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ASYMPTOTIC STABILITY OF NON-AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAYS

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ABSTRACT. We consider the integro differential equation

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} \lambda(s)x(s)\,\mathrm{d}s, \quad o \le a(t),\ 0 \le t < \infty,$$

where $a, b : \mathbb{R}_+ \to \mathbb{R}$, $\lambda : [-h, \infty) \to \mathbb{R}$ are piecewise continuous functions and h is a positive constant. We establish sufficient conditions guaranteeing either asymptotic stability or uniform asymptotic stability for the zero solution. These conditions state that the instantaneous stabilizing term on the right-hand side dominates in some sense the perturbation term with delays. Our conditions not require a being bounded from above. The results are based on the method of Lyapunov functionals and Razumikhin functions.

1. Introduction

Consider the scalar functional differential equation (FDE)

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} \lambda(s)x(s)\,\mathrm{d}s, \quad 0 \le a(t)\ t \in \mathbb{R}_+ := [0, \infty), \tag{1.1}$$

where $a, b : \mathbb{R}_+ \to \mathbb{R}$, $\lambda : [-h, \infty) \to \mathbb{R}$ are piecewise continuous and everywhere continuous from the right, $0 < h$ is a constant. We will the standard notation [4]: C is the Banach space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}$ with the maximum norm $\|\varphi\| := \max_{-h \le \theta \le 0} |\varphi(\theta)|$; C_H denotes the open ball of radius $H > 0$ in C around $\varphi = 0$. As is usual, if $x : [-h, \beta) \to \mathbb{R}$ ($\beta > 0$), then $x_t(\theta) := x(t + \theta)$ for $-h \le \theta \le 0$, $0 \le t < \beta$. Let $x(\cdot; t_0, \varphi) : [t_0 - h, t_0 + \alpha) \to \mathbb{R}$ denote a solution of (1.1) satisfying the initial condition $x_{t_0}(\cdot; t_0, \varphi) = \varphi$. It is known [4] that for each $t_0 \in \mathbb{R}_+$ and $\varphi \in C$ there is exactly one solution $x(\cdot; t_0, \varphi) : [t_0 - h, \infty) \to \mathbb{R}$.

Equation (1.1) is the model of a system in which there act an instantaneous negative feedback stabilizing the equilibrium $x = 0$ and a perturbation with distributed delays on the interval $[t - h, t]$. We look for sufficient conditions guaranteeing asymptotic stability for the zero solution of (1.1). To this end we have to suppose that the stabilizing term on the right-hand side dominates in some sense the perturbation term. In fact, if $a(t) \equiv b(t)\int_{t-h}^{t} \lambda$, then every constant is a solution of the equation, so the zero solution is not asymptotically stable.

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The parameters of the system are varying in time, so (1.1) is non-autonomous. In the stability theory of the non-autonomous FDE's

$$\mathbf{x}'(t) = f(t, x_t) \tag{1.2}$$

it is typically supposed that $f : [-h, \infty) \times C \to \mathbb{R}$ maps sets $[-h, \infty) \times C_H$ into bounded sets of C (see, e.g., [4, Theorem 5.2.1]). If we apply theorems of this type to (1.1), then we have to require boundedness on $\mathbb{R}_+$ also of function a (see a similar situation in [4, equation (5.2.25)]). However, in the case of (1.1) this is not a natural condition because, obviously, the larger $a(t)$ is the better from the point of view of the asymptotic stability. So we refine the techniques and allow only conditions not requiring any boundedness type conditions from above of function a.

Equation (1.1) is not only a model, it is also an important "test equation" in the stability theory: authors often choose it as an example to illustrate their general theorems on (1.2) [1, 2, 6, 14, 15]. In these applications the conditions on a, b, λ are accorded with the requirements of the general theorems. In this article, equation (1.1) is in the focus of the investigation, and we look for optimal conditions of asymptotic stability finding the most adequate methods and techniques.

Equation (1.1) is linear, but we will not use the consequences of this fact deeply; the results can be easily transformed for the more general equation

$$x'(t) = -a(t)f(x) + b(t)\int_{t-h}^{t} \lambda(s)g(x(s))\,\mathrm{d}s, \quad 0 \le a(t),\ 0\ (t \in \mathbb{R}_+)$$

treated in [2, Section 5]. The reason of the choice $f(x) \equiv g(x) \equiv x$ is that the main ideas can be demonstrated well by this special case so that the formulation of the results is essentially simpler.

In Section 2 we give theorems based upon the combination of the method of Lyapunov functionals [4, 1, 2] and the annulus argument [5]; in Section 3 we use the Lyapunov-Razumikhin method [4, 9, 11, 12, 16, 17].

2. A Lyapunov functional

The following stability concepts are standard [1, 4].

Definition 2.1. The zero solution of (1.1) is:

(a) *stable* if for every $\varepsilon > 0$ and $t_0 \ge 0$ there is a $\delta(\varepsilon, t_0) > 0$ such that $\|\varphi\| < \delta$, $t \ge t_0$ imply that $|x(t; t_0, \varphi)| < \varepsilon$.

(b) *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\|\varphi\| < \delta, t_0 \ge 0, t \ge t_0$ imply that $|x(t; t_0, \varphi)| < \varepsilon$.

(c) *asymptotically stable* if it is stable and for every $t_0 \ge 0$ there is a $\sigma(t_0) > 0$ such that $\|\varphi\| < \sigma$ implies $\lim_{t\to\infty} x(t; t_0, \varphi) = 0$.

(d) *uniformly asymptotically stable* (UAS) if it is uniformly stable and there is a $D > 0$ and for each $\mu > 0$ there is a $T(\mu)$ such that $t_0 \in \mathbb{R}_+$, $\|\varphi\| < D$, $t \ge t_0 + T$ imply that $|x(t; t_0, \varphi)| < \mu$.

It can be seen that the zero solution is stable if and only if all solutions are bounded on $\mathbb{R}_+$, and it is asymptotically stable if and only if all solutions tend to zero as t tends to infinity.

A continuous functional $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$ which is locally Lipschitz in φ is called a *Lyapunov functional* if its right derivative with respect to system (1.1) is non-positive:

$$V'_{(1.1)}(t,\varphi) = V'(t,\varphi) := \limsup_{\delta\to 0+0}\Big(\frac{1}{\delta}\big(V(t+\delta, x_{t+\delta}(\cdot,t,\varphi)) - V(t,\varphi)\big)\Big) \le 0.$$

We consider the functional

$$\begin{aligned} V_1(t,\varphi) :&= |\varphi(0)| + \int_{-h}^0 \int_\theta^0 |b(t+\vartheta-\theta)|\,|\lambda(t+\vartheta)||\varphi(\vartheta)|\,\mathrm{d}\vartheta\,\mathrm{d}\theta \\ &= V_2(t,\varphi) + V_3(t,\varphi), \quad V_2(t,\varphi) := |\varphi(0)|. \end{aligned} \tag{2.1}$$

The following lemma says that V_1 is a Lyapunov functional under appropriate conditions.

Lemma 2.2. *The derivative of the functional in* (2.1) *satisfies the inequality*

$$V_1'(t,\varphi) \le -\Big(a(t) - |\lambda(t)|\int_t^{t+h}|b|\Big)|\varphi(0)| \quad (t\in\mathbb{R}_+). \tag{2.2}$$

Proof. Let us consider the solution $t\mapsto x(t) = x(t;t_0,\varphi)$. If $\varphi(0)\neq 0$, then

$$\begin{aligned} V_2'(t_0,\varphi) &= \operatorname{sign}(\varphi(0))x'(t_0) \\ &= -a(t_0)|\varphi(0)| + \operatorname{sign}(\varphi(0))b(t_0)\int_{t_0-h}^{t_0}\lambda(s)\varphi(s-t_0)\,\mathrm{d}s \\ &\le -a(t_0)|\varphi(0)| + |b(t_0)|\int_{t_0-h}^{t_0}|\lambda(s)||\varphi(s-t_0)|\,\mathrm{d}s. \end{aligned} \tag{2.3}$$

If $\varphi(0)=0$, then

$$\begin{aligned} V_2'(t_0,\varphi) &= \limsup_{\delta\to 0+0}\Big(\frac{1}{\delta}\left(|x(t_0+\delta)| - |x(t_0)|\right)\Big) \\ &\le \limsup_{\delta\to 0+0}\Big|\frac{x(t_0+\delta)-x(t_0)}{\delta}\Big| \\ &= |x'(t_0)| \le -a(t_0)|\varphi(0)| + |b(t_0)|\int_{t_0-h}^{t_0}|\lambda(s)||\varphi(s-t_0)|\,\mathrm{d}s \end{aligned} \tag{2.4}$$

because $\varphi(0)=0$.

On the other hand,

$$V_3(t,x_t) = \int_{-h}^0\int_{t+\theta}^t |b(s-\theta)|\,|\lambda(s)||x(s)|\,\mathrm{d}s\,\mathrm{d}\theta,$$

therefore,

$$\begin{aligned} &V_3'(t_0,\varphi) \\ &= \limsup_{\delta\to 0+0}\Big(\frac{1}{\delta}\Big(\int_{-h}^0\Big(\int_{t_0}^{t_0+\delta}|b(s-\theta)||\lambda(s)||x(s)|\,\mathrm{d}s \\ &\quad - \int_{t_0+\theta}^{t_0+\theta+\delta}|b(s-\theta)||\lambda(s)||x(s)|\,\mathrm{d}s\Big)\,\mathrm{d}\theta\Big)\Big) \\ &= |\lambda(t_0)|\Big(\int_{-h}^0|b(t_0-\theta)|\,\mathrm{d}\theta\Big)|\varphi(0)| - |b(t_0)|\int_{t_0-h}^{t_0}|\lambda(s)||\varphi(s-t_0)|\,\mathrm{d}s. \end{aligned}$$

From $V_1'(t_0,\varphi)\le V_2'(t_0,\varphi)+V_3'(t_0,\varphi)$ we obtain (2.2). ☐

Lemma 2.2 suggests the first condition on the dominance of the negative feedback:

(A1) $a(t) - |\lambda(t)| \int_t^{t+h} |b| \geq 0 \quad (t \in \mathbb{R}_+)$.

Lemma 2.3. *If condition* (A1) *is satisfied, then the zero solution of* (1.1) *is stable. If, in addition,*

$$\Big(\int_t^{t+h} |b|\Big)\Big(\int_{t-h}^{t} |\lambda|^2\Big)^{1/2} \leq K_1 \quad (t \in R_+) \tag{2.5}$$

with some constant K_1, then the zero solution is uniformly stable.

Proof. By Lemma 2.2 and condition (A1), for any solution $x(\cdot; t_0, \varphi)$ the function $t \mapsto V_1(t, x_t)$ is non-increasing. Since $x(t) \leq V_1(t, x_t)$, every solution is bounded, which proves stability.

To prove uniform stability, let us estimate functional in (2.1):

$$\begin{aligned} V_1(t,\varphi) &= |\varphi(0)| + \int_{-h}^{0} |\lambda(t+\vartheta)||\varphi(\vartheta)|\Big(\int_{-h}^{\vartheta} |b(t+\vartheta-\theta)|\,\mathrm{d}\theta\Big)\,\mathrm{d}\vartheta \\ &\leq |\varphi(0)| + \Big(\int_t^{t+h} |b|\Big)\int_{-h}^{0} |\lambda(t+\vartheta)||\varphi(\vartheta)|\,\mathrm{d}\vartheta \\ &\leq |\varphi(0)| + \Big(\int_t^{t+h} |b|\Big)\Big(\int_{t-h}^{t} |\lambda|^2\Big)^{1/2}\Big(\int_{-h}^{0} |\varphi|^2\Big)^{1/2}. \end{aligned}$$

For arbitrarily fixed $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, let us define $\delta(\varepsilon) := \varepsilon/(1 + K_1)$. If $\|x_{t_0}\| < \delta(\varepsilon)$, then

$$|x_t| \leq V_1(t, x_t) \leq V_1(t_0, x_{t_0}) < \varepsilon \quad (t \geq t_0)),$$

which proves uniformity. □

The following concept is widely used in stability theory of non-autonomous differential equations [2, 7, 8, 9, 10, 13].

Definition 2.4. A locally integrable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is called *integrally positive* if for every $\delta > 0$ the inequality

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} \eta > 0 \tag{2.6}$$

holds.

Now we need this concept in a more general form.

Definition 2.5. Let $\eta, M : \mathbb{R}_+ \to \mathbb{R}_+$ be locally integrable, and for any $\nu > 0$ define

$$\Delta_M(t,\nu) := \inf\{\tau > 0 : \int_{t-\tau}^{t} M = \nu\}. \tag{2.7}$$

Function η is called *integrally positive with respect to M* if for every $\nu > 0$

$$\liminf_{t\to\infty} \int_{t-\Delta_M(t,\nu)}^{t} \eta > 0, \tag{2.8}$$

i.e., for every $\nu > 0$ there exists $W(\nu) > 0$ and $t_*(\nu)$ such that if $t > t_*(\nu)$, then $\int_{t-\Delta_M(t,\nu)}^{t} \eta > W(\nu)$.

Remark 2.6. If $M(t) \equiv c = \text{const.}$, then $\Delta_M(t,\nu) = \nu/c$, and η is integrally positive with respect to M if and only if it is integrally positive. Furthermore, if $t \mapsto \int_0^t M$ is uniformly continuous, then for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|t' - t''| < \delta(\varepsilon)$ implies $\int_{t'}^{t''} M < \varepsilon$, therefore $\Delta_M(t,\nu) \geq \delta(\nu)$. Consequently, integral positivity of η implies integral positivity of η with respect to M. It may be also interesting that every function is integrally positive with respect to itself. More generally, if $\eta \geq M$, then $\Delta_\eta(t,\nu) \leq \Delta_M(t,\nu)$ for all t, from which it follows that η is integrally positive with respect to M. Of course, the converse assertion is not true.

The following lemma is based on the method of annulus argument [5].

Lemma 2.7. *Suppose that condition* (A1) *is satisfied. If, in addition, either* $M(t) := |b(t)| \int_{t-h}^t |\lambda|$ *is integrable on* $\mathbb{R}_+$, *or function* $\eta_1(t) := a(t) - |\lambda(t)| \int_t^{t+h} |b|$ *is integrally positive with respect to* M, *then every solution of* (1.1) *has a finite limit as* t *goes to infinity.*

Proof. Suppose that the statement is not true, i.e., there exists a solution x having no limit at infinity. We can suppose that $|x(t)| \leq 1$ $(t \geq t_0)$, so there exist $0 < \varepsilon_1 < \varepsilon_2$ and a sequence $\{r_i, s_i\}_{i=1}^\infty$ such that

$$
\begin{gathered}
r_1 \geq h \quad r_i < s_i < r_{i+1}; \\
|x(r_i)| = \varepsilon_1, \quad |x(s_i)| = \varepsilon_2, \quad \varepsilon_1 \leq |x(t)| \leq \varepsilon_2 \ (r_i \leq t \leq s_i) \quad (i \in \mathbb{N}).
\end{gathered}
$$

Using the notation $v_1(t) := V_1(t, x_t)$, $v_2(t) := V_2(t, x_t) = |x(t)|$ and inequalities (2.2)–(2.4) we obtain the estimate

$$
v_1'(t) \leq -v_2'(t) + M(t). \tag{2.9}
$$

For every i there are two possibilities: (a) $\int_{r_i}^{s_i} M < (\varepsilon_2 - \varepsilon_1)/2$, and (b) $\int_{r_i}^{s_i} M \geq (\varepsilon_2 - \varepsilon_1)/2$.

In case (a) we integrate (2.9) and get

$$
v_1(s_i) - v_1(r_i) \leq -(\varepsilon_2 - \varepsilon_1) + \frac{\varepsilon_2 - \varepsilon_1}{2} = -\frac{\varepsilon_2 - \varepsilon_1}{2} < 0.
$$

In case (b) we have $s_i - r_i \geq \Delta_M(s_i, (\varepsilon_2 - \varepsilon_1)/2)$, therefore, by (2.2) and (2.7)

$$
v_1(s_i) - v_1(r_i) \leq -\varepsilon_1 \int_{r_i}^{s_i} \eta_1 \leq -\varepsilon_1 \int_{s_i - \Delta_M(s_i, (\varepsilon_2 - \varepsilon_1)/2)}^{s_i} \eta_1 \leq -\varepsilon_1 W\Big(\frac{\varepsilon_2 - \varepsilon_1}{2}\Big),
$$

provided that $s_i > t_*(\varepsilon_2 - \varepsilon_1)/2)$. At least one of the last two inequalities is satisfied for infinitely many i's, which means that $v_1(t) \to -\infty$ as $t \to \infty$, but this is a contradiction. ∎

Theorem 2.8. *Suppose that condition* (A1) *is satisfied. If*

$$
\int^\infty \Big(a(t) - |\lambda(t)| \int_t^{t+h} |b|\Big)\,\mathrm{d}t = \infty, \tag{2.10}
$$

and either $M(t) := |b(t)| \int_{t-h}^t |\lambda|$ *is integrable on* $\mathbb{R}_+$, *or function* $\eta_1(t) := a(t) - |\lambda(t)| \int_t^{t+h} |b|$ *is integrally positive with respect to* M, *then the zero solution of* (1.1) *is asymptotically stable.*

If, in addition, conditions (2.5) *and*

$$\lim_{S\to\infty}\int_t^{t+S}\Big(a(t)-|\lambda(t)|\int_t^{t+h}|b|\Big)\,\mathrm{d}t=\infty \quad \textit{uniformly w.r.t. } t\in\mathbb{R}_+ \tag{2.11}$$

are assumed, then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. Lemma 2.7 guaranties the existence of a finite limit for every solution of (1.1). If this limit is not equal to zero for a solution x, then (2.2) and (2.10) imply $\lim_{t\to\infty}V_1(t,x_t)=-\infty$, which is a contradiction.

To proof UAS we start off uniform stability; remember that $\delta(\varepsilon)$ corresponds to ε in this definition. Setting $D=\delta(1)$ and fixing $\mu>0$ we are looking for $T(\mu)$ introduced in the definition of UAS. Let $t_0\ge 0,\varphi$ ($\|\varphi\|<\delta(1)$) be arbitrary. According to the definition of uniform stability it is enough to proof that there exists a $T(\mu)$ such that $\|x_{t_0+R}(\cdot;t_0,\varphi)\|<\delta(\mu)$ for some $R\in[t_0,t_0+T(\mu)]$. If this is not true, then there is a $\overline{\mu}>0$ such that for every T there are t_0,φ such that $\|x_{t_0+R}(\cdot;t_0,\varphi)\|\ge\delta(\overline{\mu})$ for all $R\in[0,T]$. Let us choose a T arbitrarily and fix such t_0,φ to it. Then $|x(t)|=|x(t;t_0,\varphi)|$ takes a value not less than $\delta(\overline{\mu})$ in every subinterval of length h of the interval $[t_0,t_0+T]$.

On the other hand, we can prove that $|x|$ often takes values less than $\delta(\overline{\mu})/2$. In fact, by (2.11) there exists $T_1(\overline{\mu})$ such that

$$\int_t^{t+T_1(\overline{\mu})}\eta_1>\frac{2(1+K_1)\delta(1)}{\delta(\overline{\mu})}\quad(t\in\mathbb{R}_+).$$

If $|x(t)|\ge\delta(\overline{\mu})/2$ on an interval $[t',t'']\subset[t_0,t_0+T]$, then

$$\begin{aligned}-(1+K_1)\delta(1)&\le -v_1(t_0)\le v_1(t'')-v_1(t')\\&\le-\int_{t'}^{t''}\eta_1(t)|x(t)|\,\mathrm{d}t\le-\frac{\delta(\overline{\mu})}{2}\int_{t'}^{t''}\eta_1,\end{aligned}$$

hence $\int_{t'}^{t''}\eta_1\le 2(1+K_1)\delta(1)/\delta(\overline{\mu})$. Therefore $|t''-t'|\le T_1(\overline{\mu})$, i.e., $|x|$ takes values less than $\delta(\overline{\mu})/2$ in every subinterval of length $T_1(\overline{\mu})$ in the interval $[t_0,t_0+T]$.

According to the latter two paragraphs there is a sequence $\{r_i,s_i\}_{i=1}^N$ such that

$$\begin{gathered}r_1\ge t_0,\quad r_i<s_i<r_{i+1},\quad s_N\le t_0+T;\\ s_i-r_i\le h,\quad r_{i+1}-s_i\le T_1(\overline{\mu})+h;\\ |x(r_i)|=\frac{\delta(\overline{\mu})}{2},\ |x(s_i)|=\delta(\overline{\mu}),\ \frac{\delta(\overline{\mu})}{2}\le|x(t)|\ (r_i\le t\le s_i).\end{gathered}\tag{2.12}$$

If we suppose that N is the largest natural number with these properties, then $N\to\infty$ as $T\to\infty$ because of (2.12). Integrating (2.9) and repeating the reasoning after (2.9) in the proof of Lemma 2.7 we obtain that for every i ($1\le i\le N$) either

$$v_1(s_i)-v_1(r_i)\le-\frac{\delta(\overline{\mu})}{4}<0$$

or

$$v_1(s_i)-v_1(r_i)\le-\frac{\delta(\overline{\mu})}{2}W\big(\frac{\delta(\overline{\mu})}{4}\big)<0.$$

Therefore

$$\begin{aligned}-v_1(t_0)&\le v_1(s_N)-v_1(r_1)\\&\le -N\frac{\delta(\overline{\mu})}{4}\min\big\{1;W\big(\frac{\delta(\overline{\mu})}{4}\big)\big\}\end{aligned}$$

$$\to -\infty \quad (N \to \infty).$$

If $T \to \infty$, then $N \to \infty$, so T cannot be arbitrarily large, which is a contradiction. □

Using Remark 2.6 we obtain the following corollary.

Corollary 2.9. *Suppose that conditions* (A1) *and* (2.5) *are satisfied. If the function* $t \mapsto \int_0^t |b(u)| \int_{u-h}^u |\lambda| \, du$ *is uniformly continuous on* $\mathbb{R}_+$*, and* $t \to a(t) - |\lambda(t)| \int_t^{t+h} |b|$ *is integrally positive, then the zero solution of* (1.1) *is uniformly asymptotically stable.*

In the following theorem we can weaken the condition of the integral positivity requiring it only along a sequence $\{t_i\}$.

Theorem 2.10. *Suppose that* (A1) *holds and the following conditions are satisfied:*

- (i) *there is a sequence* $\{t_i\}_{i=1}^\infty$ $(t_{i+1} - t_i \geq h)$ *such that*
$$\Big(\int_{t_i}^{t_i+h} |b|\Big)\Big(\int_{t_i-h}^{t_i} |\lambda|^2\Big)^{1/2} \leq K_1 \quad (i \in \mathbb{N}) \tag{2.13}$$
with some constant K_1*;*
- (ii) *for every* $\varepsilon > 0$ *and for every sequence* $\{s_i\}$ $(s_i \in [t_i - h, t_i])$ *there holds*
$$\sum_{i=1}^\infty \int_{\max\{t_{i-1}; s_i - \Delta(s_i,\varepsilon)\}} \eta_1 = \infty. \tag{2.14}$$

Then the zero solution of (1.1) *is asymptotically stable.*

Suppose that, instead of (2.13) *estimate* (2.5) *holds, and instead of* (2.14)*,*

- (ii') *there are a sequence* $\{t_i\}_{i=1}^\infty$ *and a constant* K_2 $(h \leq t_{i+1} - t_i \leq K_2)$ *such that for every* $\varepsilon > 0$ *there is a* $\kappa_1(\varepsilon) > 0$ *with*
$$\int_{t-\Delta_M(t,\varepsilon)\}}^t \eta_1 \geq \kappa_1(\varepsilon) \quad (t \in [t_i - h, t_i]); \tag{2.15}$$

moreover there is a constant $\kappa_2 > 0$ *with*

- (iii') $\int_{t_{i-1}}^{t_i-h} \eta \geq \kappa_2 \quad (i \in \mathbb{N})$.

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. First we show that under conditions (i)-(ii) every solution x tends to zero as $t \to \infty$. Since $V_1(t_i, x_{t_i}) \leq (1+K_1)\|x_{t_i}\|$, it is sufficient to prove $\liminf_{i\to\infty} \|x_{t_i}\| = 0$. If this is not true, then we may suppose without any loss of the generality that $\|x_{t_i}\| \geq 3\varepsilon > 0$ for all $i \in \mathbb{N}$ with some $\varepsilon > 0$. This means that there exists a sequence $\{s_i \in [t_i - h, t_i]\}$ having the properties $v_2(s_i) := |x(s_i)| \geq 3\varepsilon$. Then for any i, either

- (a) $v_2(t) \geq \varepsilon$ in the interval $[t_{i-1}, s_i]$, or
- (b) there exists $r_i \in [t_{i-1}, s_i]$ such that $v_2(r_i) = \varepsilon$.

In case (a) we integrate (2.2) and obtain
$$v_1(t_i) - v_1(t_{i-1}) \leq v_1(s_i) - v_1(t_{i-1}) \leq -\varepsilon \int_{t_{i-1}}^{s_i} \eta_1 \leq -\varepsilon \int_{t_{i-1}}^{t_i-h} \eta_1. \tag{2.16}$$

In case (b) either we have

(b/1) $\Delta_M(s_i,\varepsilon) \ge s_i - r_i$ and
$$v_1(t_i) - v_1(t_{i-1}) \le v_1(s_i) - v_1(r_i) \le -2\varepsilon + \varepsilon = -\varepsilon < 0, \tag{2.17}$$
or

(b/2) $\Delta_M(s_i,\varepsilon) < s_i - r_i$, when
$$v_1(t_i) - v_1(t_{i-1}) \le -\varepsilon \int_{s_i-\Delta_M(s_i,\varepsilon)}^{s_i} \eta_1. \tag{2.18}$$

If case (b/1) occurs infinitely many times, then $\lim_{t\to\infty} v_1(t) = -\infty$, which is a contradiction. Otherwise, there is a natural number I such that for any $i > I$ either (a) or (b/2) is satisfied. Then from condition (2.14) we obtain $\sum_{i=I}^{\infty}(v_1(t_i) - v_1(t_{i-1})) = -\infty$, which results in a contradiction again. This concludes the proof of asymptotic stability.

Now turn to the proof of UAS. Conditions (A1) and (2.5) imply uniform stability; let $\delta(\varepsilon)$ correspond to ε in the sense of the definition of this property. For $t_0 \in \mathbb{R}_+$, φ ($\|\varphi\| < 1/(1+K_1)$) we have $v_2(t) \le v_1(t) \le 1$ for all $t \ge t_0$. Let i_0 denote the natural number of the property $t_{i_0-1} < t_0 \le t_{i_0}$. To prove UAS, for any $\mu > 0$ we will show the existence of an $I(\mu) \in \mathbb{N}$ such that $\max\{x(t_{i_0+I(\mu)} + \theta) : -h \le \theta \le 0\} < \delta(\mu)$. Suppose that for a fixed $\mu > 0$ such an index does not exists amongst $i_0, i_0+1, \dots, i_*$. Then one of the possibilities (a), (b/1), (b/2) occurs for every $i_0 \le i \le i_*$ with $\varepsilon := \delta(\mu)/3$. By the estimates (2.16)-(2.18), now these cases have the consequences

(a) $v_1(t_i) - v_1(t_{i-1}) \le -\varepsilon\kappa_2(\varepsilon)$;

(b/1) $v_1(t_i) - v_1(t_{i-1}) \le -\varepsilon$;

(b/2) $v_1(t_i) - v_1(t_{i-1}) \le -\varepsilon\kappa_1(\varepsilon)$.

Consequently, there is a $\kappa(\varepsilon) > 0$ such that
$$-1 \le v_1(t_{i_*}) - v_1(t_0) \le v_1(t_{i_*}) - v_1(t_{i_0}) \le -\kappa(\varepsilon)(i_* - i_0).$$
In other words, i_* must not be arbitrarily large; namely, $i_* - i_0 \le 1/\kappa(\varepsilon)$. This means that the choice $I(\mu) := [1/\kappa(\delta(\mu)/3)]$ is appropriate, where $[\alpha]$ denotes the fractional part of $\alpha \in \mathbb{R}$. □

Now we consider a modification of functional (2.1):
$$\begin{aligned} V_\beta(t,\varphi) :&= |\varphi(0)| + \beta \int_{-h}^{0}\int_{\theta}^{0} |b(t+\vartheta-\theta)||\lambda(t+\vartheta)||\varphi(\vartheta)|\,\mathrm{d}\vartheta\,\mathrm{d}\theta \\ &= V_2(t,\varphi) + \beta V_3(t,\varphi), \quad V_2(t,\varphi) := |\varphi(0)| \quad (\beta > 1). \end{aligned} \tag{2.19}$$
It is a consequence of Lemma 2.2 that
$$\begin{aligned} V_\beta'(t,\varphi) \le &-\Big(a(t) - \beta|\lambda(t)|\int_t^{t+h} |b|\Big)|\varphi(0)| \\ &- (\beta-1)|b(t)|\int_{-h}^{0} |\lambda(t+\theta)||\varphi(\theta)|\,\mathrm{d}\theta \quad (t \in \mathbb{R}_+). \end{aligned} \tag{2.20}$$
To make V_β non-increasing along the solutions of (1.1), we assume a stronger dominance condition

(A2) $a(t) - \beta|\lambda(t)|\int_t^{t+h} |b| \ge 0$ with some $\beta > 1$ ($t \in \mathbb{R}_+$).

This condition alone guaranties the existence of limits of solutions.

Lemma 2.11. *Condition* (A2) *implies that every solution of* (1.1) *has a finite limit as* t *goes to infinity.*

Proof. Suppose that the statement is not true, and consider a solution x and the sequence $\{r_i, s_i\}_{i=1}^{\infty}$ with properties (2.8) from the proof of the analogous Lemma 2.7. By using (2.3), (2.4), and (2.20), for the function $v_\beta(t) := V_\beta(t, x_t)$ we obtain the inequality

$$v'_\beta \le -(\beta - 1)v'_2(t).$$

Integrating this inequality we obtain

$$v_\beta(s_i) - v_\beta(r_i) \le -(\beta - 1)(\varepsilon_2 - \varepsilon_1) < 0 \quad (i \in \mathbb{N}),$$

which means that $v_\beta(t) \to -\infty$ as $t \to \infty$, but this is a contradiction. □

Theorem 2.12. *Suppose that* (A2) *is satisfied. If condition* (2.10) *also holds, then the zero solution of* (1.1) *is asymptotically stable.*

If, in addition, we assume (2.5) *and* (2.11)*, then the zero solution is uniformly asymptotically stable.*

Proof. We have to repeat the proof of Theorem 2.8 with the only modification that, instead of (2.9), now we have $v'_\beta(t) \le -v'_2(t)$. From this estimate it follows that

$$v_\beta(s_i) - v_\beta(r_i) \le -\frac{\delta(\overline{\mu})}{2} \quad (i = 1, 2, \dots, N).$$

Now this also implies $\lim_{t\to\infty} v_\beta(t) = -\infty$, a contradiction. □

2.1. **The case of $\lambda(t) \equiv 1$.** Consider the equation

$$x'(t) = -a(t)x + b(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s, \quad a(t) \ge 0 \ (t \in \mathbb{R}_+). \tag{2.21}$$

This equation was investigated by Ting Xiu Wang in [15, pp. 849–853, Theorems 3.1 and 3.2], where he applied his abstract theorems from [14] and proved very useful sufficient conditions for the asymptotic stability and uniform asymptotic stability of the zero solution of (2.21). Our Theorem 2.8 becomes more comparable to these results if we deduce a more explicit form of integral positivity for this special case.

Theorem 2.13. *Suppose that condition* (A1) *with $\lambda(t) \equiv 1$ is satisfied. If*

$$\int^{\infty} \Big(a(t) - \int_t^{t+h} |b|\Big)\,\mathrm{d}t = \infty, \tag{2.22}$$

and either $|b|$ is integrable on $\mathbb{R}_+$, or $t \mapsto \int_0^t |b|$ is uniformly continuous on $\mathbb{R}_+$ and for every $\kappa > 0$ there exist $\gamma(\kappa) > 0$ and $t_{**}(\kappa)$ such that

$$\frac{1}{\kappa}\int_t^{t+\kappa} a - \int_t^{t+\kappa+h} |b| \ge \gamma(\kappa) \quad (t \ge t_{**}(\kappa)), \tag{2.23}$$

then the zero solution of (2.21) *is asymptotically stable.*

If, in addition,

$$\lim_{S\to\infty}\int_t^{t+S} \Big(a(t) - \int_t^{t+h} |b|\Big)\,\mathrm{d}t = \infty \quad \textit{uniformly w.r.t. } t \in \mathbb{R}_+, \tag{2.24}$$

then the zero solution of (2.21) *is uniformly asymptotically stable.*

Proof. The assertions follow from Theorem 2.8 if we prove that (2.23) is sufficient for the integral positivity of $t \to a(t) - \int_t^{t+h} |b|$ with respect to $|b|$. Let $\nu > 0$ be arbitrarily fixed. Since $t \mapsto \int_0^t |b|$ is uniformly continuous, there exists a $\delta(\nu) > 0$ such that $\Delta = \Delta_{|b|}(t, \nu) > \delta(\nu)$. Therefore, changing the order of the successive integration and using (2.23) we obtain

$$\begin{aligned}
&\int_t^{t+\Delta} a - \int_t^{t+\Delta} \int_s^{s+h} |b(u)| \,\mathrm{d}u \,\mathrm{d}s \\
&\geq \int_t^{t+\Delta} a - \int_t^{t+\Delta} \int_t^{t+\Delta+h} |b(u)| \,\mathrm{d}u \,\mathrm{d}s \\
&= \int_t^{t+\Delta} a - \Delta \int_t^{t+\Delta+h} |b| \\
&= \Delta\Big(\frac{1}{\Delta} \int_t^{t+\Delta} a - \int_t^{t+\Delta+h} |b|\Big) \\
&\geq \delta(\nu)\gamma(\Delta_{|b|}(t,\nu)) =: W(\nu) > 0 \quad \text{for } t \geq t_{**}(\Delta_{|b|}(t,\nu)) =: t_*(\nu),
\end{aligned}$$

which means the desired integral positivity. □

Parts of Wang's results (Theorem 3.1 (a), Theorem 3.2 (a)) are consequences of Theorems 2.8 and 2.12, another part are independent of our theorems. Theorem 2.13 does not require some key conditions of Wang's theorem (for example, function $t \to a(t) - \int_t^{t+h} |b|$ is non-decreasing).

3. Lyapunov-Razumikhin method

This method uses *functions* $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ instead of *functionals* $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$. For example, if we define the function $V(\varphi(0)) := |\varphi(0)|$ ($\varphi \in C$) to equation (1.1), then, by (2.3) and (2.4), the derivative of V with respect to (1.1) allows the estimate

$$V'(t,\varphi) \leq -a(t)|\varphi(0)| + |b(t)| \int_{t-h}^{t} |\lambda(s)||\varphi(s-t)| \,\mathrm{d}s. \tag{3.1}$$

In the Lyapunov-Razumikhin method function V is called a *Lyapunov function* if

$$V'(t,\varphi) \leq 0, \text{ provided that } V(t+\theta, \varphi(\theta)) \leq V(t, \varphi(0)) \ (-h \leq \theta \leq 0). \tag{3.2}$$

So $V(\varphi(0)) := |\varphi(0)|$ will be a Lyapunov function to (1.1) if we require

(A3) $a(t) - |b(t)| \int_{t-h}^{t} |\lambda| \geq 0$ ($t \in \mathbb{R}_+$),

which is a new dominance condition to equation (1.1). The following lemma can be proved easily (see, e.g., [4, Theorem 5.4.1]).

Lemma 3.1. *If* (A3) *is satisfied, and x is a solution of* (1.1)*, then the functional* $\overline{V}(t, x_t) := \sup\{V(x(t+\theta)) : -h \leq \theta \leq 0\}$ *is non-increasing in $\mathbb{R}_+$, and the zero solution of* (1.1) *is uniformly stable.*

This lemma shows that the Lyapunov-Razumikhin method can be very effective: the dominance condition (A3) alone guaranties uniform stability without any boundedness condition of type (2.5).

Lemma 3.2. *If*

(A4) $a(t) - \beta|b(t)| \int_{t-h}^{t} |\lambda| \geq 0$ *($t \in \mathbb{R}_+$) with some constant $\beta > 1$,* *then every solution has a finite limit as t appraoches infinity.*

Proof. Expression (3.1) and condition (A4) imply that

$$V'(t,\varphi) \le -\Big(a(t) - \beta|b(t)|\int_{t-h}^{t}|\lambda|\Big)|\varphi(0)| \le 0, \tag{3.3}$$

provided that $V(\varphi(\theta)) \le \beta V(\varphi(0))$ $(-h \le \theta \le 0)$.

For an arbitrary solution x introduce the notation

$$\begin{gathered} v(t) := V(x(t)) = |x(t)|, \quad \overline{v}(t) := \sup\{v(t+\theta) : -h \le \theta \le 0\}, \\ \overline{v}_0 := \lim_{t\to\infty} \overline{v}(t). \end{gathered} \tag{3.4}$$

For any $\varepsilon > 0$ there is $t_*(\varepsilon)$ such that $t > t_*(\varepsilon)$ implies $\overline{v}_0 \le \overline{v}(t) < \overline{v}_0 + \varepsilon$. If x has no limit, then $x(t) \not\to \overline{v}_0$, and there are ε $(0 < \varepsilon < \overline{v}_0(\beta-1)/(\beta+1))$ and r, s with the properties

$$t_*(\varepsilon) < r < s, \quad v(r) = \overline{v}_0 - \varepsilon, \quad v(s) = \overline{v}_0, \quad \overline{v}_0 - \varepsilon \le v(t) \le \overline{v}_0$$

if $r \le t \le s$. Therefore, if $r \le t \le s$, then

$$\beta v(t) > \beta(\overline{v}_0 - \varepsilon) > \overline{v}_0 + \varepsilon > v(t+\theta) \quad (-h \le \theta \le 0), \tag{3.5}$$

hence $v'(t) \le 0$, which is a contradiction. □

Theorem 3.3. *If* (A4) *is satisfied, and*

$$\int^{\infty} \Big(a(t) - \beta|b(t)|\int_{t-h}^{t}|\lambda|\Big)\,\mathrm{d}t = \infty, \tag{3.6}$$

then the zero solution of (1.1) *is asymptotically stable.*

Proof. If x is an arbitrary solution, then, by Lemma 3.2, $\lim_{t\to\infty}|x(t)| = \overline{v}_0$. We have to show that $\overline{v}_0 = 0$. Suppose the contrary, i.e., $\overline{v}_0 > 0$. Then for every ε $(0 < \varepsilon < \overline{v}_0(\beta-1)/(\beta+1))$ there is a $t_*(\varepsilon)$ such that (3.5) holds for all $t > t_*(\varepsilon)$. Consequently, (3.3) yields

$$\begin{gathered} v'(t) \le -\eta_4(t)(\overline{v}_0 - \varepsilon) \quad \eta_4(t) := a(t) - \beta|b(t)|\int_{t-h}^{t}|\lambda|, \\ v(t) - v(t_*) \le -(\overline{v}_0 - \varepsilon)\int_{t_*}^{t}\eta_4 \to -\infty \quad (t\to\infty), \end{gathered}$$

a contradiction. □

Theorem 3.4. *If* (A4) *is satisfied and*

$$\lim_{S\to\infty}\int_{t}^{t+S}\Big(a(t) - \beta|b(t)|\int_{t-h}^{t}|\lambda|\Big)\,\mathrm{d}t = \infty \quad \textit{uniformly w.r.t. } t \in \mathbb{R}_+, \tag{3.7}$$

then the zero solution is uniformly asymptotically stable.

Proof. Let $\delta(\varepsilon)$ belong to ε in the sense of uniform stability and let $t_0 \ge 0, \varphi$ $(\|\varphi\| < \delta(1))$ be arbitrary. It is sufficient to prove that for every $\mu > 0$ there exists a $T(\mu)$ such that $\|x_{t_0+T(\mu)}(\cdot; t_0, \varphi)\| < \delta(\mu)$.

Given $\mu > 0$ introduce some notation. Let $N = N(\mu)$ denote the smallest natural number for which $1/\beta^N < \delta(\mu)$ holds. Condition (3.7) guaranties the existence of an $S_*(\mu)$ such that

$$\int_{t}^{t+S_*(\mu)}\eta_4 > (\beta-1)\beta^{N-1}, \quad \eta_4(t) := a(t) - \beta|b(t)|\int_{t-h}^{t}|\lambda|. \tag{3.8}$$

Finally, set $t_i := t_0 + i(S_*(\mu) + h)$ $(i = 1, 2, \dots, N)$.

We state that

$$\text{if } t > t_i - h, \text{ then } v(t) := |x(t)| \le \frac{1}{\beta^i} \text{ for } i = 0, 1, \dots, N. \tag{3.9}$$

In fact, the assertion is true for $i = 0$. Using the method of the mathematical induction, we assume that the assertion is true for some i $(0 \le i < N)$ and prove that it is also true for $i + 1$.

If $t \ge t_i$ and $v(t) \ge 1/\beta^{i+1}$, then

$$\beta v(t) \ge \frac{1}{\beta^i} \ge v(t + \theta) \quad (-h \le \theta \le 0),$$

hence, by (A4), we have

$$v'(t) \le -\eta_4(t) v(t) \le 0.$$

This means that if $s > t_i$ and $v(s) \le 1/\beta^{i+1}$, then $v(s + \tau) \le 1/\beta^{i+1}$ for all $\tau \ge 0$. Therefore, if $t \ge t_i$, $v(t) > 1/\beta^{i+1}$, then $v(s) > 1/\beta^{i+1}$ for $s \in [t_i, t]$ and

$$v'(s) \le -\eta_4(s) v(s) \le -\frac{1}{\beta^{i+1}} \eta_4(s) \le -\frac{1}{\beta^N} \eta_4(s).$$

Consequently, if $t \ge t_i + S_*(\mu)$ and $v(t) > 1/\beta^{i+1}$, then

$$v(t) - v(t_i) \le -\frac{1}{\beta^N} \int_{t_i}^{t_i + S_*(\mu)} \eta_4 < -\frac{1}{\beta^N} (\beta - 1) \beta^{N-1} = -(1 - \frac{1}{\beta}).$$

On the other hand,

$$v(t) - v(t_i) \ge \frac{1}{\beta^{i+1}} - \frac{1}{\beta^i} = -\frac{1}{\beta^i}(1 - \frac{1}{\beta}) \ge -(1 - \frac{1}{\beta}),$$

which is a contradiction. Consequently, if we define $t_{i+1} := t_i + S_*(\mu) + h$, and if $t > t_{i+1} - h$, then $v(t) \le 1/\beta^{i+1}$, i.e., assertion (3.9) is true for $i + 1$. By the rule of the mathematical induction, assertion (3.9) is true for $i = N$. In other words, if $t > t_N$, then $v(t) \le 1/\beta^N < \delta(\mu)$. So the definition $T(\mu) := N(\mu)(S_*(\mu) + h)$ completes the proof. □

In some applications it may happen that dominance condition (A4) is not satisfied with a uniform (independent of t) coefficient $\beta > 1$, but it is satisfied with $\beta(t) \ge 1$. The remaining part of the section is devoted to this situation. Suppose that

- (A5) $a(t) - \beta(t)|b(t)| \int_{t-h}^{t} |\lambda| \ge 0$ $(t \in \mathbb{R}_+)$ with some piecewise continuous function $\beta : \mathbb{R}_+ \to [1, \infty)$.

This condition and (3.1) imply

$$V'(t, \varphi) \le -\Big(a(t) - \beta(t)|b(t)| \int_{t-h}^{t} |\lambda|\Big) |\varphi(0)| \le 0, \tag{3.10}$$

provided that $V(\varphi(\theta)) \le \beta(t) V(\varphi(0))$ $(-h \le \theta \le 0)$ instead of (3.3).

Now we import two lemmas from Kato [9], in which he transformed estimate (3.10) into a comparison statement. Let $U : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be measurable with respect to the first variable and locally Lipschitz with respect to the second one, and denote by $u(\cdot; t_*, \alpha) : [t_* - h, \infty) \to \mathbb{R}$ the solution of the initial value problem

$$u' = U(t, u), \quad u(t_*) = \alpha. \tag{3.11}$$

If $v : [t_0 - h, \infty) \to \mathbb{R}$ $(t_0 \ge 0)$ is a continuous function, then define
$$v'(t) := \limsup_{\delta \to 0+0} \Big(\frac{1}{\delta}(v(t+\delta) - v(t)) \Big).$$

Lemma 3.5 ([9, Lemma 1]). *Suppose that*
$$\begin{gathered} v'(t) \le U(t, v(t)) \quad \text{if} \quad t \ge t_* \text{ and} \\ v(t+\theta) \le u(t+\theta; t, v(t)) \quad (-h \le \theta \le 0). \end{gathered} \tag{3.12}$$
Then
$$v(t) \le u(t; t_*, \alpha) \quad (t \ge t_*),$$
provided that $v(t_* + \theta) \le u(t_* + \theta; t_*, \alpha)$ $(-h \le \theta \le 0)$.

As a consequence of (3.10), from [9, Lemma 2] we obtain the following lemma.

Lemma 3.6. *If* (A5) *is satisfied, then there exists a function* $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_-$ *such that for every solution* x *of* (1.1) *the function* $v(t) := |x(t)|$ *satisfies the differential inequality* (3.12) *with* $t_* = t_0$. *Namely, the choice*
$$\begin{gathered} U(t,u) := -\min\Big\{ \frac{1}{3h}; \frac{2}{3h} \min\{\beta(t+\tau) - 1 : 0 \le \tau \le h\}; \eta_5(t) \Big\} u \\ (u > 0) \quad \Big(\eta_5(t) := a(t) - \beta(t)|b(t)| \int_{t-h}^{t} |\lambda| \ge 0 \Big) \end{gathered} \tag{3.13}$$
is appropriate.

Theorem 3.7. *Suppose that* (A5) *is satisfied. If*
$$\int^{\infty} \gamma = \infty \tag{3.14}$$
$$\Big(\gamma(t) := \min\big\{ 1; \min\{\beta(t+\tau) : 0 \le \tau \le h\} - 1; a(t) - \beta(t)|b(t)| \int_{t-h}^{t} |\lambda| \big\} \Big),$$
then the zero solution of (1.1) *is asymptotically stable. If*
$$\lim_{S \to \infty} \int_t^{t+S} \gamma = \infty \quad \text{uniformly w.r.t. } t \in \mathbb{R}_+, \tag{3.15}$$
then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. If $U(t,u) = -\Gamma(t)u$ $(\Gamma(t) \ge 0)$ in (3.13), then the solution of the initial value problem (3.11) is
$$u(t; t_*, \alpha) = \alpha \exp\big[-\int_{t_*}^{t} \Gamma \big].$$

It is easy to see that for arbitrary measurable functions $A, B : \mathbb{R}_+ \to \mathbb{R}_+$ the divergence
$$\int^{\infty} \min\{A(t); B(t)\}\, \mathrm{d}t = \infty$$
is true if and only if
$$\int^{\infty} \min\{A(t); qB(t)\}\, \mathrm{d}t = \infty$$
holds for all $q > 0$. Therefore (3.14) implies $\lim_{t\to\infty} u(t; t_*, \alpha) = 0$ for all t_*, α.

For arbitrary t_0, φ, applying Lemma 3.5 to $v(t) = |x(t; t_0, \varphi)|$ with $t_* = t_0$, $\alpha = \|\varphi\|$ we obtain $|x(t; t_0, \varphi)| \le u(t; t_0, \|\varphi\|) \to 0$ if $t \to \infty$, which proves asymptotic stability.

For UAS, take $\delta(1) > 0$ from the definition of the uniform stability. To prove UAS we have to show that for every $\mu > 0$ there is a $T(\mu)$ such that $[t_0 \in \mathbb{R}_+$, $\|\varphi\| < \delta(1)$, $t \geq t_0 + T(\mu)]$ imply $|x(t; t_0, \varphi)| < \mu$. We show that the choice of $T(\mu)$ with the properties

$$\int_{t_0}^{t_0+T(\mu)} \gamma > \ln \frac{\delta(1)}{\mu}$$

is appropriate. In fact, by Lemma 3.5 we have

$$\begin{aligned} |x(t; t_0, \varphi)| &\leq u(t; t_0, \|\varphi\|) \leq u(t_0 + T(\mu); t_0, \delta(1)) \\ &\leq \delta(1) \exp\Big[- \int_{t_0}^{t_0+T(\mu)} \gamma \Big] < \delta(1) \exp\big[- \ln \frac{\delta(1)}{\mu} \big] = \mu. \end{aligned} \tag{3.16}$$

□

4. Examples

Example 4.1. Let a sequence $\{\bar{t}_i\}_{i=1}^{\infty}$ and a number δ $(0 < \delta < h,\ \bar{t}_{i+1} \geq \bar{t}_i + (2h + \delta))$ be given and define the functions

$$a(t) :\equiv 1; \quad b(t) := \begin{cases} 0 & \text{if } \bar{t}_i \leq t < \bar{t}_i + \delta, \\ \frac{1}{h} & \text{otherwise}; \end{cases} \quad \lambda(t) :\equiv 1 \tag{4.1}$$

for the coefficients in (1.1).

The zero solution of equation (1.1) with coefficients (4.1) is asymptotically stable. If there is a constant K_2 such that $\bar{t}_{i+1} - \bar{t}_i \leq K_2$, then the asymptotic stability is uniform.

Define $t_i := \bar{t}_i + \delta$ and apply Theorem 2.10. It is easy to see that $\eta_1(t) = a(t) - \int_t^{t+h} |b|$ and $M(t) = h|b(t)|$ have the properties

$$\begin{gathered} \Delta_M(t, \varepsilon) \geq \varepsilon \quad (t \in \mathbb{R}_+), \\ \int_{t-\Delta_M(t,\varepsilon)}^{t} \eta_1 \geq \frac{\varepsilon^2}{2h} =: \kappa_1(\varepsilon) \quad \text{if } t \in [t_i - h, t_i]; \\ \int_{t_i}^{t_{i+1}-h} \eta_1 \geq \frac{\delta^2}{h} =: \kappa_2 \quad (i \in \mathbb{N}), \end{gathered}$$

from which the assertion follows.

Interpreting the result we can say that if sometimes the delayed perturbation underacts with respect to the balance $a = hb$ for an arbitrarily short time, then the instantaneous stabilizer can stabilize the equilibrium.

It may be noticed that the other results of this paper and Wang's theorems [14, 15] cannot be applied to this example. The same can be noticed also in the cases of the further examples.

Example 4.2. For a given sequence $\{p_i\}_{i=1}^{\infty}$ $(0 < p_i \leq 1)$ let us define the coefficients

$$\begin{gathered} a(t) := \begin{cases} 1 & \text{if } (2i-2)h \leq t < (2i-1)h, \\ 0 & \text{otherwise}; \end{cases} \\ b(t) := \begin{cases} \frac{p_i}{h} & \text{if } (2i-2)h \leq t < (2i-1)h, \\ 0 & \text{otherwise} \end{cases} \quad (i \in \mathbb{N});\ \lambda(t) :\equiv 1. \end{gathered} \tag{4.2}$$

If

$$\lim_{I\to\infty}\Big(\sum_{i=i_*}^{i_*+I}(1-p_i)\Big)=\infty \quad (i_*\in\mathbb{N}), \tag{4.3}$$

then the zero solution of (1.1) with coefficients (4.2) is asymptotically sable. If the divergence in (4.3) is uniform with respect to $i_*\in\mathbb{N}$, then the asymptotic stability is uniform.

We apply Theorem 3.7. Define

$$\beta(t):=\begin{cases}\frac12(1+\frac{1}{p_i}) & \text{if } (2i-2)h\le t<(2i-1)h,\\ 2 & \text{otherwise.}\end{cases} \tag{4.4}$$

According to definitions (4.2) and (4.4), we have

$$a(t)-\beta(t)|b(t)|\int_{t-h}^{t}|\lambda|=\begin{cases}\frac12(1-p_i) & \text{if } (2i-2)h\le t<(2i-1)h,\\ 0 & \text{otherwise;}\end{cases}$$

$$\underline{\beta}(t):=\min\{\beta(t+\tau):0\le\tau\le h\}=\begin{cases}\frac12(1+\frac{1}{p_i}) & \text{if } p_i\ge\frac13,\\ 2 & \text{if } p_i<\frac13\end{cases}$$

for $(2i-2)h\le t<2i$, whence for $\gamma(t)$ in (3.14) we obtain

$$\gamma(t)=\begin{cases}\frac12\,(1-p_i) & \text{if } (2i-2)h\le t<(2i-1)h,\\ 0 & \text{otherwise.}\end{cases}$$

Now the assertion is a corollary of Theorem 3.7.

Example 4.3. For a given $\alpha>1$ let us define the functions

$$a(t):=\alpha(t+\frac{h}{2});\quad b(t):=\frac{t}{h};\quad \lambda(t):=1\quad (t\in\mathbb{R}_+). \tag{4.5}$$

The zero solution of (1.1) with coefficients (4.5) is asymptotically sable.

We have $\eta_1(t):=a(t)-\int_t^{t+h}|b|=(\alpha-1)(t+h/2)$ and $a(t)-\alpha\int_t^{t+h}|b|\equiv 0$, so Theorem 2.12 can be applied.

Let us modify the definition of λ so that

$$\lambda(t):=\begin{cases}\alpha & \text{if } i\le t\le i+\frac{1}{i^2},\\ 1 & \text{otherwise,}\end{cases}\quad (i\in\mathbb{N});\quad M(t):=|b(t)|\int_{t-h}^{t}|\lambda|\le\alpha t.$$

Then we have

$$\eta_1(t):=a(t)-|\lambda(t)|\int_t^{t+h}|b|=\begin{cases}0 & \text{if } i\le t\le i+\frac{1}{i^2},\\ (\alpha-1)(t+\frac{h}{2}) & \text{otherwise,}\end{cases}$$

and $\Delta_M(t,\nu)\ge\nu/(2\alpha t)$. Applying Theorem 2.8 we obtain that the assertion remains true. Obviously Theorem 2.12 cannot be applied.

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