$ such that $\operatorname{meas}(\{m > 0\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$ and $\phi_1\phi_2 > 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$. Applying Lemma 2.10, we get that there exists $\tau \in (z_{j_1}^2, z_{j_1+1}^2)$ such that $\phi_1(\tau) = 0$ and this contradicts $\phi_1 \in S_\rho^{k,\kappa}$.

Now, let $k_1 = \max\{l \leq k : z_j^1 = z_j^2 \text{ for } j \leq l\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$ and without loss of generality, assume that $\xi_1 = z_{k_1+1}^1 < \eta_1 = z_{k_1+1}^2$. We obtain from Lemma 2.7 that there exist two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z_{k_1+m}^1 < \eta_n = z_{k_1+n}^2 < \eta_{n+1} = z_{k_1+n+1}^2 \leq \xi_{m+1} = z_{k_1+m+1}^1$$

and from Lemma 2.8 we have

$$0 < \int_{\xi_0}^{\xi_1} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m\phi_1\phi_2, \tag{3.2}$$

$$0 > \int_{\eta_n}^{\eta_{n+1}} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2. \tag{3.3}$$

On the one hand, from (3.2) we have $\lambda_1 > \lambda_2$, and on the other hand, from (3.3) we have $\lambda_1 < \lambda_2$. This completes the proof. □

Lemma 3.5. *Let $\rho \in \Delta$, $m \in K_\rho^*$, $\alpha, \beta \in L_\rho^1$, $k \geq 1$ and $\kappa = +, -$ and assume that $(\lambda_1, \phi_1), (\lambda_2, \phi_2)$ are two solutions of (3.1) such that for $i = 1, 2$, $\phi_i \in S_\rho^{k_i,\kappa}$. If $k_2 > k_1$ then $\lambda_2 > \lambda_1$.*

Proof. For $i = 1, 2$, let $(z_j^i)_{j=0}^{j=k}$ be the sequence of zeros of ϕ_i and set $k_1 = \max\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\}$. Consider $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$. We distinguish then two cases.

(i) $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case we have

$$0 < \int_{\eta_0}^{\eta_1} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_0}^{\eta_1} m\phi_1\phi_2$$

proving that $\lambda_1 < \lambda_2$.

(ii) $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case, Lemma 2.7 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z^1_{k_1+m} < \eta_n = z^2_{k_1+n} < \eta_{n+1} = z^2_{k_1+n+1} \le \xi_{m+1} = z^1_{k_1+m+1}.$$

As above, we have

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_1 \pounds_\rho \phi_2 - \phi_2 \pounds_\rho \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2,$$

proving that $\lambda_1 < \lambda_2$. This completes the proof. □

Lemma 3.6. *Let $\rho = (\xi, \eta, p, q, a, b, 1, 0) \in \Delta$, $m \in K^+_\rho$ and $\alpha, \beta \in L^1_\rho$ and suppose that for all $\theta, \xi < \theta \le \eta$, $\lambda^\kappa_k(\rho_l(\theta), m, \alpha, \beta)$ exists where $\rho_l(\theta) = (\xi, \theta, p, q, a, b, 1, 0)$. Then, the function $\theta \to \lambda^\kappa_k(\rho_l(\theta), m, \alpha, \beta)$ is continuous and decreasing. Moreover, we have $\lim_{\theta\to\xi} \lambda^\kappa_k(\rho_l(\theta), m, \alpha, \beta) = +\infty$.*

Proof. **Step 1 (Monotonicity).** In this step, we prove that the function $\theta \to \lambda^+_k(\rho_l(\theta), m, \alpha, \beta)$ is decreasing, the case $\kappa = -$ is checked similarly. Let θ_1, θ_2 be such that $\xi < \theta_1 < \theta_2 \le \eta$ and let for $i = 1, 2$, $\lambda_i = \lambda^+_k(\rho_l(\theta_i), m, \alpha, \beta)$, and ϕ_i be the eigenfunction associated with λ_i. Denoting for $i = 1, 2$, $(z^i_j)^{j=k}_{j=0}$ as the sequence of zeros of ϕ_i, we have

$$\xi = z^1_0 < z^1_1 < \dots < z^1_k = \theta_1, \quad \xi = z^2_0 < z^2_1 < \dots < z^2_k = \theta_2.$$

For $i = 1, 2$, let $\widetilde{\rho}_l(z^i_1) = (\xi, z^i_1, p, q - \alpha, a, b, 1, 0)$, and note that $\lambda_i = \mu_1(\widetilde{\rho}_l(z^i_1), m)$. We claim that $z^1_1 < z^2_1$. Indeed, if $z^1_1 = z^2_1$, then we have from Lemma 2.16 that $\lambda_2 > \lambda_1$. Applying Lemma 2.10, we get that ϕ_2 vanishes in all intervals (z^1_j, z^1_{j+1}) for all $j = 1, \dots, k-1$. This contradicts $\phi_2 \in S^{k,\kappa}_{\rho_l(\theta_2)}$.

At the end, Lemma 2.8 leads to

$$0 < \int_\xi^{z^1_1} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = (\lambda_1 - \lambda_2) \int_\xi^{z^1_1} m\phi_1\phi_2,$$

proving that $\lambda_1 > \lambda_2$.

Step 2 (Continuity). Let $\overline{\mu} > 0$ be such that

$$\inf(\mu_1(\rho_\alpha, m), \mu_1(\rho_\beta, m), \mu_1(\rho, m)) > -\overline{\mu}$$

where $\rho_\alpha = (\xi, \eta, p, q - \alpha, a, b, 1, 0)$ and $\rho_\beta = (\xi, \eta, p, q - \beta, a, b, 1, 0)$. Consider the BVP

$$\begin{gathered} \pounds_{\widetilde{\rho}} u = \lambda m u + \alpha u^+ - \beta u^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B^l_{\widetilde{\rho}} u = B^r_{\widetilde{\rho}} u = 0, \end{gathered} \tag{3.4}$$

where $\widetilde{\rho} = (\xi, \eta, p, q + \overline{\mu} m, a, b, 1, 0)$. Clearly, if λ is a half-eigenvalue of (3.4) then $(\lambda - \overline{\mu})$ is a half eigenvalue of (3.1), and note that because of $\mu_k(\widetilde{\rho}, m) \ge \mu_1(\widetilde{\rho}, m) = \mu_1(\rho, m) + \overline{\mu} > 0$ for all integers $k \ge 1$, $G_{\widetilde{\rho}}$ exists.

Let $\theta, \xi \le \theta < \eta$ and $(\theta_n) \subset (\xi, \eta)$ such that $\lim \theta_n = \theta$. Fix $k \ge 1$ and κ and set $\lambda = \lambda_k^\kappa(\widetilde{\rho}_l(\theta), m, \alpha, \beta)$, $\lambda_n = \lambda_k^\kappa(\widetilde{\rho}_l(\theta_n), m, \alpha, \beta)$ and let for all $n \ge 1$, ϕ_n be the normalized eigenfunction corresponding to λ_n. We have that

$$\begin{aligned} \phi_n(t) = \lambda_n \int_\xi^{\theta_n} G_n(t,s) m(s)\phi_n(s) ds + \int_\xi^{\theta_n} G_n(t,s)\alpha(s)\phi_n^+(s) ds \\ - \int_\xi^{\theta_n} G_n(t,s)\beta(s)\phi_n^-(s) ds \end{aligned}$$

where $G_n = G_{\widetilde{\rho}_l(\theta_n)}$. By the change of variables $s = \sigma_n(\tau)$ with

$$\sigma_n(\tau) = \begin{cases} \tau + h_n & \text{if } \xi = -\infty \\ \varepsilon_n \tau + \omega_n & \text{if } \xi > -\infty \end{cases}$$

where $h_n = \theta_n - \theta$, $\varepsilon_n = (\theta_n - \xi)/(\theta - \xi)$ and $\omega_n = -(\theta_n - \theta)\xi/(\theta - \xi)$, we have that the function φ_n defined by $\varphi_n(t) = \phi_n(\sigma_n(t))$ satisfies

$$\begin{aligned} \varphi_n(t) = \lambda_n \int_\xi^{\theta} \widetilde{G}_n(t,\tau) m(\sigma_n(\tau))\varphi_n(\tau) d\tau + \int_\xi^{\theta} \widetilde{G}_n(t,\tau)\alpha(\sigma_n(\tau))\varphi_n^+(\tau) d\tau \\ - \int_\xi^{\theta} \widetilde{G}_n(t,\tau)\beta(\sigma_n(\tau))\varphi_n^-(\tau) d\tau \end{aligned}$$

where

$$\widetilde{G}_n(t,\tau) = \begin{cases} G_n(\sigma_n(t), \sigma_n(\tau)) & \text{if } \xi = -\infty, \\ \varepsilon_n G_n(\sigma_n(t), \sigma_n(\tau)) & \text{if } \xi > -\infty. \end{cases}$$

Then from Lemma 2.12 we have

$$\widetilde{G}_n(t,\tau) = \begin{cases} G_\rho(\sigma_n(t), \sigma_n(\tau)) - (\Big(\Psi_\rho(\theta_n)/\Phi_\rho(\theta_n)\Big) \\ \times \Phi_\rho(\sigma_n(t))\Phi_\rho(\sigma_n(\tau)) & \text{if } \xi = -\infty \\ \varepsilon_n G_\rho\big(\sigma_n(t), \sigma_n(\tau)\big) \\ -\varepsilon_n (\Psi_\rho(\theta_n)/\Phi_\rho(\theta_n))\Phi_\rho(\sigma_n(t))\Phi_\rho(\sigma_n(\tau)) & \text{if } \xi > -\infty. \end{cases}$$

Now, we need to prove that for all $\chi \in L^1_{\rho(\theta)}$, $L_{\chi,n} \to L_\chi$ in operator norm, where $L_{\chi,n}, L_\chi : C_{\widetilde{\rho}_l(\theta)} \to C_{\widetilde{\rho}_l(\theta)}$ are defined by

$$\begin{aligned} L_{\chi,n} u(t) &= \int_\xi^\theta \widetilde{G}_n(t,\tau)\chi(\sigma_n(\tau)) u(\tau) d\tau, \\ L_{\chi,\theta} u(t) &= \int_\xi^\theta G_{\widetilde{\rho}_l(\theta)}(t,\tau)\chi(\tau) u(\tau) d\tau. \end{aligned}$$

For $u \in C_{\widetilde{\rho}_l(\theta)}$ with $\|u\| = 1$, we have

$$\begin{aligned} |L_{\chi,n} u(t) - L_\chi u(t)| &\le \int_\xi^\theta |\widetilde{G}_n(t,\tau)\chi(\sigma_n(\tau)) - G_{\widetilde{\rho}_l(\theta)}(t,\tau)\chi(\tau)| d\tau \\ &\le \int_\xi^\theta |\widetilde{G}_n(t,\tau) - G_{\widetilde{\rho}_l(\theta)}(t,\tau)||\chi(\sigma_n(\tau))| d\tau \\ &\quad + \int_\xi^\theta |G_{\widetilde{\rho}_l(\theta)}(t,\tau)||\chi(\sigma_n(\tau)) - \chi(\tau)| d\tau. \end{aligned} \tag{3.5}$$

Let $\epsilon > 0$. Since in both the cases $\xi = -\infty$ and $\xi > -\infty$, $\sigma_n(\tau)$ converges uniformly to τ in (ξ, η) and the functions $\Phi_{\widetilde{\rho}}, \Psi_{\widetilde{\rho}}, G_{\widetilde{\rho}}$ are uniformly continuous, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$|\widetilde{G}_n(t,\tau) - G_{\widetilde{\rho}_l(\theta)}(t,\tau)| \leq \epsilon \quad \text{for all } t \text{ and } \tau \text{ with } \xi \leq t, \tau < \eta.$$

Moreover, we have

$$\lim \int_\xi^\theta |\chi(\sigma_n(\tau))| d\tau = \lim \int_\xi^{\theta_n} |\chi(\tau)| d\tau = \|\chi\|_{L^1_{\rho(\theta)}}$$

and from Lemma 1.1, we obtain

$$\lim \int_\xi^\theta |\chi(\sigma_n(\tau)) - \chi(\tau)| d\tau = 0.$$

Consequently, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$,

$$\int_\xi^\theta |\chi(\sigma_n(\tau)) - \chi(\tau)| d\tau \leq \epsilon \quad \text{and} \quad \int_\xi^\theta |\chi(\sigma_n(\tau))| d\tau \leq (\|\chi\|_{L^1_{\rho(\theta)}} + \epsilon)$$

and from (3.5) we obtain that for all $n \geq \max(n_1, n_2)$,

$$\sup_{t \in (\xi,\theta)} |L_{\chi,n} u(t) - L_\chi u(t)| \leq \epsilon(\|\chi\|_{L^1_{\rho(\theta)}} + \epsilon) + \sup_{t,\tau \in (\xi,\theta)} |G_{\rho(\theta)}(t,\tau)| \epsilon$$

proving that $L_{\chi,n} \to L_\chi$ in operator norm.

Let $\delta > 0$ be such that $\theta_n \in [\theta - \delta, \theta - \delta]$. We have from Step 1 that

$$\lambda_k^\kappa(\rho_l(\theta + \delta), m, \alpha, \beta) \leq \lambda_n = \lambda_k^\kappa(\rho_l(\theta_n), m, \alpha, \beta) \leq \lambda_k^\kappa(\rho_l(\theta - \delta), m, \alpha, \beta).$$

Hence, $\lambda_{\sup} = \limsup \lambda_n$ and $\lambda_{\inf} = \liminf \lambda_n$ are finite numbers.

For all $n \in \mathbb{N}$ and $\nu = \sup$ or $\inf$, we have

$$\begin{aligned} \varphi_n &= \lambda_n L_{m,n} \varphi_n + L_{\alpha,n} I^+ \varphi_n - L_{\beta,n} I^- \varphi_n \\ &= (\lambda_n - \lambda_\nu) L_{m,n} \varphi_n + \lambda_\nu (L_{m,n} - L_{m,\theta}) \varphi_n + (L_{\alpha,n} - L_{\alpha,\theta}) I^+ \varphi_n \\ &\quad - (L_{\beta,n} - L_{\beta,\theta}) I^- \varphi_n + \lambda_\nu L_{m,\theta} \varphi_n + L_{\alpha,\theta} I^+ \varphi_n - L_{\beta,\theta} I^- \varphi_n. \end{aligned}$$

This, and the compactness of the operators L_m, L_α, L_β and the fact that $L_{m,n} \to L_m, L_{\alpha,n} \to L_\alpha, L_{\beta,n} \to L_\beta$, imply that there exist $\varphi_{\sup}, \varphi_{\inf} \in \overline{S^{k,\kappa}_{\widetilde{\rho}_l(\theta)}}$ such that for $\nu = \sup$ or $\inf$,

$$\varphi_\nu = \lambda_\nu L_m \varphi_\nu + L_\alpha I^+ \varphi_\nu - L_\beta I^- \varphi_\nu.$$

In other words, each of the pairs $(\lambda_{\sup}, \varphi_{\sup})$ and $(\lambda_{\inf}, \varphi_{\inf})$ satisfies

$$\pounds_{\widetilde{\rho}_l(\theta)} u = \lambda m u + \alpha u^+ - \beta u^- \text{ a.e. in } (\xi, \theta),$$
$$B^l_{\widetilde{\rho}_l(\theta)} u = B^r_{\widetilde{\rho}_l(\theta)} u = 0,$$

and $\varphi_{\sup}, \varphi_{\inf} \in S^{k,\kappa}_{\widetilde{\rho}_l(\theta)}$ (if $\varphi_{\sup} \in \partial S^{k,\kappa}_{\widetilde{\rho}_l(\theta)}$ then there exists $\tau, \xi \leq \tau \leq \eta$ such that $\varphi_{\sup}(\tau) = \varphi^{[p]}_{\sup}(\tau) = 0$ and by Corollary 2.4, we have $\varphi_{\sup} = 0$ contradicting $\|\varphi_{\sup}\| = 1$). At the end, we obtain from Lemma 3.4 that $\lambda_{\sup} = \lambda_{\inf} = \lambda$.

Step 3. We have

$$1 \leq \lambda_k^\kappa(\widetilde{\rho}_l(\theta), m, \alpha, \beta) \|L_{m,\theta}\| + \|L_{\alpha,\theta}\| + \|L_{\beta,\theta}\|$$

leading to

$$\lambda_k^\kappa(\widetilde{\rho}_l(\theta), m, \alpha, \beta) \geq (1 - \|L_{\alpha,\theta}\| - \|L_{\beta,\theta}\|) / \|L_{m,\theta}\|.$$

Since $\|L_{\alpha,\theta}\|, \|L_{\beta,\theta}\|, \|L_{m,\theta}\| \to 0$ as $\theta \to \xi$ (see the proof of Lemma 2.16), we have

$$\lim_{\theta\to\xi} \lambda_k^\kappa(\widetilde{\rho}_l(\theta), m, \alpha, \beta) = +\infty.$$

This completes the proof. □

Lemma 3.7. *For $\rho \in \Delta$, $m \in K_\rho^+$ and $\alpha, \beta \in L_\rho^1$, BVP* (3.1) *admits two increasing sequences of simple half-eigenvalues $(\lambda_k^+(\rho, m, \alpha, \beta))_{k\geq 1}$ and $(\lambda_k^-(\rho, m, \alpha, \beta))_{k\geq 1}$, such that for all integers $k \geq 1$ and $\kappa = +, -$, the corresponding half-line of solutions lies in $\{\lambda_k^\kappa(\rho, m, \alpha, \beta) \times S_k^\kappa\}$. Furthermore, aside from these solutions and the trivial one, there are no other solutions of* (3.1).

Proof. We proceed by induction on k. Clearly, for $k = 1$, $\lambda_1^+ = \mu_1(\widetilde{\rho}_+, m)$ and $\lambda_1^- = \mu_1(\widetilde{\rho}_-, m)$ where for all $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $\widetilde{\rho}_+ = (\xi, \eta, p, q - \alpha, a, b, c, d)$ and $\widetilde{\rho}_- = (\xi, \eta, p, q - \beta, a, b, c, d)$.

Now, assume that for all $\rho \in \Delta$, $\lambda_k^\kappa = \lambda_k^\kappa(\rho, m, \alpha, \beta)$ exists and let us prove that $\lambda_{k+1}^\kappa = \lambda_{k+1}^\kappa(\rho, m, \alpha, \beta)$ exists. Let for $\theta \in (\xi, \eta)$, $\lambda_k^\kappa(\theta) = \lambda_k^\kappa(\rho_l(\theta), m, \alpha, \beta)$ where $\rho_l(\theta) = (\xi, \theta, p, q, a, b, 1, 0)$ and let $\mu(\theta) = \mu_1(\widetilde{\rho}_r(\theta), m)$ where $\widetilde{\rho}_r(\theta) = (\theta, \eta, p, q - \alpha, 1, 0, c, d)$. From Lemmas 2.16 and 3.6, there is a unique $\theta_{k+1} \in (\xi, \eta)$ such that $\lambda_k^\kappa(\theta_0) = \mu(\theta_0)$. Let ϕ_{k,θ_0} and $\phi_{1,\theta_0} > 0$ be respectively the eigenfunction associated with the half-eigenvalue $\lambda_k^\kappa(\theta_0)$ the eigenvalue $\mu(\theta_0)$, then the function

$$\phi_{k+1} = \begin{cases} \phi_{k,\theta_0} & \text{in } (\xi, \theta_0) \\ (\phi_{k,\theta_0}^{[p]}(\theta_0)/\phi_{1,\theta_0}^{[p]}(\theta_0))\phi_{1,\theta_0} & \text{in } (\kappa_0, \eta) \end{cases}$$

belongs to $S_\rho^{k+1,\kappa}$ and the pair $(\lambda_k^\kappa(\theta_0), \phi_{k+1}) = (\mu(\theta_0), \phi_{k+1})$ satisfies the BVP

$$\begin{gathered} \pounds_\rho u = \lambda m u + \alpha u^+ - \beta u^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered}$$

Thus, we have proved that $\lambda_{k+1}^\kappa(\rho, m, \alpha, \beta)$ exists. □

Proposition 3.8. *Let $\rho \in \Delta$, $m \in K_\rho^*$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L_\rho^1$. Assume that $\lambda_k^\kappa(\rho, m, \alpha_1, \beta_1)$, $\lambda_k^\kappa(\rho, m, \alpha_2, \beta_1)$ and $\lambda_k^\kappa(\rho, m, \alpha_1, \beta_2)$ exist.*

1. *If $\alpha_1 \leq \alpha_2$ a.e. in (ξ, η), then $\lambda_k^\kappa(\rho, m, \alpha_1, \beta_1) \geq \lambda_k^\kappa(\rho, m, \alpha_2, \beta_1)$.*
2. *If $\beta_1 \leq \beta_2$ a.e. in (ξ, η), then $\lambda_k^\kappa(\rho, m, \alpha_1, \beta_1) \geq \lambda_k^\kappa(\rho, m, \alpha_1, \beta_2)$.*

Proof. We present the proof of property (1) only; Property (2) is checked similarly. Fix k, κ and set for $i = 1, 2$, $\lambda_i = \lambda_k^\kappa(\rho, m, \alpha_i, \beta_1)$ and let ϕ_i be the eigenfunction associated with λ_i having a sequence of zeros $(z_j^i)_{j=0}^{j=k}$. We distinguish two cases:

(i) $z_j^1 = z_j^2$ for all $j \in \{1, \dots, k-1\}$: Let $j_1 \in \{1, \dots, k-1\}$ be such that $\text{meas}(\{m > 0\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$ and

$$\begin{aligned} 0 &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = (\lambda_1 - \lambda_2) \int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 \\ &\quad + \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\alpha_1 \phi_1^+ \phi_2 - \alpha_2 \phi_2^+ \phi_1) + \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\beta_1 \phi_1^- \phi_2 - \beta_1 \phi_2^- \phi_1) \qquad (3.6) \\ &= (\lambda_1 - \lambda_2) \int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 + \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\alpha_1 \phi_1^+ \phi_2 - \alpha_2 \phi_2^+ \phi_1). \end{aligned}$$

Thus, from (3.6) in both the case $\phi_1,\phi_2>0$ in $(z^2_{j_1},z^2_{j_1+1})$ and the case $\phi_1,\phi_2<0$ in $(z^2_{j_1},z^2_{j_1+1})$, we obtain $\lambda_1\ge\lambda_2$.

(ii) $z^1_{j_0}\neq z^2_{j_0}$ for some j_0: In this case set $k_1=\max\{l\le k: z^1_j=z^2_j \text{ for all } j\le l\}$. If $z^1_{k_1+1}<z^2_{k_1+1}$, then

$$0<\int_{z^1_{k_1}}^{z^1_{k_1+1}}\phi_2\pounds_\rho\phi_1-\phi_1\pounds_\rho\phi_2=(\lambda_1-\lambda_2)\int_{z^1_{k_1}}^{z^1_{k_1+1}}m\phi_1\phi_2+\int_{z^1_{k_1}}^{z^1_{k_1+1}}(\alpha_1-\alpha_2)\phi_1\phi_2$$

proving that $\lambda_1>\lambda_2$ and if $z^2_{k_1+1}<z^1_{k_1+1}$ then considering the families $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ with $\xi_j=z^1_{k_1+j}$ and $\eta_j=z^2_{k_1+j}$, we obtain from Lemma 2.7 that there exist two integers $m,n\ge1$ having the same parity such that

$$\xi_m=z^2_{k_1+m}<\eta_n=z^1_{k_1+n}<\eta_{n+1}=z^1_{k_1+n+1}\le\xi_{m+1}=z^2_{k_1+m+1}.$$

Therefore, Lemma 2.8 leads to

$$0<\int_{\eta_n}^{\eta_{n+1}}\phi_2\pounds_\rho\phi_1-\phi_1\pounds_\rho\phi_2=(\lambda_1-\lambda_2)\int_{\eta_n}^{\eta_{n+1}}m\phi_1\phi_2+\int_{\eta_n}^{\eta_{n+1}}(\alpha_1-\alpha_2)\phi_1\phi_2,$$

and then $\lambda_1>\lambda_2$. This completes the proof. □

Proposition 3.9. *Let $\rho=(\xi,\eta,p,q,a,b,c,d)\in\Delta$, $m_1,m_2\in K^*_\rho$ and $\alpha,\beta\in L^1_\rho$. Assume that $m_1\le m_2$ a.e. in (ξ,η), $m_1<m_2$ in a subset of positive measure, and $\lambda^\kappa_k(\rho,m_1,\alpha,\beta)$, $\lambda^\kappa_k(\rho,m_2,\alpha,\beta)$ exist for some integer $k\ge1$ and $\kappa=+,-$. If either $\lambda^\kappa_k(\rho,m_1,\alpha,\beta)\ge0$ or $\lambda^\kappa_k(\rho,m_2,\alpha,\beta)\ge0$, then $\lambda^\kappa_k(\rho,m_1,\alpha,\beta)>\lambda^\kappa_k(\rho,m_2,\alpha,\beta)$, and if either $\lambda^\kappa_k(\rho,m_1,\alpha,\beta)\le0$ or $\lambda^\kappa_k(\rho,m_2,\alpha,\beta)\le0$, then $\lambda^\kappa_k(\rho,m_1,\alpha,\beta)<\lambda^\kappa_k(\rho,m_2,\alpha,\beta)$.*

Proof. For $i=1,2$, set $\mu_i=\mu_k(\rho,m_i)$ and let ϕ_i be the eigenfunction associated with μ_i having a sequence of zeros $(z^i_j)_{j=0}^{j=k}$. First, we claim that there exists j_0 such that $z^1_{j_0}\neq z^2_{j_0}$. Indeed, assume that $\phi_1(z^2_j)=0$ for all $j\in\{1,\dots,k-1\}$ and $\mu_1<\mu_2$ and note that there exists $j_1\in\{1,\dots,k-1\}$ such that $\text{meas}(\{m_2>m_1\}\cap(z^2_{j_1},z^2_{j_1+1}))>0$ and $\phi_1\phi_2>0$ in $(z^2_{j_1},z^2_{j_1+1})$. Applying Lemma 2.10, we get that there exists $\tau\in(z^2_{j_1},z^2_{j_1+1})$ such that $\phi_1(\tau)=0$ and this contradicts $\phi_1\in S^{k,\kappa}_\rho$.

Now, let $k_1=\max\{l\le k: z^1_j=z^2_j \text{ for all } j\le l\}$, and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j=z^1_{k_1+j}$ and $\eta_j=z^2_{k_1+j}$. We distinguish then two cases.

(i] $\xi_1=z^1_{k_1+1}<\eta_1=z^2_{k_1+1}$: In this case

$$\begin{aligned}0&<\int_{\eta_0}^{\eta_1}\phi_2\pounds_\rho\phi_1-\phi_1\pounds_\rho\phi_2=\int_{\eta_0}^{\eta_1}(\mu_1m_1-\mu_2m_2)\phi_1\phi_2\\&=\int_{\eta_0}^{\eta_1}(\mu_1-\mu_2)m_1\phi_1\phi_2+\int_{\eta_0}^{\eta_1}\mu_2(m_1-m_2)\phi_1\phi_2\\&=\int_{\eta_0}^{\eta_1}\mu_1(m_1-m_2)\phi_1\phi_2+\int_{\eta_0}^{\eta_1}(\mu_1-\mu_2)m_2\phi_1\phi_2\end{aligned}$$

and this proves that in both the cases $\mu_1\ge0$ and $\mu_2\ge0$, we have $\mu_1>\mu_2$.

(ii) $\xi_1=z^1_{k_1+1}>\eta_1=z^2_{k_1+1}$: In this case Lemma 2.7 guarantees existence of two integers m,n having the same parity such that

$$\xi_m=z^1_{k_1+m}<\eta_n=z^2_{k_1+n}<\eta_{n+1}=z^2_{k_1+n+1}\le\xi_{m+1}=z^1_{k_1+m+1}.$$

As above, we have

$$\begin{aligned}
0 &< \int_{\eta_n}^{\eta_{n+1}} \phi_2 \pounds_\rho \phi_1 - \phi_1 \pounds_\rho \phi_2 = \int_{\eta_n}^{\eta_{n+1}} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\
&= \int_{\eta_n}^{\eta_{n+1}} (\mu_1 - \mu_2) m_1 \phi_1 \phi_2 + \int_{\eta_n}^{\eta_{n+1}} \mu_2 (m_1 - m_2)\phi_1\phi_2 \\
&= \int_{\eta_n}^{\eta_{n+1}} \mu_1 (m_1 - m_2)\phi_1\phi_2 + \int_{\eta_n}^{\eta_{n+1}} (\mu_1 - \mu_2) m_2 \phi_1\phi_2
\end{aligned}$$

and this proves that in both the cases $\mu_1 \geq 0$ and $\mu_2 \geq 0$, we have $\mu_1 > \mu_2$.

The cases $\lambda_k^\kappa(\rho, m_1, \alpha, \beta) \leq 0$ and $\lambda_k^\kappa(\rho, m_2, \alpha, \beta) \leq 0$ are checked in similar way and this ends the proof. □

Theorem 3.10. *For $\rho \in \Delta$, $m \in K_\rho^*$ and $\alpha, \beta \in L_\rho^1$, BVP (3.1) admits two increasing sequences of simple half-eigenvalues $(\lambda_k^+(\rho, m, \alpha, \beta))_{k\geq 1}$ and $(\lambda_k^-(\rho, m, \alpha, \beta))_{k\geq 1}$, such that for all integers $k \geq 1$ and $\kappa = +, -$, the corresponding half-line of solutions lies in $\{\lambda_k^\kappa(\rho, m, \alpha, \beta)\} \times S_\rho^{k,\kappa}$ and $\lim_{k\to+\infty} \lambda_k^\kappa(\rho, m, \alpha, \beta) = +\infty$. Furthermore, aside from these solutions and the trivial one, there are no other solutions of (3.1).*

Proof. Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$, $\alpha, \beta \in L_\rho^1$ and (ϵ_n) be a decreasing sequence of real numbers converging to 0 and let $A > 0$ be such that

$$\min(\mu_1(\rho, m + \epsilon_1), \lambda_1^+(\rho, m + \epsilon_1, \alpha, \beta), \lambda_1^-(\rho, m + \epsilon_n, \alpha, \beta)) > -A.$$

Consider the BVP

$$\begin{gathered}
\pounds_{\widetilde{\rho}} u = \lambda m u + \alpha u^+ - \beta u^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\
B_{\widetilde{\rho}}^l u = B_{\widetilde{\rho}}^r u = 0,
\end{gathered} \tag{3.7}$$

where $\widetilde{\rho} = (\xi, \eta, p, q + Am, a, b, c, d)$ and note that λ is a half-eigenvalue of (3.7) if and only if $(\lambda - A)$ is a half-eigenvalue of (3.1). For k and κ fixed, let $\lambda_{k,n}^\kappa = \lambda_k^\kappa(\widetilde{\rho}, m + \epsilon_n, \alpha, \beta)$ be associated with a normalized eigenfunction $\phi_{k,n} \in S_\rho^{k,\kappa}$, and let $[\gamma, \delta] \subset (\xi, \eta)$ be such that $m > 0$ a.e. in (γ, δ).

First, because

$$\lambda_{k,1}^\kappa = \lambda_k^\kappa(\rho, m + \epsilon_1, \alpha, \beta) + A \geq \lambda_1^\kappa(\rho, m + \epsilon_1, \alpha, \beta) + A > 0,$$

we have from property 1 in Proposition 3.9 that for all $n \in \mathbb{N}$, $\lambda_{k,n+1}^\kappa \geq \lambda_{k,n}^\kappa \geq \lambda_{k,1}^\kappa > 0$.

Set $\widetilde{q} = -(|\alpha| + |\beta|)$, $\rho^* = (\xi, \eta, p, q + Am - \widetilde{q}, a, b, c, d)$ and $\rho_* = (\gamma, \delta, p, q + Am - \widetilde{q}, 1, 0, 1, 0)$. Then properties 2 and 3 in Proposition 3.8, Lemma 2.15 and Lemma 2.14 lead to

$$0 < \lambda_{k,n}^\kappa \leq \lambda_k^\kappa(\widetilde{\rho}, m + \epsilon_n, \widetilde{q}, \widetilde{q}) = \mu_k(\rho^*, m + \epsilon_n) \leq \mu_k(\rho_*, m + \epsilon_n) \leq \mu_k(\rho_*, m)$$

proving that $\lim \lambda_{k,n}^\kappa = \lambda_k^\kappa > 0$.

Now, if $\mu_l(\widetilde{\rho}, m)$ exists for some $l \geq 1$, then $\mu_l(\widetilde{\rho}, m) = \mu_l(\rho, m) + A$ (that is $\mu_l(\rho, m)$ exists) and

$$\mu_l(\widetilde{\rho}, m + \epsilon_1) = \mu_l(\rho, m + \epsilon_1) + A > \mu_1(\rho, m + \epsilon_1) + A > 0.$$

We obtain from Proposition 3.9 that $\mu_l(\widetilde{\rho}, m) > \mu_l(\widetilde{\rho}, m + \epsilon_1) > 0$, proving that that $G_{\widetilde{\rho}}$ exists.

At this stage, we have

$$\phi_{k,n} = \lambda_{k,n}^{\kappa} L_m \phi_{k,n} + \epsilon_n L\phi_{k,n} + \Phi(\phi_{k,n})$$

where $L_m, L, \Phi : C_\rho \to C_\rho$ are defined by

$$L_m u(t) = \int_\xi^\eta G_{\widetilde{\rho}}(t,s)m(s)u(s)ds,$$

$$Lu(t) = \int_\xi^\eta G_{\widetilde{\rho}}(t,s)u(s)ds$$

$$\Phi(u)(t) = \int_\xi^\eta G_{\widetilde{\rho}}(t,s)(\alpha(s)u^+(s) - \beta(s)u^-(s))ds.$$

Since L_m is compact, L is bounded and Φ is completely continuous, $\phi_{k,n}$ converge (up to a subsequence) to some $\phi_k \in \overline{S_\rho^{k,\kappa}}$ with $\|\phi_k\| = 1$ and we have $\phi_k = \lambda_k^\kappa L_m \phi_k + \Phi(\phi_k)$. Because of Theorem 2.2, $\phi_k \in S_k$ and λ_k^κ is a half-eigenvalue of (3.1).

Since uniqueness and simplicity of λ_k^κ follow from Lemmas 3.3 and 3.4 and the monotonicity of the the sequence (λ_k^κ) is assured by Lemma 3.5, it remains to show that $\lim_{k\to\infty} \lambda_k^\kappa = +\infty$. We have from Proposition 3.8 that

$$\lambda_k^\kappa = \lambda_k^\kappa(\widetilde{\rho}, m, \alpha, \beta) \geq \lambda_k^\kappa(\widetilde{\rho}, m + \epsilon_1, \alpha, \beta) \geq \lambda_k^\kappa(\widetilde{\rho}, m + \epsilon_1, -\widetilde{q}, -\widetilde{q}) = \mu_k(\hat{\rho}, m + \epsilon_1)$$

where $\hat{\rho} = (\xi, \eta, p, q + Am + \widetilde{q}, a, b, c, d)$. Therefore, we have from Assertion 1 of Theorem 2.13 that $\lim_{k\to\infty} \lambda_k^\kappa = +\infty$. This completes the proof. □

In the following three propositions, we present some important properties of half-eigenvalues needed in the reminder of this work.

Proposition 3.11. *Let for $i = 1, 2$, $\rho_i = (\xi, \eta, p, q_i, a, b, c, d) \in \Delta$, $m \in K_{\rho_1}^*$, $\alpha, \beta \in L_{\rho_1}^1$ and suppose that for $i = 1, 2$, $\lambda_i = \lambda_k^\kappa(\rho_i, m, \alpha, \beta)$ exists for some integer $k \geq 1$ and $\kappa = +, -$. If $q_1 \leq q_2$ a.e. in (ξ, η) then $\lambda_1 \leq \lambda_2$. Moreover, if $q_1 < q_2$ in a subset of positive measure, then $\lambda_1 < \lambda_2$.*

Proof. Since for $i = 1, 2$, $\lambda_i = \lambda_k^\kappa(\rho_i, m_2, 0, 0) = \lambda_k^\kappa(\hat{\rho}, m, -q_i, -q_i)$ with $\hat{\rho} = (\xi, \eta, p, 0, a, b, c, d)$, we have from Proposition 3.8 that if $q_1 \leq q_2$ a.e. in (ξ, η) then $\mu_1 \leq \mu_2$. Now, suppose that $q_1 < q_2$ in a subset of positive measure, and for $i = 1, 2$, let ϕ_i be the eigenfunction associated with λ_i having a sequence of zeros $(z_j^i)_{j=0}^{j=k}$. We distinguish two cases.

(i) $z_j^1 = z_j^2 = 0$ for all $j \in \{1, \dots, k-1\}$: In this case , for all j we have

$$\begin{aligned} &\int_{z_j^2}^{z_{j+1}^2} -\phi_2(\phi_1^{[p]})' + \phi_1(\phi_2^{[p]})' + \int_{z_j^2}^{z_{j+1}^2} (q_1 - q_2)\phi_1\phi_2 \\ &= \int_{z_j^2}^{z_{j+1}^2} (q_1 - q_2)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{z_j^2}^{z_{j+1}^2} m\phi_1\phi_2. \end{aligned} \tag{3.8}$$

Let $j_1 \in \{1, \dots, k-1\}$ be such that $\text{meas}(\{q_2 > q_1\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$. Then from (3.8) we have

$$0 > \int_{z_j^2}^{z_{j+1}^2} (q_1 - q_2)\phi_1\phi_2$$

$$= (\lambda_1 - \lambda_2) \int_{z_j^2}^{z_{j+1}^2} m\phi_1\phi_2$$

leading to $\lambda_2 > \lambda_1$.

(ii) $z_{j_0}^1 \neq z_{j_0}^2$ for some j_0: In this case set $k_1 = \max\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\}$. If $z_{k_1+1}^2 < z_{k_1+1}^1$ then we have

$$\begin{aligned} 0 &> \int_{z_{k_1}^2}^{z_{k_1+1}^2} -\phi_2(\phi_1^{[p]})' + \phi_1(\phi_2^{[p]})' \\ &= (\lambda_1 - \lambda_2) \int_{z_{k_1}^1}^{z_{k_1+1}^1} m\phi_1\phi_2 - \int_{z_{k_1}^1}^{z_{k_1+1}^1} (q_1 - q_2)\phi_1\phi_2 \end{aligned}$$

proving that $\lambda_2 > \lambda_1$ and if $z_{k_1+1}^1 < z_{k_1+1}^2$, then considering the families $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ with $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$, we obtain from Lemma 2.7 that there exists two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z_{k_1+m}^1 < \eta_n = z_{k_1+n}^2 < \eta_{n+1} = z_{k_1+n+1}^2 \leq \xi_{m+1} = z_{k_1+m+1}^1.$$

As above, we have

$$\begin{aligned} 0 &> \int_{\eta_n}^{\eta_{n+1}} -\phi_2(\phi_1^{[p]})' + \phi_1(\phi_2^{[p]})' \\ &= (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2 - \int_{\eta_n}^{\eta_{n+1}} (q_1 - q_2)\phi_1\phi_2 \end{aligned}$$

proving that $\lambda_2 > \lambda_1$. This proof is complete. □

Proposition 3.12. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$, $(q_n) \subset L_\rho^1$ and $(m_n) \subset K_\rho^*$ such that $q_n \to q$ and $m_n \to m$ in L_ρ^1. Set $\rho_n = (\xi, \eta, p, q_n, a, b, c, d)$. Then for all $\alpha, \beta \in L_\rho^1$, $k \geq 1$ and $\kappa = +, -$, we have $\lim_{n\to\infty} \lambda_k^\kappa(\rho_n, m_n, \alpha, \beta) = \lambda_k^\kappa(\rho, m, \alpha, \beta)$.*

Proof. **Step 1.** In this first step we fix m in K_ρ^*, α, β in L_ρ^1, the integer $k \geq 1$ and $\kappa = +, -$ and we prove the continuity of the mapping $q \to \lambda_k^\kappa(\rho(q), m, \alpha, \beta)$ on L_ρ^1. Let $\bar\lambda > 0$ such that $\lambda_k^\kappa(\bar\rho, m, \alpha, \beta) > 0$ for all $k \geq 1$, where $\bar\rho = (\xi, \eta, p, q + \bar\lambda m, a, b, c, d)$ and let $\lambda_n = \lambda_k^\kappa(\bar\rho_n, m, \alpha, \beta)$ and $\lambda = \lambda_k^\kappa(\bar\rho, m, \alpha, \beta)$, where $\bar\rho_n = (\xi, \eta, p, q_n + \bar\lambda m, a, b, c, d)$. Since $\lambda = \lambda_k^\kappa(\rho, m, \alpha, \beta) + \bar\lambda$ and $\lambda_n = \lambda_k^\kappa(\rho_n, m, \alpha, \beta) + \bar\lambda$, we have to show that $\lim \lambda_n = \lambda$. We claim now, that the sequence (λ_n) is bounded. Indeed, if this is not the case and there is a subsequence denoted also for convenience by (λ_n) such that $\lim_{n\to+\infty} |\lambda_n| = \infty$. We have then from [23, Proposition 4.11], that there is a function $\tilde q \in K_\rho^*$ and a subsequence (q_{n_l}) such that $|q_{n_l}| \leq \tilde q$. Thus, from Proposition 3.11 we have

$$\lambda_k^\kappa(\bar\rho_-, m, \alpha, \beta) \leq \lambda_{n_l} = \lambda_k^\kappa(\bar\rho_{n_l}, m, \alpha, \beta) \leq \lambda_k^\kappa(\bar\rho_+, m, \alpha, \beta)$$

where for $\nu = +, -$, $\bar\rho_\nu = (\xi, \eta, p, \nu\tilde q + \bar\lambda m, a, b, c, d)$, contradicting $\lim |\mu_{n_l}| = \infty$.

Now, let ϕ_n, ϕ be the normalized eigenfunctions associated respectively with λ_n and λ and note that $G_{\bar\rho}$ exists. Then we have

$$\phi = \lambda L_m\phi + \Phi(\phi), \quad \phi_n = \lambda_n L_m\phi_n + \Phi(\phi_n) + L_n\phi_n$$

where $L_m, L_n, \Phi : C_{\bar\rho} \to C_{\bar\rho}$ are defined by:

$$\Phi(u)(t) = \int_\xi^\eta G_{\bar\rho}(t,s)(\alpha(s)u^+(s) - \beta(s)u^-(s))ds,$$
$$L_m u(t) = \int_\xi^\eta G_{\bar\rho}(t,s)m(s)u(s)ds \text{ and}$$
$$L_n u(t) = \int_\xi^\eta G_{\bar\rho}(t,s)(q(s) - q_n(s))u(s)ds.$$

Let $\lambda_+ = \limsup \lambda_n$ and $\lambda_- = \liminf \lambda_n$, we obtain from the compactness of the operators L_m, Φ and the fact that $L_n \to 0$ in operator norm, that there exist $\psi_+, \psi_- \in \overline{S_\rho^{k,\kappa}}$ such that

$$\psi_+ = \lambda_+ L_m \psi_+ + \Phi(\psi_+), \quad \psi_- = \lambda_- L_m \psi_- + \Phi(\psi_-).$$

At the end by Theorem 2.2 we conclude that $\psi_+, \psi_- \in S_\rho^{k,\kappa}$ and the uniqueness of the half-eigenvalue leads to $\lim \lambda_n = \lambda_+ = \lambda_- = \lambda$.

Step 2. We prove the proposition, we denote $\lambda_n = \lambda_k^\kappa(\rho_n, m_n, \alpha, \beta)$ and $\lambda = \lambda_k^\kappa(\rho, m, \alpha, \beta)$ where $\rho_n = (\xi, \eta, p, q_n, a, b, c, d)$. We claim that the sequence (λ_n) is bounded. Indeed, if this is not the case, and there is a subsequence denoted also for convenience by (λ_n) such that $\lim_{n\to+\infty} |\lambda_n| = \infty$. Let ϕ_n, ϕ be the normalized eigenfunctions associated respectively with λ_n and λ, we have

$$\pounds_{\rho_n}\phi_n - \mu_n m_n \phi_n = \alpha\phi_n^+ - \beta\phi_n^- \quad \text{in } (\xi,\eta) \text{ a.e.},$$
$$B_\rho^l \phi_n = B_\rho^r \phi_n = 0$$

and

$$\pounds_\rho \phi - \mu m \phi = \alpha\phi^+ - \beta\phi^- \quad \text{in } (\xi,\eta) \text{ a.e.},$$
$$B_\rho^l \phi = B_\rho^r \phi = 0,$$

from which we obtain

$$\begin{gathered} \lambda_k^\kappa(\tilde\rho_n, m_n, \alpha, \beta) = \lambda_k^\kappa(\tilde\rho, m, \alpha, \beta) = 0, \\ \text{with } \tilde\rho_n = (\xi, \eta, p, q_n - \mu_n m_n, a, b, c, d) \text{ and} \\ \tilde\rho = (\xi, \eta, p, q - \mu m, a, b, c, d). \end{gathered} \tag{3.9}$$

Suppose now, that $\lim \lambda_n = -\infty$ and let $\omega > -\lambda$. There exists $n_0 \in \mathbb{N}$ such that $-\mu_n \ge \omega$ for all $n \ge n_0$ and we have

$$0 = \lambda_k^\kappa(\tilde\rho_n, m, \alpha, \beta) = \lambda_k^\kappa(q_n - \mu_n m_n) \ge \lambda_k^\kappa(q_n + \omega m_n) \quad \text{for all } n \ge n_0.$$

This together with Proposition 3.11 leads to the contradiction

$$\begin{aligned} 0 &= \lambda_k^\kappa(\tilde\rho, m, \alpha, \beta) = \lambda_k^\kappa(q - \mu m) \\ &< \lambda_k^\kappa(q + \omega m) = \lim \lambda_k^\kappa(q_n + \omega m_n) \le 0. \end{aligned}$$

Similarly, if $\lim \mu_n = +\infty$ and $\omega > \mu$, there exists $n_0 \in \mathbb{N}$ such that $\mu_n \ge \omega$ for all $n \ge n_0$ and we have

$$0 = \lambda_k^\kappa(\tilde\rho_n, m, \alpha, \beta) = \lambda_k^\kappa(q_n - \mu_n m_n) \le \lambda_k^\kappa(q_n - \omega m_n) \quad \text{for all } n \ge n_0.$$

This, and Proposition 3.11, leads to the contradiction

$$\begin{aligned} 0 &= \lambda_k^\kappa(\tilde\rho, m, \alpha, \beta) = \lambda_k^\kappa(q - \mu m) \\ &> \lambda_k^\kappa(q - \omega m) = \lim \lambda_k^\kappa(q_n - \omega m_n) \ge 0. \end{aligned}$$

At this stage let $\lambda_+ = \limsup \lambda_n$ and $\lambda_- = \liminf \lambda_n$. From (3.9) we obtain

$$\begin{aligned}
\lambda_k^\kappa(\tilde\rho_+, m, \alpha, \beta) &= \lambda_k^\kappa(q - \mu_+ m) = 0\\
\lambda_k^\kappa(\tilde\rho_-, m, \alpha, \beta) &= \lambda_k^\kappa(q - \mu_- m) = 0\\
\tilde\rho_+ &= (\xi, \eta, p, q - \mu_+ m, a, b, c, d)\\
\tilde\rho_- &= (\xi, \eta, p, q - \mu_- m, a, b, c, d),
\end{aligned}$$

and uniqueness of the eigenvalue $\mu = \mu_k(\rho, m)$ leads to $\lim \mu_n = \mu_+ = \mu_- = \mu$, completing the proof. □

Proposition 3.13. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$, $(\alpha_n) \subset L_\rho^1$ and $(\beta_n) \subset K_\rho^*$ such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in L_ρ^1. Then for all $k \geq 1$ and $\kappa = +, -$, we have $\lim_{n\to\infty} \lambda_k^\kappa(\rho, m, \alpha_n, \beta_n) = \lambda_k^\kappa(\rho, m, \alpha, \beta)$.*

Proof. Fix the integer $k \geq 1$ and $\kappa = +, -$ and let $\bar\lambda > 0$ such that $\lambda_k^\kappa(\bar\rho, m, \alpha, \beta) > 0$ for all $k \geq 1$, where $\bar\rho = (\xi, \eta, p, q + \bar\lambda m, a, b, c, d)$. Set $\lambda_n = \lambda_k^\kappa(\bar\rho, m, \alpha_n, \beta_n)$ and $\lambda = \lambda_k^\kappa(\bar\rho, m, \alpha, \beta)$, since $\lambda = \lambda_k^\kappa(\rho, m, \alpha, \beta) + \bar\lambda$ and $\lambda_n = \lambda_k^\kappa(\rho, m, \alpha_n, \beta_n) + \bar\lambda$, we have to show that $\lim \lambda_n = \lambda$. We claim now, that the sequence (λ_n) is bounded. Indeed, if this is not the case and there is a subsequence, denoted also for convenience by (λ_n), such that $\lim_{n\to+\infty} |\lambda_n| = \infty$, we have then from [23, Proposition 4.11], that there are two functions $\tilde\alpha, \tilde\beta \in K_\rho^*$ and subsequences (α_{n_l}), (β_{n_l}) such that $|\alpha_{n_l}| \leq \tilde\alpha$ and $|\beta_{n_l}| \leq \tilde\beta$. Thus, we have from Proposition 3.8 that

$$\lambda_k^\kappa(\bar\rho, m, \tilde\alpha, \tilde\beta) \leq \lambda_{n_l} = \lambda_k^\kappa(\bar\rho_{n_l}, m, \alpha_{n_l}, \beta_{n_l}) \leq \lambda_k^\kappa(\bar\rho, m, -\tilde\alpha, -\tilde\beta)$$

contradicting $\lim |\lambda_{n_l}| = \infty$.

Now, let ϕ_n, ϕ be the normalized eigenfunctions associated respectively with λ_n and λ and note that $G_{\bar\rho}$ exists. Then we have

$$\begin{aligned}
\phi_n &= \lambda_n L_m \phi_n + L_{\alpha_n} I^+(\phi_n) - L_{\beta_n} I^-(\phi_n)\\
&= \lambda_n L_m \phi_n + L_\alpha I^+(\phi_n) - L_\beta I^-(\phi_n) + (L_{\alpha_n} - L_\alpha) I^+(\phi_n) - (L_{\beta_n} - L_\beta) I^-(\phi_n)
\end{aligned}$$

where for $\chi \in L_\rho^1$, $L_\chi : C_{\bar\rho} \to C_{\bar\rho}$ is defined by $L_\chi u(t) = \int_\xi^\eta G_{\bar\rho}(t, s) m(s) u(s) ds$.

Let $\lambda_+ = \limsup \lambda_n$ and $\lambda_- = \liminf \lambda_n$, we obtain from the compactness of the operators L_m, L_α, L_β, and the fact that $(L_{\alpha_n} - L_\alpha), (L_{\beta_n} - L_\beta) \to 0$ in operator norm, that there exist $\psi_+, \psi_- \in \overline{S_\rho^{k,\kappa}}$ such that

$$\begin{aligned}
\psi_+ &= \lambda_+ L_m \psi_+ + L_\alpha I^+(\psi_+) - L_\beta I^-(\psi_+),\\
\psi_- &= \lambda_- L_m \psi_- + L_\alpha I^+(\psi_-) - L_\beta I^-(\psi_-).
\end{aligned}$$

At the end we conclude by Theorem 2.2 that $\psi_+, \psi_- \in S_\rho^{k,\kappa}$ and the uniqueness of the half-eigenvalue leads to $\lim \lambda_n = \lambda_+ = \lambda_- = \lambda$. This concludes the proof. □

Taking $\alpha = \beta = 0$ in (3.1), we obtain from Theorem 3.10 the following corollary which is an improvement of [41, Theorem 4.9.1].

Corollary 3.14. *For all $\rho \in \Delta$ and $m \in K_\rho^*$, BVP (2.9) admits an increasing sequences of eigenvalues $(\mu_k(\rho, m))_{k\geq 1}$ such that*

1. *$\lim \mu_k(\rho, m) = +\infty$,*
2. *$\mu_k(\rho, m)$ is simple,*
3. *If ϕ_k is an eigenvalue associated with $\mu_k(\rho, m)$, then $\phi_k \in S_\rho^k$.*

From Theorem 3.10 and Proposition 3.9 we obtain the following property for eigenvalues of (2.9).

Proposition 3.15. *Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m_1, m_2 \in K_\rho^*$ and assume that $m_1 \le m_2$ a.e. in (ξ, η) and $m_1 < m_2$ in a subset of positive measure. If for some integer $k \ge 1$, either $\mu_k(\rho, m_1) \ge 0$ or $\mu_k(\rho, m_2) \ge 0$, then $\mu_k(\rho, m_1) > \mu_k(\rho, m_2) \ge 0$.*

At the end of this section, we consider for $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$ and $h \in L^1(\xi, \eta)$ the BVP

$$\begin{gathered} \pounds_\rho u = \mu m u + h \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered} \tag{3.10}$$

where μ is a real parameter. The following result is an extension of what is known as the Fredholm alternative.

Theorem 3.16. *For all $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $m \in K_\rho^*$ and $h \in L_\rho^1$, BVP (3.10) admits*

1. *a unique solution if $\mu \neq \mu_k(\rho, m)$,*
2. *no solution if $\mu = \mu_{k_0}(\rho, m)$ for some integer $k_0 \ge 1$ and $\int_\xi^\eta \phi_{k_0} h \neq 0$,*
3. *infinitely many solutions if $\mu = \mu_{k_0}(\rho, m)$, for some integer $k_0 \ge 1$ and $\int_\xi^\eta \phi_{k_0} h = 0$.*

Proof. Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$.

(1) If $\mu \neq \mu_k(\rho, m)$ for all $k \ge 1$, then 0 is the unique solution to the BVP

$$\begin{gathered} (\pounds_\rho - \mu m) u = 0 \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered}$$

Thus, we have from Assertion 4 in Theorem 2.11, $u(t) = \int_\xi^\eta G_{\widetilde{\rho}}(t, s) h(s) ds$ is the unique solution to (3.10), where $\widetilde{\rho} = (\xi, \eta, p, q - \mu m, a, b, c, d)$.

(2) Suppose that $\mu = \mu_{k_0}(\rho, m)$ for some integer $k_0 \ge 1$ and let ϕ_{k_0} be the eigenfunction associated with $\mu = \mu_{k_0}(\rho, m)$. Therefore, if u satisfies (3.10), then

$$0 = \int_\xi^\eta \pounds_\rho u - u \pounds_\rho \phi_{k_0} = \int_\xi^\eta \phi_{k_0} h.$$

This proves that if $\int_\xi^\eta \phi_{k_0} h \neq 0$ then (3.10) has no solution.

(3) Now, suppose that $\int_\xi^\eta \phi_{k_0} h = 0$ and let ψ be such that $\{\phi_{k_0}, \psi\}$ form a fundamental system for the differential equation $(\pounds_\rho - m) u = 0$. Then $Wr = \phi_{k_0} \psi^{[p]} - \psi \phi_{k_0}^{[p]}$ is constant on (ξ, η) and $B_\rho^l \psi B_\rho^r \psi \neq 0$. Therefore, for all $\sigma \in \mathbb{R}$, the function

$$u(t) = \Big(\sigma + \frac{1}{Wr} \int_\xi^t h(s) \psi(s) ds\Big) \phi_{k_0} + \Big(\frac{1}{Wr} \int_\xi^t h(s) \phi_{k_0}(s) ds\Big) \psi(t)$$

solves (3.10). The proof is complete. □

Now for $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ consider the BVP

$$\begin{gathered} \pounds_\rho u = \lambda(\alpha u^+ - \beta u^-) \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered} \tag{3.11}$$

where $\alpha, \beta \in K_\rho^*$. Note that the nonlinearity in (3.11) is the same as in (3.1), positively 1-homogeneous and so we can define the concept of half-eigenvalue as it is done in Definition 3.1. In [4], the authors proved in the case, where $q = 0$, $a, -b, c, d \in [0, +\infty)$ with $\Delta = ad + ac\int_\xi^\eta \frac{d\tau}{p(\tau)} - bc > 0$ and $\alpha, \beta \in K_\rho^*$, that (3.11) admits two sequences of half-eigenvalues having the same properties as that in Theorem 3.10. At the end of this section, we prove that Theorem 3.10 holds for (3.11).

Theorem 3.17. *For all $\rho \in \Delta$, and $\alpha, \beta \in K_\rho^*$ with $\alpha\beta \in K_\rho^*$, BVP* (3.11) *admits two increasing sequences of simple half-eigenvalues $(\lambda_k^+(\rho, m, \alpha, \beta))_{k\geq 1}$ and $(\lambda_k^-(\rho, m, \alpha, \beta))_{k\geq 1}$, such that for all integer $k \geq 1$ and $\kappa = +, -$, the corresponding half-line of solutions lies in $\{\lambda_k^\kappa(\rho, m, \alpha, \beta)\} \times S_\rho^{k,\kappa}$ and $\lim_{k\to+\infty} \lambda_k^\kappa(\rho, m, \alpha, \beta) = +\infty$. Furthermore, aside from these solutions and the trivial one, there are no other solutions of* (3.11).

Proof. Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and m be an arbitrary function in K_ρ^* and consider the BVP

$$\begin{gathered} \pounds_\rho u = \lambda m u + \theta\alpha u^+ - \theta\beta u^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered}$$

where θ is a real parameter.

Fix $k \geq 1$ and $\kappa = +, -$ and set $\lambda(\theta) = \lambda_k^\kappa(\rho, m, \theta\alpha, \theta\beta)$. Note that because of Proposition 3.8, the mapping $\lambda(\cdot)$ is non-increasing and if for some $\theta_0 \in \mathbb{R}$, $\lambda(\theta_0) = 0$, then θ_0 is a half-eigenvalue of (3.11) having an eigenfunction in $S_\rho^{k,\kappa}$. Therefore, we have to prove that $\lim_{\theta\to-\infty} \lambda(\theta) = +\infty$ and $\lim_{\theta\to+\infty} \lambda(\theta) = -\infty$. Moreover, since $\lambda(\theta) \leq \lambda_k^\kappa(\rho, m, \theta\psi, \theta\psi)$ for $\theta < 0$ and $\lambda(\mu) \geq \lambda_k^\kappa(\rho, m, \theta\psi, \theta\psi)$ for $\theta \geq 0$, where $\psi = \sup(\alpha, \beta)$, we have to check that $\lim_{\theta\to-\infty} \mu_k(\theta) = +\infty$ and $\lim_{\theta\to+\infty} \mu_k(\theta) = -\infty$, where $\mu_k(\theta) = \lambda_k^\kappa(\rho, m, \theta\psi, \theta\psi) = \mu_k(\rho(\theta), m)$ and $\rho(\theta) = (\xi, \eta, p, q - \theta\psi, a, b, c, d)$. We present in what follows the proof of $\lim_{\theta\to-\infty} \mu_k(\theta) = +\infty$, the other limit is checked similarly.

To the contrary, suppose that $\lim_{\theta\to-\infty} \mu_k(\theta) = \mu^\infty < +\infty$ and let $\epsilon_0 > 0$ be fixed. There exists $\theta_0 > 0$ such that for all $\theta \leq -\theta_0$, $(\mu^\infty - \epsilon_0) < \mu_k(\theta) < \mu^\infty$. Let $\mu_0 = \mu_k(\rho_\infty, \psi)$ where $\rho_\infty = (\xi, \eta, p, q - (\mu^\infty - \epsilon_0)m, a, b, c, d)$ and $\phi, \phi_\theta \in S_\rho^k$ such that

$$\begin{gathered} \pounds_\rho \phi - (\mu^\infty - \epsilon_0)m\phi - \mu_0\psi\phi = 0 \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l \phi = B_\rho^r \phi = 0. \end{gathered} \tag{3.12}$$

For $\theta > \max(\theta_0, \mu_0)$ let $\phi_\theta \in S_\rho^k$ be such that

$$\begin{gathered} \pounds_\rho \phi_\theta - \mu_k(\theta)m\phi_\theta - \theta\psi\phi_\theta = 0 \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B_\rho^l \phi_\theta = B_\rho^r \phi_\theta = 0. \end{gathered} \tag{3.13}$$

Then from (3.12) and (3.13) we have $\mu_k(\tilde{\rho}_{\epsilon_0}, m) = 0 = \mu_k(\tilde{\rho}(\theta), m)$ where $\tilde{\rho}_{\epsilon_0} = (\xi, \eta, p, q - (\mu^\infty - \epsilon_0)m - \mu_0\psi, a, b, c, d)$ and $\tilde{\rho}(\theta) = (\xi, \eta, p, q - \mu_k(\theta)m - \theta\psi, a, b, c, d)$. Since $(\mu^\infty - \epsilon_0)m + \mu_0\psi \leq \mu_k(\theta)m + \theta\psi$ a.e. in (ξ, η) and $(\mu^\infty - \epsilon_0)m + \mu_0\psi < \mu_k(\theta)m + \theta\psi$ in a subset of positive measure; from Proposition 3.11 we have the contradiction

$$0 = \mu_k(\tilde{\rho}_{\epsilon_0}, m) > \mu_k(\tilde{\rho}(\theta), m) = 0.$$

Since Proposition 3.13 guarantees that $\lambda(\cdot)$ is a continuous function, we conclude that there exists θ_k^κ such that $\lambda(\theta_k^\kappa) = 0$; namely, θ_k^κ is a half-eigenvalue of BVP (3.11) having an eigenfunction in $S_\rho^{k,\kappa}$. This completes the proof. □

4. Bifurcation diagram for an asymptotically linear Sturm-Liouville BVP

Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and $m \in K_\rho^*$ and consider in this section, the BVP

$$\begin{gathered} \pounds_\rho u = \lambda m u + u f(t,u), \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered} \tag{4.1}$$

where λ is a real parameter and $f : (\xi,\eta)\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

We assume throughout this section that

$$f(t,0) = 0 \text{ a.e. } t \in (\xi,\eta), \tag{4.2}$$

and also that there exist $\alpha, \beta, \gamma \in K_\rho^*$ such that

$$\lim_{u\to-\infty} f(t,u) = \beta(t) \quad \text{a.e. } t \in (\xi,\eta), \tag{4.3}$$

$$\lim_{u\to+\infty} f(t,u) = \alpha(t) \quad \text{a.e. } t \in (\xi,\eta), \tag{4.4}$$

$$|f(t,u)| \le \gamma(t) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } t \in (\xi,\eta. \tag{4.5}$$

For the statement of the main result of this section and its proof, it is useful to introduce the following notation. For $k \ge 1$ and $\kappa = +,-$, denote $\lambda_k^\kappa = \lambda_k^\kappa(\rho, m, \alpha, \beta)$ and $\mu_k = \mu_k(\rho, m)$. Without loss of generality, assume that $\mu_k \ne 0$ for all $k \ge 1$ (otherwise consider $\widetilde{\rho} = (\xi, \eta, p, q + Am, a, b, c, d)$ with A sufficiently large). Thus, G_ρ exists and $(\lambda, u) \in \mathbb{R} \times \tilde{W}_\rho$ is a solution to (4.1) if and only if $u = T(\lambda, u)$, where $T : \mathbb{R} \times C_\rho \to C_\rho$ is defined by $T = i \circ L_\rho \circ F$, $F : \mathbb{R} \times C_\rho \to L_\rho^1$ is the Nymetski operator defined for $u \in C_\rho$ by $F(\lambda, u)(t) = \lambda m(t)u(t) + u f(t,u)$, and i is the compact embedding of $\tilde{W}_\rho$ in C_ρ.

Let $H, K : C_\rho \to C_\rho$ be defined by $Hu(t) = \int_\xi^\eta G_\rho(t,s)u(s)f(s,u(s))ds$ and $Ku(t) = \int_\xi^\eta G_\rho(t,s)\widetilde{f}(s,u(s))ds$, where $\widetilde{f}(s,u) = uf(s,u) - \alpha(s)u^+ + \beta(s)u^-$. Then we have

$$\begin{gathered} T(\lambda,u) = \lambda L_m u + Hu, \\ T(\lambda,u) = \lambda L_m u + L_\alpha I^+ u - L_\beta I^- u + Ku \end{gathered} \tag{4.6}$$

where for $\chi \in L_\rho^1$, $L_\chi : C_\rho \to C_\rho$ is defined by $L_\chi u(t) = \int_\xi^\eta G_\rho(t,s)\chi(s)u(s)ds$. Clearly, L_χ is compact for all $\chi \in L_\rho^1$, and H and K are completely continuous.

Lemma 4.1. *Assume that* (4.2) *and* (4.5) *hold. Then* $H(u) = o(\|u\|)$ *near* 0.

Proof. Let $(u_n) \subset C_\rho$ be such $\lim \|u_n\| = 0$. Because of the inequality

$$|Hu_n(t)|/\|u_n\| \le \int_\xi^\eta R_n(s)ds, \text{ where } R_n(s) = \|G_\rho\|_\infty |f(s,u_n(s))|,$$

it suffices to prove that $\int_\xi^\eta R_n(s)ds \to 0$ as $n \to \infty$.

Hypothesis (4.3) implies that $R_n(s) \to 0$ as $n \to +\infty$, a.e. $s \in (\xi,\eta)$ and Hypothesis (4.5) implies

$$R_n(s) = \|G_\rho\|_\infty |f(s,u_n(s))| \le \|G_\rho\|_\infty \gamma(s) \quad \text{a.e. } s \in (\xi,\eta).$$

Thus, by the Lebesgue dominated convergence theorem, we conclude that $H(u) = o(\|u\|)$ at 0. □

Lemma 4.2. *Assume that* (4.3)–(4.5) *hold. Then* $K(u) = o(\|u\|)$ *near* ∞.

Proof. Let $(u_n) \subset C_\rho$ be such $\lim \|u_n\| = \infty$. Because of the inequality

$$|Ku_n(t)|/\|u_n\| \leq \int_\xi^\eta R_n(s)ds,$$

where

$$P_n(s) = \|G_\rho\|_\infty |\frac{u_n(s)}{\|u_n\|} f(s, u_n(s)) - \alpha(s)\frac{u_n^+(s)}{\|u_n\|} + \beta(s)\frac{u_n^-(s)}{\|u_n\|}|,$$

it suffices to prove that $\int_\xi^\eta P_n(s)(s)ds \to 0$ as $n \to \infty$.

From (4.5) we have

$$P_n(s) = \|G_\rho\|_\infty(\gamma(s) + \alpha(s) + \beta(s)) \quad \text{a.e. } s \in (\xi, \eta).$$

It remains to prove that $\lim P_n(s) = 0$ for a.e. $s \in (\xi, \eta)$. Let $s \in (\xi, \eta)$. We distinguish the following cases:

(i) $\lim u_n(s) = +\infty$: In this case,

$$P_n(s) \leq \|G_\rho\|_\infty |(f(s, u_n(s))) - \alpha(s)| \to 0 \quad \text{as } n \to +\infty.$$

(ii) $\lim u_n(s) = -\infty$: in this case,

$$P_n(s) \leq \|G_\rho\|_\infty |(f(s, u_n(s))) - \beta(s)| \to 0 \quad \text{as } n \to +\infty.$$

(iii) $\lim u_n(s) \neq \pm\infty$: in this case there may exist subsequences $(u_{n_k^1}(s))$ and $(u_{n_k^2}(s))$ such that $(u_{n_k^1}(s))$ is bounded and $\lim u_{n_k^2}(s) = \pm\infty$. Arguing as in the above two cases we get $\lim P_{n_k^2}(s) = 0$ and we have

$$P_{n_k^1}(s) \leq G(s,s)(\gamma(s) + \delta(s) + \alpha(s) + \beta(s))\big(|u_{n_k^1}(s)|/\|u_{n_k^1}\|\big) \to 0 \quad \text{as } k \to +\infty.$$

Thus, we have $\lim P_n(s) = 0$ for a.e. $s \in (\xi, \eta)$. By the Lebesgue dominated convergence theorem, we conclude that $Ku_n = o(\|u_n\|)$ near ∞. □

Theorem 4.3. *Assume that* (4.2) *and* (4.5) *hold. Then for all integers* $k \geq 1$ *and* $\kappa = +, -$, *BVP* (4.1) *admits an unbounded component* ζ_k^κ *of solutions bifurcating from* $(\mu_k, 0)$ *such that* $\zeta_k^\kappa \subset \mathbb{R} \times S_\rho^{k,\kappa}$. *Moreover, if* (4.3) *and* (4.4) *hold, then* ζ_k^κ *rejoins the point* $(\lambda_k^\kappa, \infty)$.

Proof. **Step 1.** Note that the set of characteristic values of L_m consists of the sequence $(\mu_k)_{k\geq 1}$. So, we need to prove that for all integers $k \geq 1, \mu_k$ is algebraically simple. Choose $u \in \mathcal{N}\big((\mu_k L_m - I)^2\big)$ and set $v = (\mu_k L_m - I)(u) = \mu_k L_m u - u$. We have $\mu_k L_m v - v = 0$ and the geometric simplicity of μ_k implies $v = x\phi_k$, and then $\mu_k L_m u - u = x\phi_k$ where $\phi_k \in S_\rho^{k,+}$ is the normalized eigenfunction associated with μ_k. In other words, we have that u satisfies the BVP

$$\begin{gathered} \pounds_\rho u = \mu_k m u - x\mu_k m\phi_k \quad \text{in } \xi, \eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{4.7}$$

Multiplying the differential equation (4.7) by ϕ_k and integrating by parts on (ξ, η), we obtain $x\mu_k \int_\xi^\eta m\phi_k^2 = 0$ leading to $x = 0$ and $u = \omega\phi_k$ for some $\omega \in \mathbb{R}$.

Since $Hu_n = o(\|u_n\|)$ near 0, we conclude from [16, Theorem 2] that for all integer $k \geq 1, \mu_k$ is a bifurcation point of two components ζ_k^+ and ζ_k^- of non trivial solutions and either ζ_k^+ and ζ_k^- are unbounded or $\zeta_k^+ \cap \zeta_k^- \neq \{(\mu_k, 0)\}$. Moreover, we have from [34, Theorem 1.25 and Lemma 1.24] that, if $\epsilon > 0$ is sufficiently small and $(\lambda, u) \in \zeta_k^\kappa \cap B_\epsilon$, where $B_\epsilon = \{(\theta, v) \in \mathbb{R} \times C_\rho : |\theta| + \|v\| < \epsilon\}$, then $|\lambda - \mu_k| < \varsigma$

and $u = \alpha\phi_k + \omega$ where $\kappa\alpha > \kappa\|u\|_\infty$, $w = o(|\alpha|)$ near 0, $\varsigma > 0$, $\kappa \in (\xi, \eta)$ and $\kappa = +, -$. Thus, considering the fact that $S_\rho^{k,+}$ and $S_\rho^{k,-}$ are open sets, we obtain from $\lim_{\alpha\to 0}(u/\alpha) = \phi_k$ that $\zeta_k^\kappa \cap B_\epsilon \subset S_\rho^{k,\kappa}$ for $\kappa = +, -$. In fact, ζ_k^κ does not leave $S_\rho^{k,\kappa}$. Indeed, if this occurs then there will exist a pair $(\overline{\lambda}, \overline{u}) \in \zeta_k^\kappa$ such that $\overline{u} \in \partial S_\rho^{k,\kappa}$, and in this case, there is $\tau, \xi \le \tau \le \eta$ such that $\overline{u}(\tau) = \overline{u}^{[p]}(\tau)$ and then we have from Corollary 2.4, $\overline{u} = 0$ and $\overline{\lambda} = \mu_l(m)$ for some $l \neq k$. This is impossible since near $(\mu_l, 0)$ the possible solutions (λ, u) are in $\mathbb{R} \times S_\rho^{l,\kappa}$. Finally, we conclude from $\zeta_k^\kappa \subset S_\rho^{k,\kappa}$ that ζ_k^κ is unbounded.

Step 2. Now, assume that (4.3) and (4.4) hold and let us prove first that for all $k \ge 1$ and $\kappa = +, -$, the projection of ζ_k^κ onto the real axis is bounded. To this aim, for $\kappa = +, -$, let $\psi_{k,\kappa}$ be the eigenfunction associated with $\mu_k(\kappa\gamma) = \mu_k(\rho_{\kappa\gamma}, m)$ where $\rho_{\kappa\gamma} = (\xi, \eta, p, q + \kappa\gamma, a, b, c, d)$ and $(\lambda, u) \in \zeta_k^\kappa$. We have from Lemma 2.9 that there exist two intervals (ξ_1, η_1) and (ξ_2, η_2) such that $u\psi_{k,\kappa} \ge 0$ for $\kappa =, -$, $\int_{\xi_1}^{\eta_1} \psi_{k,+}\pounds_\rho u - u\pounds_\rho\psi_{k,+} \le 0$ and $\int_{\xi_2}^{\eta_2} \psi_{k,-}\pounds_\rho u - u\pounds_\rho\psi_{k,-} \ge 0$. We have then from Hypothesis (4.5),

$$
\begin{aligned}
0 &\ge \int_{\xi_1}^{\eta_1} \psi_{k,+}\pounds_\rho u - u\pounds_\rho\psi_{k,+} \\
&= \int_{\xi_1}^{\eta_1} (\lambda - \mu_k(\gamma))m\psi_{k,+}u + (f(s,u) + \gamma)u\psi_{k,+} \\
&\ge (\lambda - \mu_k(\gamma)) \int_{\xi_1}^{\eta_1} m\phi_k^+ u ds
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \int_{\xi_2}^{\eta_2} \psi_{k,-}\pounds_\rho u - u\pounds_\rho\psi_{k,-} \\
&= \int_{\xi_2}^{\eta_2} ((\lambda - \mu_k(-\gamma))m\psi_{k,-}u + (f(s,u) - \gamma)u\psi_{k,-})ds \\
&\le (\lambda - \mu_k(-\gamma)) \int_{\xi_2}^{\eta_2} m\psi_{k,-}u ds
\end{aligned}
$$

leading to $\mu_k(-\gamma) \le \lambda \le \mu_k(\gamma)$.

Step 3. Let (λ_n, u_n) be sequence in ζ_k^κ such that $\lim_{n\to\infty}\|u_n\|_\infty = +\infty$. Set $v_n = \frac{u_n}{\|u_n\|_\infty}$ and note that $\|v_n\| = 1$ and

$$
\begin{gathered}
\pounds_\rho v_n = \lambda_n m v_n + \alpha v_n^+ - \beta v_n^- + (\widetilde{f}(t, u_n)/\|u_n\|) \quad \text{in } (\xi, \eta) \text{ a.e.}, \\
a v_n(\xi) + b v_n^{[p]}(\xi) = c v_n(\eta) + d v_n^{[p]}(\eta) = 0.
\end{gathered}
$$

Clearly, the above equation is equivalent to the equation

$$
v_n = \lambda_n L_m v_n + L_\alpha I^+ v_n - L_\beta I^- v_n + (K u_n/\|u_n\|). \tag{4.8}
$$

Because of the compactness of L_m, L_α, L_β, boundedness of (λ_n), and the fact that $Ku = \circ(\|u\|)$ at ∞, we have, up to subsequences, $v_n \to v \in \overline{S_\rho^{k,\kappa}}$, and $\lambda_n \to \lambda$, and the pair (λ, v) satisfies

$$
\begin{gathered}
\pounds_\rho v = \lambda m v + \alpha v^+ - \beta v^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\
B_\rho^l v = B_\rho^r v = 0.
\end{gathered} \tag{4.9}
$$

Since $\|v\| = \lim \|v_n\| = 1$, from Theorem 2.2 we have $v \in S_\rho^{k,\kappa}$, and from (4.9) we conclude that $\lambda = \lambda_k^\kappa$. The proof is complete. $\square$

5. Multiplicity results for an asymptotically linear Sturm-Liouville BVP

Let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$ and consider the BVP

$$\begin{gathered} \pounds_\rho u = ug(t,u) \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered} \tag{5.1}$$

where $g : (\xi,\eta) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

The main result of this section will be obtained under the following conditions on the function g: There exist $m, \alpha, \beta, \gamma \in K_\rho^*$ such that

$$\begin{gathered} \lim_{u\to 0} g(t,u) = m(t) \quad \text{a.e. } t \in (\xi,\eta), \\ \lim_{u\to +\infty} g(t,u) = \alpha(t) \quad \text{a.e. } t \in (\xi,\eta), \\ \lim_{u\to -\infty} g(t,u) = \beta(t) \quad \text{a.e. } t \in (\xi,\eta), \\ |g(t,u)| \le \gamma(t) \text{a.e. } t \in (\xi,\eta)\,. \end{gathered} \tag{5.2}$$

Set $\varphi = \inf(\alpha,\beta), \psi = \sup(\alpha,\beta)$ and for $k \ge 1$, $\mu_k(m) = \mu_k(\rho,m)$, $\mu_k(\alpha) = \mu_k(\rho,\alpha)$, $\mu_k(\beta) = \mu_k(\rho,\beta)$, $\mu_k(\psi) = \mu_k(\rho,\psi)$ and $\mu_k(\varphi) = \mu_k(\rho,\varphi)$ if $\varphi \in K_\rho^*$.

Theorem 5.1. *Assume that* (5.2) *is fulfilled.*

(1) *If $\varphi \in K_\rho^*$ and there exist two integers $i \ge j \ge 1$ such that*
$$\mu_i(\varphi) < 1 < \mu_j(m), \tag{5.3}$$
then (5.1) *admits, in each of $S_\rho^{j,+}, \dots, S_\rho^{i,+}, S_\rho^{j,-}, \dots, S_\rho^{i,-}$, a solution.*

(2) *If there exist two integers $i \ge j \ge 1$ such that*
$$\mu_i(m) < 1 < \mu_j(\psi), \tag{5.4}$$
then (5.1) *admits, in each of $S_\rho^{j,+}, \dots, S_\rho^{i,+}, S_\rho^{j,-}, \dots, S_\rho^{i,-}$, a solution.*

(3) *If there exist two integers $i \ge j \ge 1$ with $i \ge 2j-1$ such that one of the situations* (5.5) *or* (5.6)*, where*
$$\mu_i(m) < 1 < \mu_j(\beta) \tag{5.5}$$
$$\mu_i(\beta) < 1 < \mu_j(m) \tag{5.6}$$
holds true, then (5.1) *admits, in each of $S_\rho^{2j,+}, \dots, S_\rho^{i,+}, S_\rho^{2j-1,-}, \dots, S_\rho^{i,-}$, a solution.*

(4) *If there exist two integer $i \ge j \ge 1$ with $i \ge 2j-1$ such that one of the situation* (5.7) *or* (5.8)*, where*
$$\mu_i(m) < 1 < \mu_j(\alpha) \tag{5.7}$$
$$\mu_i(\alpha) < 1 < \mu_j(m) \tag{5.8}$$
holds true, then (5.1) *admits, in each of $S_\rho^{2j-1,+}, \dots, S_\rho^{i,+}, S_\rho^{2j,-}, \dots, S_\rho^{i,-}$, a solution.*

Proof. Set $f(x,u) = g(x,u) - m(x)u$ and consider the BVP

$$\begin{gathered} \pounds_\rho u = \lambda m u + f(t,u) \qquad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{5.9}$$

Note that if $(1, u)$ is a solution to (5.9) then u is solution to (5.1). Let $(\lambda_l^+)_{l\geq 1}$ and $(\lambda_l^-)_{l\geq 1}$ be the sequences of half-eigenvalue of the problem

$$\mathcal{L}_\rho u = \lambda m u + (\alpha - m)u^+ - (\beta - m)u^- \quad \text{in } (\xi, \eta) \text{ a.e.},$$
$$B_\rho^l u = B_\rho^r u = 0.$$

Since the function f satisfies Hypotheses (4.2)–(4.5), from Theorem 4.3 we have that for all integers $k \geq 1$ and $\kappa = +, -$, the component ζ_k^κ of nontrivial solutions of (5.9), which bifurcate from $\mu_k(\rho, m)$, rejoins the point $(\lambda_k^\kappa, \infty)$. Thus we have to compute, for each of the Cases 1–4, the number of components ζ_k^κ crossing the hyperplane $\{1\} \times C_\rho$. To be brief, we present the proofs of Case 1 and Case 3 with $\mu_p(m) < 1 < \mu_j(\beta)$.

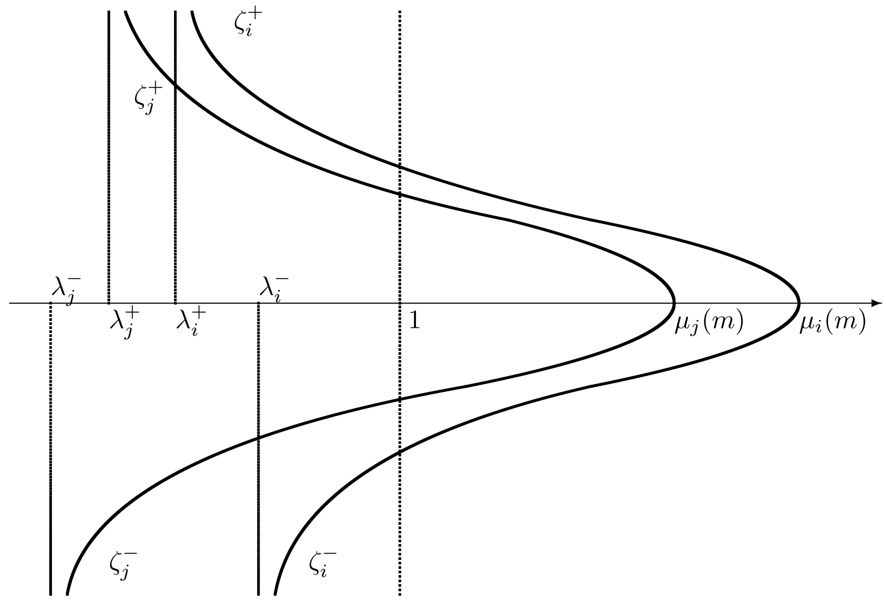

FIGURE 2. $\mu_i(\varphi) < 1 < \mu(m_j)$

(1) Suppose that $\mu_i(\rho, \varphi) < 1 < \mu_j(\rho, m)$ and let

$$\overline{\rho} = (\xi, \eta, p, q + m - \varphi, a, b, c, d)$$
$$\mu_i^* = \mu_i(\overline{\rho}, \varphi),$$
$$\widetilde{\rho} = (\xi, \eta, p, q + (1 - \mu_i^*)m, a, b, c, d).$$

Then we have

$$\lambda_i^\kappa = \lambda_i^\kappa(\rho, m, \alpha - m, \beta - m) \leq \lambda_i^\kappa(\rho, m, \varphi - m, \varphi - m) = \mu_i^*.$$

Let u be such that

$$\mathcal{L}_\rho u + (1 - \mu_i^*)mu = \varphi u \quad \text{in } (\xi, \eta) \text{ a.e.},$$
$$B_\rho^l u = B_\rho^r u = 0.$$

We conclude from the above BVP that $\mu_i(\widetilde{\rho},\varphi)=1$. Thus, if $\mu_i^*\geq 1$, from Proposition 3.8 we have the contradiction

$$1=\mu_i(\widetilde{\rho},\varphi)=\lambda_i^{\kappa}(\rho,\varphi,(\mu_i^*-1)m,(\mu_i^*-1)m)\leq \mu_i(\rho,\varphi,0,0)=\mu_i(\rho,\varphi)<1.$$

We have proved that for all integers $k\in\{j,\dots,i\}$ and $\kappa=+,-$, ζ_k^{κ} crosses the hyperplane $\{1\}\times C_\rho$ (see 2).

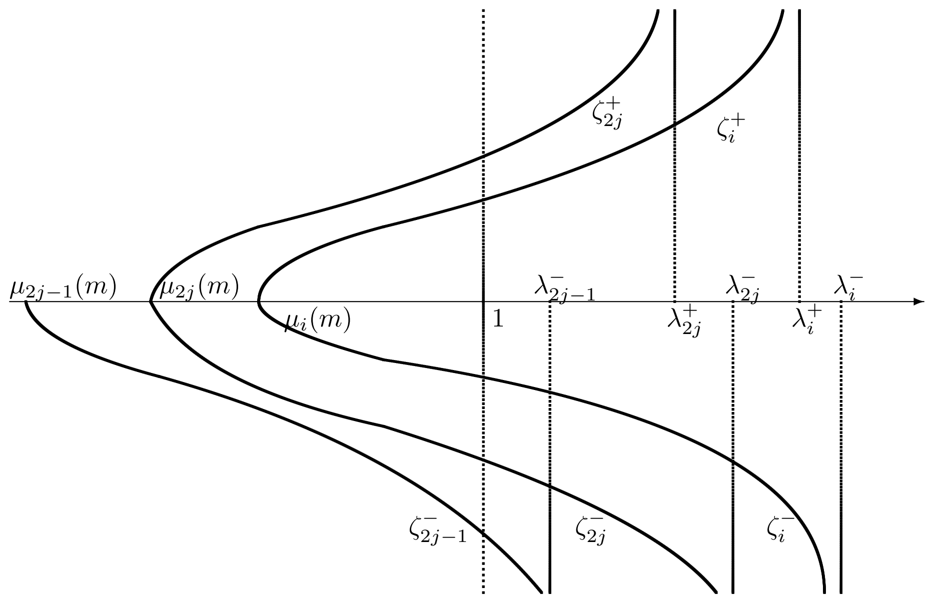

FIGURE 3. $\mu_i(m)<1<\mu_j(\beta)$

(2) Suppose that $\mu_i(m)<1<\mu_j(\beta)$. We claim also that $\lambda_{2j}^+>1$ and $\lambda_{2j-1}^->1$. Indeed if $\lambda_{2j}^+\leq 1$ (we check $\lambda_{2j-1}^->1$ in the same way) and u,v satisfy respectively

$$\begin{gathered}\pounds_\rho u=(\lambda_{2j}^+-1)mu+\alpha u^+-\beta u^-\quad \text{in } (\xi,\eta)\text{ a.e.},\\ B_\rho^l u=B_\rho^r u=0,\end{gathered}$$

and

$$\begin{gathered}\pounds_\rho v=\mu_j(\beta)\beta v\quad \text{in } (\xi,\eta)\text{ a.e.},\\ B_\rho^l v=B_\rho^r v=0,\end{gathered}$$

we let $(z_l)_{l=0}^{l=2j}$ be the sequence of zeros of u. We have for all $l=0,\dots,j-1$,

$$\begin{aligned}0&\leq Wr(u,v)(z_{2l+2})-Wr(u,v)(z_{2l+1})\\ &=\int_{z_{2l+1}}^{z_{2l+2}}((\lambda_{2j}^+-1)m+(1-\mu_j(\beta))\beta)uv.\end{aligned}$$

This equality implies that in each of the intervals $[z_{2l+1},z_{2l+2}]$, $l=0,\dots,j-2$, and $[z_{2j-1},\eta)$, v vanishes at least once. This means that v admits at least j zeros in (ξ,η), contradicting $v\in S_\rho^j$. Thus, we have proved that $\lambda_{2j}^+>1$. Thus, ζ_k^+ crosses the hyperplane $\{1\}\times C_\rho$ for all integers $k\in\{2j,\dots,i\}$, and ζ_k^- crosses the hyperplane $\{1\}\times C_\rho$ for all integers $k\in\{2j-1,\dots,i\}$ (see Figure 3). □

Now, consider the boundary value problem

$$\begin{gathered} \pounds_\rho u = \omega(t)uh(u) \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B^l_\rho u = B^r_\rho u = 0, \end{gathered} \tag{5.10}$$

where $\omega \in K^*_\rho$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{u\to 0} h(u) = h_0 > 0, \quad \lim_{u\to+\infty} h(u) = h_+ > 0, \quad \lim_{u\to-\infty} h(u) = h_- > 0. \tag{5.11}$$

Theorem 5.1 yields the following result.

Corollary 5.2. *Assume that* (5.11) *is fulfilled.*

(1) If there exist two integers $i \ge j \ge 1$ such that one of the following two conditions holds,

$$h_0 < \mu_j(\omega) < \mu_i(\omega) < \min(h_+, h_-), \tag{5.12}$$

$$\max(h_+, h_-) < \mu_j(\omega) < \mu_i(\omega) < h_0, \tag{5.13}$$

then (5.10) *admits, in each of the sets $S^{j,+}_\rho, \dots, S^{i,+}_\rho, S^{j,-}_\rho, \dots, S^{i,-}_\rho$, a solution.*

(2) If there exist two integers $i \ge j \ge 1$ with $i \ge 2(j-1)$ and such that one of the following two conditions holds,

$$h_- < \mu_j(\omega) < \mu_i(\omega) < h_0, \tag{5.14}$$

$$h_0 < \mu_j(\omega) < \mu_i(\omega) < h_-, \tag{5.15}$$

then (5.10) *admits, in each of the sets $S^{2j,+}_\rho, \dots, S^{i,+}_\rho, S^{(2j-1),-}_\rho, \dots, S^{i,-}_\rho$, a solution.*

(3) If there exist two integers $i \ge j \ge 1$ with $i \ge 2(j-1)$ and such that one of the two conditions holds,

$$h_+ < \mu_j(\omega) < \mu_i(\omega) < h_0, \tag{5.16}$$

$$h_0 < \mu_j(\omega) < \mu_i(\omega) < h_+, \tag{5.17}$$

then (5.10) *admits, in each of the sets $S^{(2j-1),+}_\rho, \dots, S^{i,+}_\rho, S^{2j,-}_\rho, \dots, S^{i,-}_\rho$, a solution.*

Proof. Set $g(t,u) = \omega(t)uh(u)$. Then condition (5.2) is satisfied for $m(t) = h_0\omega(t)$, $\alpha(t) = h_+\omega(t)$, $\beta(t) = h_-\omega(t)$. For all integers $i \ge 1$, we have

$$\begin{gathered} \mu_i(m) = \mu_i(\omega)/h_0, \mu_i(\alpha) = \mu_i(\omega)/h_+, \quad \mu_i(\beta) = \mu_i(\omega)/h_- \\ \mu_i(\varphi) = \mu_i(\omega)/\min(h_+, h_-), \quad \mu_i(\psi) = \mu_i(\omega)/\max(h_+, h_-). \end{gathered}$$

Therefore, Assertions 1, 2 and 3 of Corollary 5.2 follow from Assertions 1-4 of Theorem 5.1. □

Remark 5.3. Assertion 1 in Corollary 5.2 shows that Assertion 1 of Theorem 5.1 implies the case $0 < f_0, f_\infty < \infty$ of the [33, Theorems 2 and 3] and extends to a more general situation, since here the operator $-d^2/dx^2$ is replaced by the differential operator $\pounds_\rho$, f is not necessarily a separated variable function, no condition on the parity of f is imposed and f is not locally Lipschitzian. Theorem 5.1 extends in some manner, [30, Theorems 1 and 2 in], [31, Theorem 1.1] and [13, Theorem 3.3].

Example 5.4. Let $\rho = (0, \pi, 1, 0, 1, 0, 1, 0)$, $f_0, f_-, f_+ \in (0, +\infty)$, and let i, j, k be integers such that $1 \le j \le i \le k$. Consider the BVP

$$\begin{gathered} -u'' = f(u) \quad \text{in } (0,\pi) \\ u(0) = u(\pi) = 0 \end{gathered} \tag{5.18}$$

where

$$f(u) = f_0 u e^{-|u|} + \frac{f_+ u^2 e^u}{1 + |u| e^u} + \frac{f_- u^2 e^{-u}}{1 + |u| e^{-u}}.$$

We have

$$\lim_{u\to 0} \frac{f(u)}{u} = f_0, \quad \lim_{u\to -\infty} \frac{f(u)}{u} = f_-, \quad \lim_{u\to +\infty} \frac{f(u)}{u} = f_+.$$

We deduce from Corollary 5.2 the following results. (1) Suppose that

$$(j-1)^2 < f_0 < j^2 \le \dots \le i^2 < f_- < (i+1)^2 \le \dots \le k^2 < f_+ < (k+1)^2$$

and $k \ge 2(j-1)$. From Part 1 of Corollary 5.2 BVP (5.18) admits one solution in each of the sets $S_\rho^{j,+}, \dots, S_\rho^{i,+}, S_\rho^{j,-}, \dots, S_\rho^{i,-}$, and from Part 3 of Corollary 5.2, BVP (5.18) admits one solution in each of the sets $S_\rho^{(2j-1),+}, \dots, S_\rho^{k,+}, S_\rho^{2j,-}, \dots, S_\rho^{k,-}$. We conclude that: If $i < 2j-1$ then (5.18) admits $2k + 2i - 6j + 5$ solutions. If $i \ge 2j-1$ then (5.18) admits $2k - 2j + 2$ solutions.

(2) Suppose that

$$(j-1)^2 < f_- < j^2 \le \dots \le i^2 < f_0 < (i+1)^2 \le \dots \le k^2 < f_+ < (k+1)^2,$$

$k \ge 2i$ and $i \ge 2(j-1)$. From Part 2 of Corollary 5.2, BVP (5.18) admits one solution in each of the sets $S_\rho^{2j,+}, \dots, S_\rho^{i,+}, S_\rho^{(2j-1),-}, \dots, S_\rho^{i,-}$, and from Part **3** of Corollary 5.2, BVP (5.18) admits one solution in each of the sets $S_\rho^{(2i+1),+}, \dots, S_\rho^{k,+}, S_\rho^{2i+2,-}, \dots, S_\rho^{k,-}$. We conclude that (5.18) admits $2k - 2i - 4j + 2$ solutions.

6. Sturm-Liouville BVP with jumping nonlinearities

6.1. **General setting.** Throughout this section, we let $\rho = (\xi, \eta, p, q, a, b, c, d) \in \Delta$, $\alpha, \hat{\alpha}, \beta, \gamma, \omega \in K_\rho^*$, $h, \phi \in L_\rho^1$, θ is a real parameter, $\chi \in C^1(\mathbb{R})$, $\hat{g} : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function satisfying $\lim_{u\to+\infty} \hat{g}(u) = 0$ and $g : (\xi, \eta) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $\frac{\partial g}{\partial u}(t, \cdot)$ exists for a.e. $t \in (\xi, \eta)$ and $\frac{\partial g}{\partial u}$ is a Caratheodory function.

Set $\varphi = \inf(\alpha, \beta)$, $\psi = \sup(\alpha, \beta)$ and for all $k \ge 1$, $\mu_k(\alpha) = \mu_k(\rho, \alpha)$, $\mu_k(\beta) = \mu_k(\rho, \beta)$, $\mu_k(\psi) = \mu_k(\rho, \psi)$, $\mu_k(\omega) = \mu_k(\rho, \omega)$, and $\mu_k(\varphi) = \mu_k(\rho, \varphi)$ if $\varphi \in K_\rho^*$.

Also, throughout this section, we assume that

$$\Big|\frac{\partial g}{\partial u}(t, u)\Big| \le \gamma(t) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } t \in (\xi, \eta); \tag{6.1}$$

$$\lim_{u\to-\infty} g(t, u)/u = \beta(t) \quad \text{a.e. } t \in (\xi, \eta); \tag{6.2}$$

$$\lim_{u\to+\infty} g(t, u)/u = \alpha(t) \quad \text{a.e. } t \in (\xi, \eta); \tag{6.3}$$

$$\lim_{u\to-\infty} \chi'(u) = \chi_-, \quad \lim_{u\to+\infty} \chi'(u) = \chi_+, \quad \chi_-, \quad \chi_+ \in \mathbb{R}. \tag{6.4}$$

Also, we set in this section,

$$a_\infty(b) = \begin{cases} -a & \text{if } b > 0 \\ 1 & \text{if } b = 0, \end{cases} \qquad c_\infty(d) = \begin{cases} -c & \text{if } d > 0 \\ 1 & \text{if } d = 0. \end{cases}$$

and let $v_{-\infty}$ and $v_{+\infty}$ be respectively the unique solutions of

$$\begin{gathered} \pounds_\rho v = \chi_+ \omega v^+ - \chi_- \omega v^-, \\ v(\xi) = b, \\ v^{[p]}(\xi) = a_\infty(b), \end{gathered}$$

and

$$\begin{gathered} \pounds_\rho v = \chi_+ \omega v^+ - \chi_- \omega v^-, \\ v(\xi) = -b, \\ v^{[p]}(\xi) = -a_\infty(b). \end{gathered}$$

6.2. **Nonlinearities without jump.** We are concerned here, with the BVP

$$\begin{gathered} \pounds_\rho u = g(x,u) + h \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{6.5}$$

The main result of this subsection, Theorem 6.1, is an extension of the results obtained in [20] and [18].

Theorem 6.1. *In addition to* (6.1)*,* (6.2)*,* (6.3)*, assume that* $\varphi \in K_\rho^*$ *and there exists* $j \geq 1$ *such that*

$$\mu_j(\varphi) < 1 < \mu_{j+1}(\psi) \quad \text{or} \quad \mu_1(\psi) > 1. \tag{6.6}$$

Then (6.5) *admits at least one solution. Moreover, if*

$$\varphi(t) \leq \frac{\partial g}{\partial u}(t,u) \leq \psi(t) \quad \text{for all } u \in \mathbb{R} \text{ and } t \text{ in } (\xi,\eta) \text{ a.e.}, \tag{6.7}$$

then (6.5) *admits a unique solution.*

In fact, (6.5) under Hypotheses (6.6) and (6.7) is a perturbation of (3.10) in Case 1 of Theorem 3.16. The proof of Theorem 6.1 uses the following lemma.

Lemma 6.2. *Assume that* $\varphi \in K_\rho^*$ *and* (6.6) *holds. Then for all* $\gamma, \delta \in K_\rho^*$ *with* $\varphi \leq \gamma, \delta \leq \psi$*, the trivial function is the unique solution of the BVP*

$$\begin{gathered} \pounds_\rho u = \gamma u^+ - \delta u^- \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered} \tag{6.8}$$

Proof. To the contrary, suppose (6.8) admits a nontrivial solution ϕ. In this case there is an integer $l \geq 1$ and $\kappa = +,-$ such that $\lambda_l^\kappa(m,\gamma,\delta) = 0$ for an arbitrary $m \in K_\rho^*$. Since $\varphi \leq \gamma, \delta \leq \psi$, Proposition 3.8 leads to

$$\lambda_1 = \lambda_l^\kappa(m,\psi,\psi) \leq \lambda_l^\kappa(m,\gamma,\delta) = 0 \leq \lambda_l^\kappa(m,\varphi,\varphi) = \lambda_2. \tag{6.9}$$

Let, for $i = 1,2$, $\phi_i \in S_\rho^{l,\kappa}$ be the eignfunction associated with λ_i and note that

$$\begin{gathered} \pounds_\rho \phi_1 = (\psi + \lambda_1 m)\phi_1 \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered}$$

and

$$\begin{gathered} \pounds_\rho \phi_2 = (\varphi + \lambda_2 m)\phi_2 \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0. \end{gathered}$$

From the above BVPs, we obtain that $\lambda_l^\kappa(\psi,\lambda_1 m,\lambda_1 m) = 1 = \lambda_l^\kappa(\varphi,\lambda_2 m,\lambda_2 m)$. Then taking into account (6.9), from Proposition 3.8 we obtain

$$\mu_l(\varphi) = \lambda_l^\kappa(\varphi,0,0) \geq \lambda_l^\kappa(\varphi,\lambda_2 m,\lambda_2 m) = 1, \tag{6.10}$$

$$\mu_l(\psi) = \lambda_l^\kappa(\psi,0,0) \leq \lambda_l^\kappa(\psi,\lambda_1 m,\lambda_1 m) = 1. \tag{6.11}$$

Therefore, when $\mu_1(\psi) > 1$, from (6.11), the contradiction $1 \geq \mu_l(\psi) \geq \mu_1(\psi) > 1$, and when $\mu_j(\varphi) < 1 < \mu_{j+1}(\psi)$ for some integer $j \geq 1$, if $l \leq j$, we have from (6.10) the contradiction $1 > \mu_j(\varphi) \geq \mu_l(\varphi) \geq 1$, and if $l \geq j+1$, we have from (6.11) the contradiction $1 \geq \mu_l(\psi) \geq \mu_{j+1}(\psi) > 1$. This completes the proof. □

Proof of Theorem 6.1.
Step 1 (Existence). For $\kappa \in [0,1]$ consider the BVP

$$\begin{gathered} \pounds_\rho u = \kappa(g(x,u) + \theta\phi + h) + (1-\kappa)\frac{\alpha+\beta}{2}u \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l u = B_\rho^r u = 0, \end{gathered} \tag{6.12}$$

and note that $u \in \tilde{W}_\rho$ is a solution to (6.12) if and only if

$$\begin{aligned} u &= \kappa T u + (1-\kappa)Lu \\ &= \kappa L_\alpha I^+ u - \kappa L_\beta I^- u + \widetilde{T}u + (1-\kappa)Lu \end{aligned} \tag{6.13}$$

where for $u \in C_\rho$,

$$\begin{gathered} Tu(t) = \int_\xi^\eta (G_\rho(t,s)g(s,u(s)) + \theta\phi(s) + h(s))ds, \\ Lu(t) = \int_\xi^\eta G_\rho(t,s)(\frac{\alpha(s)+\beta(s)}{2})ds, \\ \widetilde{T}u(t) = \int_\xi^\eta (G_\rho(t,s)\widetilde{g}(s,u(s)) + \theta\phi(s) + h(s))ds, \\ \widetilde{g}(s,u) = g(s,u) - \alpha(s)u^+ + \beta(s)u^-. \end{gathered}$$

Now, we claim that there exists $R > 0$ large such that Equation (6.13) has no solution in $\partial B(0,R)$. Indeed, if this is not the case and for all $n \in \mathbb{N}$ there exist $\kappa_n \in [0,1]$ and $u_n \in \partial B(0,n)$ such that the pair (κ_n, u_n) satisfies (6.12), then the pair (κ_n, v_n) with $v_n = u_n/\|u_n\|$, satisfies

$$v_n = \kappa_n L_\alpha I^+ v_n - \kappa L_\beta I^- v_n + \big(\widetilde{T}u_n/\|u_n\|\big) + (1-\kappa)Lv_n.$$

Arguing as in the proof of Lemma 4.2, we obtain that $\widetilde{T}u_n = \circ(\|u_n\|)$ at ∞ and then we obtain from the compactness of the operators L_α, L_β and L_0 that there is a pair (κ, v), with $\kappa \in [0,1]$ and $\|v\| = 1$, satisfying the equation

$$u = \kappa L_\alpha I^+ u - \kappa L_\beta I^- u + (1-\kappa)Lu.$$

In other words, we have

$$\begin{gathered} \pounds_\rho v = A_\kappa v^+ - B_\kappa v^- \quad \text{in } (\xi,\eta) \text{ a.e.}, \\ B_\rho^l v = B_\rho^r v = 0, \end{gathered}$$

where

$$A_\kappa = \frac{1+\kappa}{2}\alpha + \frac{1-\kappa}{2}\beta, \quad B_\kappa = \frac{1-\kappa}{2}\alpha + \frac{1+\kappa}{2}\beta, \quad \varphi \leq A_\kappa, B_\kappa \leq \psi.$$

This contradicts Lemma 6.2 and proves existence of $R > 0$ large such that Equation (6.13) has no solution. For such a radius $R > 0$, we have from homotopy property of the degree and Lemma 6.2 that

$$d(I - T, B(0, R), 0) = d(I - L, B(0, R), 0) = (-1)^{\varepsilon} \neq 0$$

where ε is the sum of algebraic multiplicities of characteristic values of L contained in $(0, 1)$. Clearly, this shows that (6.15) admits a solution.

Step 2 (Uniqueness). Assume that (6.7) holds and (6.15) admits two solutions ϕ_1, ϕ_2. Set $\phi = \phi_1 - \phi_2$ and

$$q(x) = \begin{cases} \frac{g(t,\phi_1(t)) - g(t,\phi_1(t))}{\phi_1(t) - \phi_2(t)} & \text{if } \phi_1(t) \neq \phi_2(t), \\ \frac{\partial g}{\partial u}(t, \phi_1(t)) & \text{if } \phi_1(t) = \phi_2(t). \end{cases}$$

Then ϕ is a solution of

$$\begin{gathered} \pounds_\rho u = qu = qu^+ - qu^- \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B^l_\rho u = B^r_\rho u = 0, \end{gathered}$$

with $\varphi \leq q \leq \psi$. This contradicts Lemma 6.2, and completes the proof. □

Consider now the separated variable case of BVP (6.5)

$$\begin{gathered} \pounds_\rho u = \omega(t)\chi(u) + h \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B^l_\rho u = B^r_\rho u = 0. \end{gathered} \tag{6.14}$$

Setting $\alpha(t) = \chi_+\omega(t)$ and $\beta(t) = \chi_-\omega(t)$, we have $\varphi = \min(\chi_-, \chi_+)\omega$ and $\psi = \max(\chi_-, \chi_+)\omega$ and for all $k \geq 1$, $\mu_k(\alpha) = \mu_k(\omega)/\chi_+$, $\mu_j(\beta) = \mu_j(\omega)/\chi_-$. Therefore, from Theorem 6.1 we obtain the following corollary.

Corollary 6.3. *In addition to* (6.4)*, assume that* $\chi_-, \chi_+ < \mu_1(\omega)$*, or that there exists an integer* $j \geq 1$ *such that* $\mu_j(\omega) < \chi_-, \chi_+ < \mu_{j+1}(\omega)$*. Then* (6.14) *admits at least one solution. Moreover, if* $\min(\chi_+, \chi_-) \leq \chi'(u) \leq \max(\chi_+, \chi_-)$*, then* (6.14) *admits a unique solution.*

6.3. **Nonlinearities with jump.** Now we consider the BVP

$$\begin{gathered} \pounds_\rho u = g(x, u) - \theta\phi + h \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B^l_\rho u = B^r_\rho u = 0, \end{gathered} \tag{6.15}$$

and we assume the following conditions: The BVP

$$\begin{gathered} \pounds_\rho u = \alpha u - \phi \quad \text{in } (\xi, \eta) \text{ a.e.}, \\ B^l_\rho u = B^r_\rho u = 0, \end{gathered} \tag{6.16}$$

admits a unique solution $\Phi \in S^{1,+}_\rho$ and

$$\big|\frac{\partial g}{\partial u}(t, u) - \alpha(t)\big| \leq \hat{\alpha}(t)\hat{g}(u) \quad \text{in } (\xi, \eta) \text{ a.e. and } u \geq 0. \tag{6.17}$$

Remark 6.4. From Hypothesis (6.17) we obtain that

$$|g(t, u) - \alpha(t)u| \leq \hat{\alpha}(t)\hat{g}(u) + |g(t, 0)| \quad \text{for all } u \geq 0 \text{ and a.e. } t \in (\xi, \eta).$$

Remark 6.5. Note that (6.16) implies $\mu_l(\widetilde{\rho}, \alpha) \neq 1$, for all $l \geq 1$, and then $G_{\widetilde{\rho}}$ exists where $\widetilde{\rho} = (\xi, \eta, p, q - \alpha, a, b, c, d)$.

Remark 6.6. Because that $S_\rho^{1,+}$ is an open set in E_ρ and $\Phi \in S_\rho^{1,+}$, there exists $r_0 > 0$ small enough such that

$$\overline{B}_{E_\rho} = \{u \in E_\rho, \|u - \Phi\|_1 \leq r_0\} \subset S_\rho^{1,+}.$$

The following theorem is the main result of this subsection. It gives a lower bound of the number of solutions to (6.15) when the real parameter θ is large.

Theorem 6.7. *In addition to*(6.1), (6.2), (6.3), (6.16) *and* (6.17), *assume that there exist two integers $i, j \geq 1$ with $i > 2(j-1)$ such that $\mu_i(\alpha) < 1 < \mu_j(\beta)$. Then there exists $\overline{\theta} > 0$ such that* (6.15) *admits* $2(i - 2(j-1))$ *solutions for all $\theta \geq \overline{\theta}$.*

The proof of Theorem 6.7 uses the following lemmas.

Lemma 6.8. *Assume that* (6.1), (6.3), (6.16) *and* (6.17) *hold. Then there exists $\theta_1 > 0$ such that* (6.15) *admits a positive solution for all $\theta > \theta_1$*

Proof. Set $\widetilde{g}(x,u) = g(x,u) - \alpha(x)u$ and for $\theta \neq 0$ consider the operator $A_\theta : E \to E$, defined for $u \in E$ by

$$A_\theta u(x) = \frac{1}{\theta}\int_\xi^\eta G_{\widetilde{\rho}}(x,s)(\tilde{g}(s, \theta(u(s) + \Phi(s))) + h(s))ds,$$

where $\widetilde{\rho}$ is that in Remark 6.5. Clearly, A_θ is a completely continuous operator. We claim that there exists $\theta_1 > 0$ such that $A_\theta(\Omega) \subset \Omega$ for all $\theta \geq \theta_1$ where $\Omega = \overline{B}_{E_\rho}(0, r_0)$ and r_0 is the real number in Remark 6.6. Indeed, let

$$\overline{G_{\widetilde{\rho}}} = \Big(\|G_{\widetilde{\rho}}\|_\infty + \sup_{t,s \in (\xi,\eta)} \big|p(t)\frac{\partial G_{\widetilde{\rho}}}{\partial t}(t,s)\big|\Big),$$

and we obtain from Remark 6.4 the following estimate for all $u \in \Omega$,

$$\|A_\theta u\|_2 \leq (\overline{G_{\widetilde{\rho}}}/\theta)\big(\|h\|_{L^1_\rho} + \|g(.,0)\|_{L^1_\rho}\Big) + \|\hat{\alpha}\|_{L^1_\rho}(r_0 + \|\Phi\|_1)\hat{g}(\theta(r_0 + \|\Phi\|_1)).$$

This together with the fact that $\lim_{x \to +\infty} \hat{g}(x) = 0$, leads to $\sup_{u \in \Omega} \|A_\theta u\|_2 \to 0$ as $\theta \to +\infty$, proving our claim.

At the end we conclude by Schauder's fixed point theorem that for all $\theta > \theta_1$, A_θ admits a fixed point u_θ and $U_\theta = \theta(u_\theta + \Phi)$ is a positive solution of (6.15). □

We need to introduce the following notation. For $\theta \geq \theta_1 > 0$ set

$$q_\theta(t) = \frac{\partial g}{\partial u}(t, U_\theta(t)), \ \hat{\rho} = (\xi, \eta, p, q + q_\theta^-, a, b, c, d),$$
$$g_\theta(t,u) = \begin{cases} \frac{g(t,u+U_\theta) - g(t,U_\theta)}{u} + q_\theta^-(t) & \text{if } u \neq 0 \\ q_\theta^+(t) & \text{if } u = 0. \end{cases}$$

From (6.1)-(6.16) we have

$$\lim_{u \to 0} g_\theta(t,u) = q_\theta^+(t) \quad \text{in } (\xi,\eta) \text{ a.e.},$$
$$\lim_{u \to -\infty} g_\theta(t,u) = \beta(t) + q_\theta^-(t) \quad \text{in } (\xi,\eta) \text{ a.e.}$$

Lemma 6.9. *Assume that* (6.1), (6.2), (6.3), (6.16) *and* (6.17) *hold. Then there exists $\theta_2 \geq \theta_1$ such that $\mu_i(\hat{\rho}_\theta, q_\theta^+) < 1 < \mu_j(\hat{\rho}_\theta, \beta_\theta)$.*

Proof. From (6.17) we have

$$\begin{aligned}\int_\xi^\eta |q_\theta(t)-\alpha(t)|dt &= \int_\xi^\eta \big|\frac{\partial g}{\partial u}(t,U_\theta(t))-\alpha(t)\big|dt\\ &\le \int_\xi^\eta \hat\alpha(t)\hat g(\theta(u_\theta+\Phi(t)))dt\\ &\le \Big(\int_\xi^\eta \hat\alpha(t)dt\Big)\hat g(\theta(r_0+\|\Phi\|))\\ &\to 0 \quad \text{as } \theta\to+\infty.\end{aligned}$$

This shows that $q_\theta \to \alpha$ in L^1_ρ as $\theta\to+\infty$ and because of inequalities (2.1), we have $q_\theta^+\to\alpha$ and $q_\theta^-\to 0$ in L^1_ρ. Therefore, we deduce from Proposition 3.12 that

$$\lim_{\theta\to+\infty}\mu_i(\hat\rho_\theta,q_\theta^+)=\mu_i(\alpha)<1<\mu_i(\beta)=\lim_{\theta\to+\infty}\mu_j(\hat\rho_\theta,\beta_\theta)$$

and there exists $\theta_2\ge\theta_1$ such that for all $\theta\ge\theta_2$, $\mu_i(\hat\rho_\theta,q_\theta^+)<1<\mu_j(\hat\rho_\theta,\beta_\theta)$, completing the proof. □

Proof of Theorem 6.7. For $\theta>\theta_2$, we consider the BVP

$$\begin{gathered}\pounds_\rho u = ug_\theta(t,u)\quad \text{in } (\xi,\eta) \text{ a.e.},\\ B^l_\rho u = B^r_\rho u = 0,\end{gathered} \tag{6.18}$$

and note that if u is a solution to (6.18) then $u+U_\theta$ is a solution of (6.15). In addition to $\mu_i(\hat\rho_\theta,q_\theta^+)<1<\mu_j(\hat\rho_\theta,\beta_\theta)$, from hypothesis (6.1) we have that $|g_\theta(t,u)|\le\gamma+q_\theta^+$. This shows that all conditions of Part 3 in Theorem 5.1 are satisfied and in addition to the trivial solution, (6.18) admits for $l=1,\dots,i-2j+1$, a solution $u_l^+\in S_\rho^{2j-1+l,+}$ and for $l=1,\dots,i-2j+2$, a solution $u_l^-\in S_\rho^{2j-2+l,-}$. We conclude that $U_\theta, U_\theta+u_l^+$, for $l=1,\dots,i-2j+1$, and $U_\theta+u_l^-$, for $l=1,\dots,i-2j+2$, are solutions to (6.3). □

Now we consider the separated variables case of (6.15),

$$\begin{gathered}\pounds_\rho u = \omega(t)\chi(u)-\theta\phi+h\quad \text{in } (\xi,\eta) \text{ a.e.},\\ B^l_\rho u = B^r_\rho u = 0,\end{gathered} \tag{6.19}$$

and suppose that the BVP

$$\begin{gathered}\pounds_\rho u = \chi_+\omega u-\phi\quad \text{in } (\xi,\eta) \text{ a.e.},\\ B^l_\rho u = B^r_\rho u = 0,\end{gathered} \tag{6.20}$$

admits a unique solution $\Phi\in S_\rho^{1,+}$.

Remark 6.10. Let $\phi_1\in S_\rho^{1,+}$ be the eigenfunction associated with $\mu_1(\omega)$, $\phi=-\omega\phi_1$ and $\mu_1(\omega)<\chi_+\ne\mu_k(\omega)$ for all $k\ge2$ (to get uniqueness from Theorem 3.16) is a typical example where (6.20) is satisfied with $\Phi=\phi_1/(\chi_+-\mu_1(\omega))$.

Setting $\alpha(t)=\chi_+\omega(t)$ and $\beta(t)=\chi_-\omega(t)$, we have $\varphi=\min(\chi_-,\chi_+)\omega$, $\psi=\max(\chi_-,\chi_+)\omega$ and if $\chi_-,\chi_+>0$, then for all $k\ge1$, $\mu_k(\alpha)=\mu_k(\omega)/\chi_+$, $\mu_j(\beta)=\mu_j(\omega)/\chi_-$. Therefore, from Theorem 6.7 we obtain the following corollary.

Corollary 6.11. *In addition to* (6.4) *and* (6.20)*, assume that there exist two integers $i,j\ge1$ with $i\ge2(j-1)$ such that $\chi_+>\mu_i(\omega)>\mu_j(\omega)>\chi_->0$. Then there exists $\overline\theta>0$ such that the* (6.19) *admits* $2(i-2(j-1))$ *solutions for all $\theta\ge\overline\theta$.*

Example 6.12. Let $\rho = (0, \pi, 1, 0, 1, 0, 1, 0)$, $g_-, g_+ \in (0, +\infty)$ and let i, j be integers such that $1 \leq j \leq i$ and $i \geq 2(j-1)$. Consider the BVP

$$\begin{gathered} -u'' = g(u) - \theta\phi + h \quad \text{in } (0,\pi) \\ u(0) = u(\pi) = 0 \end{gathered} \tag{6.21}$$

where $h \in L^1_\rho$ and

$$g(u) = \frac{g_+ u^2 e^u}{1 + |u|e^u} + \frac{g_- u^2 e^{-u}}{1 + |u|e^{-u}}.$$

We have

$$\lim_{u \to -\infty} \frac{g(u)}{u} = g_-, \quad \lim_{u \to +\infty} \frac{g(u)}{u} = g_+.$$

Example 6.13. Suppose that $0 < g_- < 1 < g_+ < 4$. Denote by Φ the unique solution of the BVP

$$\begin{gathered} -u'' = g_+ u - \phi \quad \text{in } (0,\pi) \\ u(0) = u(\pi) = 0. \end{gathered}$$

(1) If $\phi(t) = 1$, then

$$\begin{aligned} \Phi(t) &= \frac{1}{g_+}\Big(-\cos(\sqrt{g_+}t) - \frac{1 - \cos(\sqrt{g_+}\pi)}{\sin(\sqrt{g_+}\pi)} \sin(\sqrt{g_+}t) + 1\Big) \\ &= \frac{-2\sin(\sqrt{g_+}(\pi - t)/2)}{g_+ \sin(\sqrt{g_+}\pi)} \sin(\pi\sqrt{g_+}/2)\sin(t\sqrt{g_+}/2) \end{aligned}$$

and $\Phi \in S^{1,+}_\rho$. Therefore, from Corollary 6.11 we deduce that (6.21) admits at least 2 nontrivial solutions for θ large.

(2) If $\phi(t) = t$, then

$$\begin{aligned} \Phi(t) &= \frac{-\pi}{g_+ \sin(\sqrt{g_+}\pi)} \sin(\sqrt{g_+}t) + \frac{t}{g_+} \\ &= \frac{-1}{g_+ \sin(\sqrt{g_+}\pi)} (\pi \sin(\sqrt{g_+}t) - t(\sqrt{g_+}\pi)). \end{aligned}$$

It is easy to see that there exists $\alpha^* \in (1, \frac{9}{4})$ such that $\Phi \in S^{1,+}_\rho$ whenever $g_+ \in (1, \frac{9}{4})$. Therefore, we deduce from Corollary 6.11 that (6.21) admits at least 2 nontrivial solutions for θ large and $g_+ \in (1, \frac{9}{4})$.

6.4. **Ambrosetti-Prodi situation.**

Theorem 6.14. *In addition to* (6.4) *and* (6.20)*, assume that* $\chi \in C^2(\mathbb{R})$*,* $\chi'' > 0$ *and* $\chi_- < \mu_1(\omega) < \chi_+ < \mu_2(\omega)$*. Then there exists a real number* θ^* *such that* (6.19) *admits*

i) *no solution if* $\theta < \theta^*$*,*
ii) *a unique solution if* $\theta = \theta^*$*, and*
iii) *exactly two solutions if* $\theta > \theta^*$*.*

The proof of the above theorem uses the following lemmas.

Lemma 6.15. *In addition to* (6.4) *and* (6.20)*, assume that* $\chi_- < \mu_1(\omega) < \chi_+$*. Then there exists a real number* θ_3 *such that* (6.19) *admits no solutions.*

Proof. Let $\epsilon > 0$ be such that $\epsilon < \min((\chi_+ - \chi_-)/2, \chi_+ - \mu_1(m), \chi_- - \mu_1(m))$. We claim that there exist two positive constants C_1 and C_2 such that

$$\chi(u) \geq (\chi_+ - \epsilon)u - C_1 \quad \text{for all } u \in \mathbb{R}, \tag{6.22}$$

$$\chi(u) \geq (\chi_- + \epsilon)u - C_2 \quad \text{for all } u \in \mathbb{R}. \tag{6.23}$$

Indeed, for such a real number ϵ there exists $A > 0$ such that

$$\chi(u) \geq (\chi_+ - \epsilon)u \geq (\chi_- + \epsilon)u \quad \text{for all } u \geq A,$$
$$\chi(u) \geq (\chi_- + \epsilon)u \geq (\chi_+ - \epsilon)u \quad \text{for all } u \leq -A.$$

This leads to existence of positive constants C_1 and C_2 such that $\chi(u) \geq (\chi_+ - \epsilon)u - C_1$ and $\chi(u) \geq (\chi_- + \epsilon)u - C_2$ for all $u \in \mathbb{R}$.

Now, let $u \in W_\rho$ be a solution of (6.19). Then

$$0 = \int_\xi^\eta \phi_1 \pounds_\rho u - u \pounds_\rho \phi_1 = \int_\xi^\eta (\chi(u) - \mu_1(\omega)u)\omega\phi_1 - \theta \int_\xi^\eta \phi_1 \phi + \int_\xi^\eta \phi_1 h \tag{6.24}$$

and

$$\begin{aligned}\int_\xi^\eta \phi_1 \phi &= (\chi_+ - \mu_1(\omega)) \int_\xi^\eta \omega\phi_1 \Phi - \int_\xi^\eta \phi_1 \pounds_\rho \Phi - \Phi \pounds_\rho \phi_1 \\ &= (\chi_+ - \mu_1(\omega)) \int_\xi^\eta \omega\phi_1 \Phi > 0.\end{aligned}$$

Therefore, if $\int_\xi^\eta \omega\phi_1 u \leq 0$, then inserting (6.23) into (6.24), we obtain

$$\theta \int_\xi^\eta \phi_1 \phi \geq ((\chi_- + \epsilon) - \mu_1(\omega)) \int_\xi^\eta \omega\phi_1 u + \int_\xi^\eta \phi_1 h \geq \int_\xi^\eta \phi_1 h$$

leading to $\theta \geq \int_\xi^\eta \phi_1 h / \int_\xi^\eta \phi_1 \phi$, and if $\int_\xi^\eta \omega\phi_1 u > 0$, then inserting (6.22) into (6.24), we obtain

$$\theta \int_\xi^\eta \phi_1 \phi \geq ((\chi_+ - \epsilon) - \mu_1(\omega)) \int_\xi^\eta \omega\phi_1 u + \int_\xi^\eta \phi_1 h \geq \int_\xi^\eta \phi_1 h,$$

leading also to $\theta \geq \int_\xi^\eta \phi_1 h / \int_\xi^\eta \phi_1 \phi = \theta_3$. This shows that if $\theta < \theta_3$, BVP (6.19) has no solution. The proof is complete. □

In what follows and without loss of generality, we assume that the real parameters b and d are nonnegative.

Lemma 6.16. *Suppose that $\chi_- < \mu_1(\omega) < \chi_+ < \mu_2(\omega)$. Then*

$$c_\infty(d)v_{+\infty}(\eta) + dv_{+\infty}^{[p]}(\eta) < 0 < c_\infty(d)v_{-\infty}(\eta) + dv_{-\infty}^{[p]}(\eta).$$

Proof. We present the proof for $v_{+\infty}$; the proof for $v_{-\infty}$ is similar. First, we claim that $v_{+\infty}$ admits at most one zero, Indeed, if there are $\xi < x_1 < x_2 \leq \eta$ such that $v_{+\infty}(x_1) = v_{+\infty}(x_2) = 0$, then for $\phi_1 \in S_\rho^{1,+}$ we have an eigenfunction associated with $\mu_1(\omega)$, yielding the contradiction

$$\begin{aligned}0 &< -\phi_1(x_2)v_{+\infty}^{[p]}(x_2) + \phi_1(x_1)v_{+\infty}^{[p]}(x_1) \\ &= \int_{x_1}^{x_2} \phi_1 \pounds_\rho v_{+\infty} - v_{+\infty} \pounds_\rho \phi_1 \\ &= (\mu_1(\omega) - \chi_-) \int_{x_1}^{x_2} \omega\phi_1 v_{+\infty} < 0.\end{aligned}$$

Therefore, we distinguish two cases:

(i) $v_{+\infty} > 0$ in (ξ, η): In this case,

$$\begin{aligned} -\phi_1(\eta) v_{+\infty}^{[p]}(\eta) + \phi_1^{[p]}(\eta) v_{+\infty}(\eta) &= \int_\xi^\eta \phi_1 \pounds_\rho v_{+\infty} - v_{+\infty} \pounds_\rho \phi_1 \\ &= (\chi_- - \mu_1(\omega)) \int_\xi^\eta \omega \phi_1 v_{+\infty} > 0 \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} &- \phi_1(\eta) v_{+\infty}^{[p]}(\eta) + \phi_1^{[p]}(\eta) v_{+\infty}(\eta) \\ &= \begin{cases} -\frac{\phi_1(\eta)}{d}(c v_{+\infty}(\eta) + d v_{+\infty}^{[p]}(\eta)) & \text{if } d > 0 \\ \phi_1^{[p]}(\eta) v_{+\infty}(\eta) & \text{if } d = 0. \end{cases} \end{aligned} \tag{6.26}$$

Since $\phi_1(\eta) > 0$ if $d > 0$, and $\phi_1^{[p]}(\eta) < 0$ if $d = 0$, from (6.25) and (6.26) we obtain $c_\infty(d) v_{+\infty}(\eta) + d v_{+\infty}^{[p]}(\eta) < 0$.

(ii) $v_{+\infty}(x_1) = 0$ for some $x_1 \in (\xi, \eta)$: In this case we have $v_{+\infty}^{[p]}(x_1) < 0$ and

$$\begin{aligned} &- \phi_1(\eta) v_{+\infty}^{[p]}(\eta) + \phi_1^{[p]}(\eta) v_{+\infty}(\eta) \\ &= \int_{x_1}^\eta \phi_1 \pounds_\rho v_{+\infty} - v_{+\infty} \pounds_\rho \phi_1 \\ &= -\phi_1(x_1) v_{+\infty}^{[p]}(x_1) + (\chi_- - \mu_1(\omega)) \int_{x_1}^\eta \omega \phi_1 v_{+\infty} > 0. \end{aligned} \tag{6.27}$$

As with the above case, from (6.27) and (6.26) we obtain $c_\infty(d) v_{+\infty}(\eta) + d v_{+\infty}^{[p]}(\eta) < 0$. This completes the proof. $\square$

Lemma 6.17. *Let for $\sigma \in \mathbb{R}$, $v_\sigma = v(\cdot, \sigma, \theta)$ be the unique solution of*

$$\begin{gathered} \pounds_\rho v = \omega \frac{\chi(\sigma v)}{\sigma} - \theta \frac{\phi}{\sigma} + \frac{h}{\sigma} \\ v(\xi) = b \\ v^{[p]}(\xi) = a_\infty(b) \end{gathered}$$

and assume that (6.4) *holds. Then* $\lim_{\sigma \to -\infty} v_\sigma = v_{-\infty}$ *and* $\lim_{\sigma \to +\infty} v_\sigma = v_{+\infty}$ *in* $\tilde{W}_\rho$.

Proof. We prove that $\lim_{\sigma \to +\infty} v_\sigma = v_{+\infty}$ in $\tilde{W}_\rho$; the other limit is checked similarly. Let $\tilde{\chi}(u) = \chi(u) - \chi_+ u^+ + \chi_- u^-$ and note that there exists $M > 0$ such that $|\chi(u)| \leq M$. For $\sigma > 0$, let $w_\sigma = v_\sigma - v_{+\infty}$ and observe that w_σ satisfies

$$\begin{gathered} \pounds_\rho w_\sigma = \hat{\chi}(s, w_\sigma) \\ w_\sigma(\xi) = 0, \\ w_\sigma^{[p]}(\xi) = 0, \end{gathered}$$

where

$$\begin{aligned} \hat{\chi}(s, u) = \omega \chi_+ ((u + v_{+\infty}(s))^+ - v_{+\infty}^+(s)) - \chi_- \omega ((u + v_{+\infty}(s))^- - v_{+\infty}^-(s)) \\ + \omega \frac{\tilde{\chi}(\sigma(u + v_{+\infty}(s)))}{\sigma} - \theta \frac{\phi(s)}{\sigma} + \frac{h(s)}{\sigma}. \end{aligned}$$

Set $W_\sigma = (w_\sigma, w_\sigma^{[p]})$, then W_σ satisfies $W_\sigma' = F(s, W_\sigma)$ and $W_\sigma(\xi) = (0,0)$ where for $X = (x,y)$, $F(s,X) = (\frac{1}{p}y, q(s)u - \hat{\chi}(s,u))$ and $W_\sigma(t) = \int_\xi^t F(s, W_\sigma(s))ds$. From (2.1) we obtain the estimates

$$\begin{aligned}
&|F(s,W_\sigma(s))|\\
&\le |q(s)||w_\sigma(s)| + \frac{|w_\sigma^{[p]}(s)|}{p(s)} + \theta\frac{|\phi(s)|}{\sigma} + \frac{|h(s)|}{\sigma}\\
&\quad + \omega(s)\frac{|\tilde{\chi}(\sigma(w_\sigma(s) + v_{+\infty}(s)))|}{\sigma} + \chi_+\omega(s)|(w_\sigma(s) + v_{+\infty}(s))^+ - v_{+\infty}^+(s)|\\
&\quad + \chi_-\omega(s)|(w_\sigma(s) + v_{+\infty}(s))^- - v_{+\infty}^-(s)|\\
&\le \frac{|w_\sigma^{[p]}(s)|}{p(s)} + (\chi_+ + \chi_-)\omega(s)|w_\sigma(s)| + \omega(s)\frac{M}{\sigma} + \theta\frac{|\phi(s)|}{\sigma} + \frac{|h(s)|}{\sigma}\\
&\le \varpi(s)(|w_\sigma^{[p]}(s)| + |w_\sigma(s)|) + \omega(s)\frac{M}{\sigma} + \theta\frac{|\phi(s)|}{\sigma} + \frac{|h(s)|}{\sigma}
\end{aligned}$$

where $\varpi(s) = (1/p(s)) + |q(s)| + (\chi_+ + \chi_-)\omega(s)$. Let $\kappa > 1$. The above estimates lead to

$$\begin{aligned}
&\exp(-\kappa\int_\xi^t \varpi(r)dr)|W_\sigma(t)|\\
&\le \int_\xi^t |F(s,W_\sigma(s))| \exp(-\kappa\int_\xi^s \varpi(r)dr)\exp(-\kappa\int_s^t \varpi(r)dr)ds\\
&\le \|W_\sigma\|_\xi \int_\xi^t \varpi(s)\exp(-\kappa\int_s^t \varpi(r)dr)ds + \frac{1}{\sigma}(M + \theta\|\phi\|_1 + \|h\|_1)\\
&\le \frac{1}{\kappa}\|W_\sigma\|_\xi + \frac{1}{\sigma}(M + \theta\|\phi\|_1 + \|h\|_1),
\end{aligned}$$

and then

$$(1 - \frac{1}{\kappa})\|W_\sigma\|_\xi \le \frac{1}{\sigma}(M + \theta\|\phi\|_1 + \|h\|_1) \to 0 \quad \text{as } \sigma \to +\infty.$$

Thus, we have proved that $w_\sigma \to 0$ in $\tilde{W}_\rho$; the proof is complete. □

Proof of Theorem 6.14. Without loss of generality, suppose that $b, d \ge 0$. For $\sigma \in \mathbb{R}$, let $u(\cdot,\sigma,\theta)$ be the unique solution given by Theorem 2.3 of the IVP

$$\begin{gathered}
\pounds_\rho u = \omega\chi(u) - \theta\phi + h\\
u(\xi) = b\sigma\\
u^{[p]}(\xi) = a_\infty(b)\sigma.
\end{gathered}$$

Consider the function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\gamma(E,\theta) = B_\rho^r u(\eta,\sigma,\theta) = c_\infty(d)u(\eta,\sigma,\theta) + du^{[p]}(\eta,\sigma,\theta).$$

Fix θ and let $\gamma_\theta(\sigma) = \gamma(\sigma,\theta)$. We have that $\lim_{\sigma\to-\infty}\gamma_\theta(\sigma) = \lim_{\sigma\to+\infty}\gamma_\theta(\sigma) = -\infty$. We present the proof of $\lim_{\sigma\to+\infty}\gamma_\theta(\sigma) = -\infty$; the other limit is checked similarly. For $\sigma > 0$, let $v_\sigma = u/\sigma$, and note that v_σ satisfies the IVP

$$\begin{gathered}
\pounds_\rho u = \omega\frac{\chi(\sigma u)}{\sigma} - \theta\frac{\phi}{\sigma} + \frac{h}{\sigma}\\
u(\xi) = b
\end{gathered}$$

$$u^{[p]}(\xi) = a_\infty(b).$$

From Lemma 6.17, we have $\lim_{\sigma\to+\infty} v_\sigma = v_{+\infty}$ in $\tilde{W}_\rho$. In particular, we have $\lim_{\sigma\to+\infty} \frac{u(\eta,\sigma,\theta)}{\sigma} = v_{+\infty}(\eta)$ and $\lim_{\sigma\to+\infty} \frac{u^{[p]}(\eta,\sigma,\theta)}{\sigma} = v_{+\infty}^{[p]}(\eta)$. Then taking into account Lemma 6.16, we obtain

$$\lim_{\sigma\to+\infty} \frac{\gamma_\theta(\sigma)}{\sigma} = c_\infty(d)v_{+\infty}(\eta) + dv_{+\infty}^{[p]}(\eta) < 0$$

and obviously, $\lim_{\sigma\to+\infty} \gamma_\theta(\sigma) = -\infty$.

Now, we claim that the mapping γ_θ admits a unique critical point at which it reaches its maximum value. Let σ^* be such that $\gamma_\theta'(\sigma^*) = 0$ and set $u_* = u(\cdot,\sigma^*,\theta)$, $v_* = \frac{\partial u}{\partial\sigma}(\cdot,\sigma^*,\theta)$ and $w_* = \frac{\partial^2 u}{\partial\sigma^2}(\cdot,\sigma^*,\theta)$ and note that

$$\begin{gathered} \pounds_\rho v_* = \omega\chi'(u^*)v_* \\ B_\rho^l v_* = B_\rho^r v_* = 0 \end{gathered} \tag{6.28}$$

and

$$\begin{gathered} \pounds_\rho w_* = \omega\chi''(u_*)(v_*)^2 + \omega\chi'(u^*)w_* \\ w_*(\xi) = 0 \\ w_*^{[p]}(\xi) = 0. \end{gathered} \tag{6.29}$$

We have that $v_* \in S_\rho^{1,+}$, indeed, from BVP (6.28) we obtain $\mu_l(q - \omega\chi'(u^*)) = \mu_l(\rho^*, m) = 0$ for some integer $l \geq 1$ and arbitrary $m \in K_\rho^*$, where $\rho^* = (\xi,\eta,p,q - \omega\chi'(u^*),a,b,c,d)$. Let ϕ_1,ϕ_2 be respectively eigenfunctions associated with $\mu_1(\omega)$, $\mu_2(\omega)$ and note that we obtain also that

$$\mu_1(q - \mu_1(\omega)\omega) = \mu_1(\rho_1, m) = \mu_2(q - \mu_2(\omega)\omega) = \mu_2(\rho_2, m) = 0$$

where for $i = 1,2$, $\rho_i = (\xi,\eta,p,q - \mu_i(\omega)\omega,a,b,c,d)$.

Since $\chi_- < \chi'(u^*) < \chi_+ < \mu_2(\omega)$, from Proposition 3.11 for $l \geq 2$, we have the contradiction

$$0 = \mu_l(q - \omega\chi'(u^*)) > \mu_l(q - \mu_2(\omega)\omega) \geq \mu_2(q - \mu_2(\omega)\omega) = 0.$$

This shows that $l = 1$ and since $v_*(\xi) = b \geq 0$ and $v_*^{[p]}(\xi) = 1$ if $b = 0$, we have $v_* \in S_\rho^{1,+}$.

At this stage, we have

$$\begin{aligned} -w_*^{[p]}(\eta)v_*(\eta) + w_*(\eta)v_*^{[p]}(\eta) &= \int_\xi^\eta v_*\pounds_\rho w_* - w_*\pounds_\rho v_* \\ &= \int_\xi^\eta \omega\chi''(u^*)(v_*)^3 > 0 \end{aligned} \tag{6.30}$$

and since

$$-w_*^{[p]}(\eta)v_*(\eta) + w_*(\eta)v_*^{[p]}(\eta) = \begin{cases} -\frac{v_*(\eta)}{d}\gamma_\theta''(\sigma^*) & \text{if } d > 0 \\ v_*^{[p]}(\eta)\gamma_\theta''(\sigma^*) & \text{if } d > 0 \end{cases}$$

and $v_*(\eta) > 0$ if $d > 0$, and $v_*^{[p]}(\eta) < 0$ if $d = 0$, from (6.30) we conclude that $\gamma_\theta''(\sigma^*) < 0$ and γ_θ reaches at σ^* its maximum value.

Now, let for $\theta \in \mathbb{R}$, $\Gamma(\theta) = \gamma(\sigma(\theta),\theta)$ where $\sigma(\theta)$ is the unique critical point of the mapping γ and $z = \frac{\partial u}{\partial\theta}(\cdot,\sigma,\theta)$. Then

$$\Gamma'(\theta) = \frac{\partial\gamma}{\partial\sigma}(\sigma(\theta),\theta)\sigma'(\theta) + \frac{\partial\gamma}{\partial\theta}(\sigma(\theta),\theta) = \frac{\partial\gamma}{\partial\theta}(\sigma(\theta),\theta)$$

and

$$\pounds_\rho z = \omega\chi'(u_*)z - \phi$$
$$z(\xi) = 0$$
$$z^{[p]}(\xi) = 0.$$

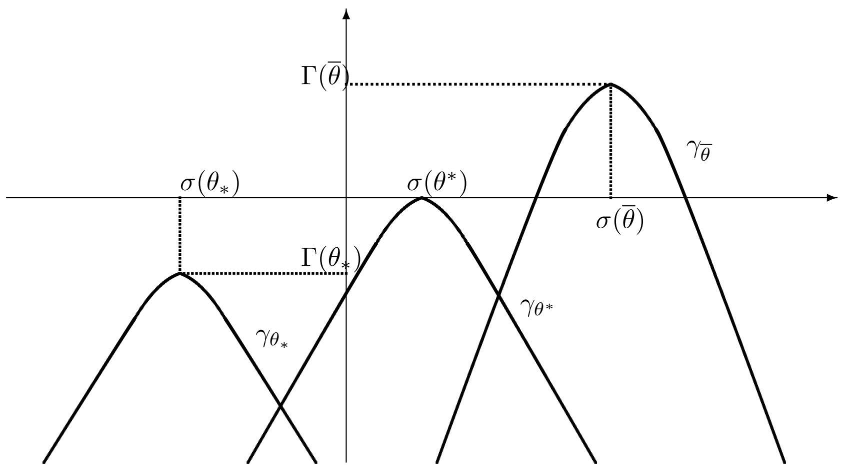

FIGURE 4. The mapping γ_θ

Similar calculations lead to

$$\begin{aligned}\int_\xi^\eta v_*\phi &= \int_\xi^\eta \Phi\pounds_\rho v_* - v_*\pounds_\rho\Phi - \int_\xi^\eta (\chi'(u_*) - \chi_+)\omega\Phi v_* \\ &= \int_\xi^\eta (\chi_+ - \chi'(u_*))\omega\Phi v_* > 0\end{aligned}$$

and

$$\begin{aligned}-v_*(\eta)z^{[p]}(\eta) + v_*^{[p]}(\eta)z_*(\eta) &= \begin{cases} -\frac{v_*(\eta)}{d}\frac{\partial\gamma}{\partial\theta}(\sigma(\theta),\theta) & \text{if } d > 0 \\ v_*^{[p]}(\eta)\frac{\partial\gamma}{\partial\theta}(\sigma(\theta),\theta) & \text{if } d = 0 \end{cases} \\ &= \int_\xi^\eta v_*\pounds_\rho w_* - w_*\pounds_\rho v_* \\ &= -\int_\xi^\eta v_*\phi < 0.\end{aligned}$$

This shows that the mapping Γ is increasing.From Lemma 6.15 we have $\Gamma(\theta_3) < 0$, and from Theorem 6.7 we have $\Gamma(\bar{\theta}) > 0$, then there exists a unique $\theta^* \in \mathbb{R}$ such that $\Gamma(\theta^*) = 0$ and consequently (6.19) has no solution if $\theta < \theta^*$, a unique solution $(u(\cdot,\sigma(\theta^*),\theta^*))$ if $\theta = \theta^*$, and exactly two solutions $(u(\cdot,\sigma_1,\theta)$ and $u(\cdot,\sigma_2,\theta)$ with $\gamma(\sigma_1,\theta) = \gamma(\sigma_2,\theta) = 0$ and $\sigma_1 < \sigma(\theta) < \sigma_2)$ if $\theta > \theta^*$. The proof is complete. □

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Abdelhamid Benmezaï
Faculty of Mathematics, USTHB, Algiers, Algeria
E-mail address: aehbenmezai@gmail.com

Wassila Esserhane
Graduate School of Statistics and Applied Economics, P.O. Box 11, Doudou Mokhtar, Ben-Aknoun Algiers, Algeria
E-mail address: ewassila@gmail.com

Johnny Henderson
Department of Mathematics, Baylor University, Waco, Texas 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu