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BOUNDARY-VALUE PROBLEMS WITH INTEGRAL CONDITIONS FOR A SYSTEM OF LAMÉ EQUATIONS IN THE SPACE OF ALMOST PERIODIC FUNCTIONS

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Abstract. We study a problem with integral boundary conditions in the time coordinate for a system of Lamé equations of dynamic elasticity theory of an arbitrary dimension. We find necessary and sufficient conditions for the existence and uniqueness of solution in the class of almost periodic functions in the spatial variables. To solve the problem of small denominators arising while constructing solutions, we use the metric approach.

1. Introduction

Many physical, biological and other processes can be described using problems with nonlocal conditions for partial differential equations. As a particular case of such conditions, there are integral conditions, which may be interpreted as measuring the solution's mean values (local conditions are interpreted as measuring at certain points).

Problems with integral conditions for partial differential equations have been studied by many research; see the references in this article and their references. Generally speaking, such problems are ill-posed in the sense of Hadamard, and their solvability in corresponding spaces of functions is connected with lower estimates of small denominators with complex nonlinear structure [8, 15].

The problem of finding almost periodic solutions in spatial coordinates for the system of equations of dynamic elasticity theory with conditions in time variable that are linear combinations of moment-type integral conditions and local boundary conditions on the time interval $[0, T]$, was studied in [7, 12, 17]. The Cauchy problem for this system was investigated in [10, 14].

In this article, we study the problem with integral boundary conditions in the time variable on $[0, T]$ for generalized systems of Lamé equations in almost periodic function spaces.

The problem statement is made in the first section of the paper. In the second one, we distinguish two specific cases of the problem and give appropriate formal solutions. In the third and fourth sections, we establish the solvability conditions of the problem for each of the two cases. The paper is finalized by giving a conclusions section.

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Problem statement. In this section, we introduce the domain in which we consider the problem, the system of partial differential equations (Lamé system) and the integral boundary conditions, as well as the spaces of almost periodic functions and the definition of solution.

In the domain $Q = [0, T] \times \mathbb{R}^p$, $p \in \mathbb{N}$, $T > 0$ of variables $(t, x) = (t, x_1, \dots, x_p)$, we consider the problem of finding an almost periodic (in vector variable x) solution with the spectrum

$$M = \{\mu_k = (\mu_{k1}, \dots, \mu_{kp}) \in \mathbb{R}^p : k \in \mathbb{Z}^p\}$$

of the system of partial differential equations

$$\sigma \partial_t^2 u = \mu^* \partial_x \partial_x^\dagger u + (\lambda^* + \mu^*) \partial_x^\dagger \partial_x u, \tag{1.1}$$

which on the interval $[0, T]$ satisfies the integral boundary conditions

$$\begin{gathered} \alpha_1 u(0, x) + \beta_1 \int_0^T \overset{r_1}{t}\, u(t, x)\, dt = \varphi_1(x), \\ \alpha_2 u(T, x) + \beta_2 \int_0^T \overset{r_2}{t}\, u(t, x)\, dt = \varphi_2(x), \end{gathered} \tag{1.2}$$

where $x = (x_1, \dots, x_p) \in \mathbb{R}^p$, $\partial_x = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}\big)$, $\partial_t = \frac{\partial}{\partial t}$ and $\overset{r}{t} = \frac{t^r}{r!}$; the system parameters σ, λ^*, μ^* are positive numbers, $\vec{\alpha} = (\alpha_1, \alpha_2)$, the vector parameters $\vec{\beta} = (\beta_1, \beta_2)$ from conditions (1.2) are complex, and vector parameter $\vec{r} = (r_1, r_2)$ is integer-valued, in particular, $|\alpha_1|^2 + |\beta_1|^2 > 0$, $|\alpha_2|^2 + |\beta_2|^2 > 0$, $r_1 \geq 0$, $r_2 \geq 0$, and † is the transposition operation. The given functions φ_1 and φ_2 and the desired solution u are p-dimensional vectors.

When $p = 3$, system (1.1) is called the Lamé system [13, 14], which describes the stress state of an isotropic homogeneous elastic solid in displacements, where σ is the medium density, λ^*, μ^* are the Lamé coefficients, t and x are the time and spatial point respectively.

If $\vec{\beta} = 0$, then the functions φ_1 and φ_2 in conditions (1.2) can be considered as the results of the function $u = u(t, \cdot)$ measurements at the $[0, T]$ interval extreme points (left and right). In the opposite case, the measurements at specified points are supplemented with integral measurements of the r_1- and r_2-order moments of function u through the whole interval $[0, T]$. Moreover, if $\vec{\alpha} = 0$, then point measurements are not conducted.

The solution of problem (1.1), (1.2) is being searched as a function of variable t, with values in the scale $\{H_M^q\}_{q \in \mathbb{R}}$ of Hilbert spaces H_M^q of almost periodic with spectrum M functions [14], obtained by the completion of the set H_M of trigonometric vector polynomials of the form

$$v(x) = \sum v_k \exp(i\mu_k, x) \equiv \sum v_k \exp(i\mu_{k1} x_1 + \dots + i\mu_{kp} x_p),$$

by the norm

$$\|v; H_M^\alpha\| = \Big(\sum_{k \in \mathbb{Z}^p} \|v_k\|^2 \big(1 + \|\mu_k\|^2\big)^\alpha \Big)^{1/2},$$

where $\mu_k = (\mu_{k1}, \dots, \mu_{kp}) \in M$, $\|\cdot\|$ is the Euclid norm. The embedding $H_M \subset H_M^q \subset H_M'$, where H_M' is the adjoint space to H_M, is continuous for all $q \in \mathbb{R}$.

Let us impose on the spectrum M of almost periodic functions from the scale of spaces H_M^α, the condition of distinctness of the spectrum elements ($\mu_{\tilde{k}} \neq \mu_{\tilde{\tilde{k}}}$ when

$\tilde{k} \neq \tilde{\tilde{k}}$) and also the increase condition

$$d_1\|k\|^{\theta_1} \leq \|\mu_k\| \leq d_2\|k\|^{\theta_2} \tag{1.3}$$

with real parameters $\vec{d} = (d_1, d_2)$ and $\vec{\theta} = (\theta_1, \theta_2)$, where $0 < d_1 \leq d_2$, $0 < \theta_1 \leq \theta_2$ (therefore $\mu_0 = 0$).

Let us denote by $H_M^{2,\alpha}$ the space of functions $u = u(t,x)$ such that $\partial_t^j u \in \mathbf{C}([0,T]; H_M^{\alpha-j})$, and assume that

$$\|u; H_M^{2,\alpha}\|^2 = \sum_{j=0}^{2} \|\partial_t^j u; \mathbf{C}([0,T]; H_M^{\alpha-j})\|^2.$$

Definition 1.1. A function u from the space $H_M^{2,q}$, where $q \in \mathbb{R}$, which satisfies on the interval $(0,T)$ equation (1.1) and conditions (1.2) in the space H_M', is called the solution of problem (1.1), (1.2).

Definition 1.1 implies that $\varphi_1 \in H_M^q$ and $\varphi_2 \in H_M^q$ are *the necessary conditions* for the existence of a solution u of problem (1.1), (1.2) in the space $H_M^{2,q}$, and, in particular

$$\|\varphi_j; H_M^q\|^2 \leq 2\big(|\alpha_j|^2 + |\beta_j|^2 \overset{r_1+1}{T}\big)\|u; H_M^{2,q}\|^2, \quad j = 1, 2.$$

Definition 1.2. If there exist such real numbers q and q' so that for two arbitrary functions φ_1 and φ_2 from the space $H_M^{q'}$ there is a unique solution for the problem (1.1), (1.2) in the space $H_M^{2,q}$, then this problem is called uniquely solvable.

The necessary conditions imply that $q \leq q'$ in definition 1.2.

2. Solution formulas

In this section, we introduce the denotations for moments of imaginary exponent, present the solution of problem (1.1), (1.2) with necessary and sufficient conditions of its solvability, and also distinguish two specific cases in conditions (1.2). Let us denote $\mathcal{I}_\pm(-1) \equiv \mathcal{I}_\pm(-1,\gamma) = 1$, and also

$$\mathcal{I}_+(r) \equiv \mathcal{I}_+(r,\gamma) = \int_0^T \overset{r}{t} e^{i\gamma t}\, dt, \quad \mathcal{I}_-(r) \equiv \mathcal{I}_-(r,\gamma) = \int_0^T \overset{r}{t} e^{-i\gamma t}\, dt, \tag{2.1}$$

$$\mathcal{I}^{t,+}(r) = 2\operatorname{Re}\big[e^{i\gamma t}\mathcal{I}_-(r)\big], \quad \mathcal{I}^{t,-}(r) = 2i\operatorname{Im}\big[e^{i\gamma t}\mathcal{I}_-(r)\big], \tag{2.2}$$

$$\mathcal{I}^+(\vec{r}) = 2\operatorname{Re}\big[\mathcal{I}_+(r_1)\mathcal{I}_-(r_2)\big], \quad \mathcal{I}^-(\vec{r}) = 2i\operatorname{Im}\big[\mathcal{I}_+(r_1)\mathcal{I}_-(r_2)\big], \tag{2.3}$$

where r, r_1, r_2 are positive integers, $\vec{r} = (r_1, r_2)$ and γ is a positive parameter.

The solution of problem (1.1), (1.2) was found in [15] in the form

$$u = \sum_{k\in\mathbb{Z}^p} e^{i(\mu_k,x)} \sum_{j=1}^{2} \Big(\frac{g_j(t,\gamma_0\|\mu_k\|)}{\Delta(\gamma_0\|\mu_k\|)}\Pi_k^0 + \frac{g_j(t,\gamma_1\|\mu_k\|)}{\Delta(\gamma_1\|\mu_k\|)}\Pi_k^1\Big)\varphi_{jk}, \tag{2.4}$$

where $\Pi_k^1 = \frac{\mu_k^\dagger \mu_k}{\|\mu_k\|^2}$ and $\Pi_k^0 = I_p - \Pi_k^1$ are the projection operators on one-dimensional space induced by the vector μ_k, and on its complement in the space $\mathbb{R}^p$ respectively, $0 < \gamma_0^2 \equiv \mu^*/\sigma < \gamma_1^2 \equiv (\lambda^* + 2\mu^*)/\sigma$,

$$\Delta(0) = \alpha_1\alpha_2 T + \alpha_1\beta_2(r_2+1)\overset{r_2+2}{T} + \alpha_2\beta_1\overset{r_1+2}{T} + \beta_1\beta_2\frac{r_2-r_1}{T}\overset{r_1+2}{T}\overset{r_2+2}{T}, \tag{2.5}$$

$$\Delta(\gamma) = -2i\alpha_1\alpha_2 \sin\gamma T + \alpha_1\beta_2\mathcal{I}^{0,-}(r_2) - \alpha_2\beta_1\mathcal{I}^{T,-}(r_1) + \beta_1\beta_2\mathcal{I}^{-}(\vec{r}), \quad \gamma > 0, \tag{2.6}$$

$$g_1(t,0) = \alpha_2(T-t) + \beta_2 \overset{r_2+1}{T}\Big(\frac{r_2+1}{r_2+2}T - t\Big), \tag{2.7}$$

$$g_2(t,0) = \alpha_1 t - \beta_1 \overset{r_1+1}{T}\Big(\frac{r_1+1}{r_1+2}T - t\Big), \tag{2.8}$$

$$g_1(t,\gamma) = 2i\alpha_2 \sin\gamma(t-T) + \beta_2\mathcal{I}^{t,-}(r_2), \quad \gamma > 0, \tag{2.9}$$

$$g_2(t,\gamma) = -2i\alpha_1 \sin\gamma t - \beta_1\mathcal{I}^{t,-}(r_1), \quad \gamma > 0. \tag{2.10}$$

For the unique solvability of problem (1.1), (1.2), it is necessary and sufficient [7, Theorem 1] that, for all $k \in Z^p$ the estimate below holds:

$$\big|\Delta(\gamma_l\|\mu_k\|)\big| \geq C_{0l}(1+\|\mu_k\|^2)^{-q_0/2}, \tag{2.11}$$

where $q_0 \geq 0$, $C_{0l} > 0$ for $l = 1,2$.

Such estimate is established in when $\alpha_1\alpha_2 \neq 0$ [7, Theorem 2] and in case when $\alpha_1 = \alpha_2 = 0$ [7, Theorems 3-5]. The other two cases:
Case (1) $\alpha_1 = 0$, $\alpha_2\beta_1 \neq 0$ and
Case (2) $\alpha_2 = 0$, $\alpha_1\beta_2 \neq 0$ are investigated in this article.

In case (1), conditions (1.2) take the form

$$\int_0^T \overset{r_1}{t}\, u(t,x)\,dt = \varphi_1^0(x), \quad u(T,x) + \beta_2^0\int_0^T \overset{r_2}{t}\, u(t,x)\,dt = \varphi_2^0(x),$$

and formulas (2.4)–(2.10) are transformed into:

$$\begin{gathered} u = \sum_{k\in\mathbb{Z}^p} e^{i(\mu_k,x)} \sum_{l=0}^{1}\Big(\frac{g_1(t,\gamma_l\|\mu_k\|)}{\Delta(\gamma_l\|\mu_k\|)}\Pi_k^l\varphi_{1k}^0 + \frac{g_2(t,\gamma_l\|\mu_k\|)}{\Delta(\gamma_l\|\mu_k\|)}\Pi_k^l\varphi_{2k}^0\Big), \\ \varphi_{1k}^0 = \frac{\varphi_{1k}}{\beta_1}, \quad \varphi_{2k}^0 = \frac{\varphi_{2k}}{\alpha_2}, \end{gathered} \tag{2.12}$$

$$\begin{gathered} \Delta(0) = \overset{r_1+2}{T} + \beta_2^0\frac{r_2-r_1}{T}\overset{r_1+2}{T}\,\overset{r_2+2}{T}, \quad \Delta(\gamma) = -\mathcal{I}^{T,-}(r_1) + \beta_2^0\mathcal{I}^{-}(\vec{r}), \\ \gamma > 0, \quad \beta_2^0 = \frac{\beta_2}{\alpha_2}, \end{gathered} \tag{2.13}$$

$$g_1(t,0) = T - t + \beta_2^0\overset{r_2+1}{T}\Big(\frac{r_2+1}{r_2+2}T - t\Big), \quad g_2(t,0) = -\overset{r_1+1}{T}\Big(\frac{r_1+1}{r_1+2}T - t\Big), \tag{2.14}$$

$$g_1(t,\gamma) = 2i\sin\gamma(t-T) + \beta_2^0\mathcal{I}^{t,-}(r_2), \quad g_2(t,\gamma) = -\mathcal{I}^{t,-}(r_1), \quad \gamma > 0. \tag{2.15}$$

The first condition from (1.2) indicates the measurement of the solution's moment, and the second indicates the measurement at the point T, which, in the case of $\beta_2^0 \neq 0$ is complemented with the moment measurement.

In the case (2) one can obtain the formulas

$$u(0,x) + \beta_1^0\int_0^T \overset{r_1}{t}\, u(t,x)\,dt = \varphi_1^0(x), \quad \int_0^T \overset{r_2}{t}\, u(t,x)\,dt = \varphi_2^0(x)$$

for the conditions (1.2). For the solution is

$$
\begin{gathered}
u=\sum_{k\in\mathbb{Z}^p} e^{i(\mu_k,x)}\sum_{l=0}^{1}\Big(\frac{g_1(t,\gamma_l\|\mu_k\|)}{\Delta(\gamma_l\|\mu_k\|)}\Pi_k^l\varphi_{1k}^0+\frac{g_2(t,\gamma_l\|\mu_k\|)}{\Delta(\gamma_l\|\mu_k\|)}\Pi_k^l\varphi_{2k}^0\Big),\\
\varphi_{1k}^0=\frac{\varphi_{1k}}{\alpha_1},\quad \varphi_{2k}^0=\frac{\varphi_{2k}}{\beta_2},
\end{gathered}
\tag{2.16}
$$

$$
\begin{gathered}
\Delta(0)=(r_2+1)\overset{r_2+2}{T}+\beta_1^0\frac{r_2-r_1}{T}\overset{r_1+2}{T}\overset{r_2+2}{T},\quad \Delta(\gamma)=\mathcal{I}^{0,-}(r_2)+\beta_1^0\mathcal{I}^{-}(\vec{r}),\\
\gamma>0,\quad \beta_1^0=\frac{\beta_1}{\alpha_1},
\end{gathered}
\tag{2.17}
$$

$$
g_1(t,0)=\overset{r_2+1}{T}\Big(\frac{r_2+1}{r_2+2}T-t\Big),\quad g_2(t,0)=t-\beta_1^0\overset{r_1+1}{T}\Big(\frac{r_1+1}{r_1+2}T-t\Big),\tag{2.18}
$$

$$
g_1(t,\gamma)=\mathcal{I}^{t,-}(r_2),\quad g_2(t,\gamma)=-2i\sin\gamma t-\beta_1^0\mathcal{I}^{t,-}(r_1),\quad \gamma>0.\tag{2.19}
$$

Herein, the second condition from (1.2) means the measurement of the solution's moment, and the first one means the measurement at the point 0, which, in the case of $\beta_1^0\neq 0$ is also complemented with the moment measurement.

We also denote the function $\zeta(q)=\sum_{k\in\mathbb{Z}^p}(1+\|\mu\|^2)^{-q/2}$, the number $\gamma'=\sqrt{1+\gamma_0^2+\gamma_1^2}$, and the $\pi(\tau)$-dimensional projection operator Π_τ for $\tau\geq 0$, which is defined by the formula $\Pi_\tau\varphi=\sum_{\|\mu_k\|<\tau}\varphi_k e^{i(\mu_k,x)}$, where

$$
\varphi=\varphi(x)=\sum_{k\in\mathbb{Z}^p}\varphi_k e^{i(\mu_k,x)}
$$

and $\pi(\tau)$ is the quantity of the solutions in integers $k_1,\dots,k_p$ of the inequality $\|\mu_k\|<\tau$.

3. Solvability conditions for case (1)

In this section, we investigate the first case for $r_1\geq 1$ and $r_1=0$ separately. We prove that the solvability of the problem in the scale of spaces $\{H_M^\alpha\}_{\alpha\in\mathbb{R}}$ for all $T\in[T_0,T_1]$ (excluding a finite number of points) when $r_1\geq 1$. Providing $r_1=0$, namely the measurement of the function mean value, the problem is ill-posed in the sense of Hadamard, and its solvability highly depends on the estimate of the present small denominators.

Theorem 3.1. *Let $\alpha_1=0$ (therefore, $\alpha_2\beta_1\neq 0$), and $\varphi_1\in H_M^{q+1}$, $\varphi_2\in H_M^q$ be given functions. If the condition $r_1\geq 1$ is satisfied, then the null space of problem (1.1), (1.2) belongs to the space $\Pi_{\tau_1^*}\mathbf{C}^2([0,T];H_M)$. This space consists of almost periodic polynomials, and there exists a set $\mathcal{T}_0$ such that $[T_0,T_1]\setminus\mathcal{T}_0$ is a finite set. Also for all $T\in\mathcal{T}_0$ there exists the unique solution (2.16) of problem (1.1), (1.2) in the space $H_M^{2,q}$, for which the following inequality holds*

$$
\begin{aligned}
&\|(I-\Pi_{\tau_1})u;H_M^{2,q}\|^2\\
&\leq 2\Big(\frac{C_{11}^2}{|\beta_1|^2}\|(I-\Pi_{\tau_1})\varphi_1;H_M^{q+1}\|^2+\frac{C_{12}^2}{|\alpha_2|^2}\|(I-\Pi_{\tau_1})\varphi_2;H_M^q\|^2\Big),
\end{aligned}
\tag{3.1}
$$

where

$$
\tau_1=\tau_1(T)=\frac{4}{\gamma_0}\Big(\frac{r_1}{T}+2|\beta_2^0|\overset{r_2}{T}\Big),\quad \tau_1^*=\max\big(\tau_1(T_0),\tau_1(T_1)\big),
$$

$$C_{11} = 3\gamma_1\gamma'/\overset{r_1}{T}, \quad C_{12} = 4\gamma'\overset{r_2}{T}/\overset{r_1}{T}.$$

Proof. It follows from (2.13), (2.15) and the formula $\mathcal{I}_\pm(r) = \frac{\pm i}{\gamma}\big(\mathcal{I}_\pm(r-1) - \overset{r}{T}e^{\pm i\gamma T}\big)$ that

$$\begin{gathered}
\Delta(\gamma) = -\frac{2i}{\gamma}\overset{r_1}{T} + \frac{2i}{\gamma}\operatorname{Re}\big[e^{i\gamma T}\mathcal{I}_-(r_1-1)\big] + \beta_2^0\mathcal{I}^-(\vec{r}), \\
g_1^{(\alpha)}(t,\gamma) = (i\gamma)^\alpha\big(\mathcal{I}^{t-T,(-1)^{\alpha+1}}(-1) + \beta_2^0\mathcal{I}^{t,(-1)^{\alpha+1}}(r_2)\big), \\
g_2^{(\alpha)}(t,\gamma) = -(i\gamma)^\alpha\mathcal{I}^{t,(-1)^{\alpha+1}}(r_1).
\end{gathered}$$

Taking into account the estimates $|\mathcal{I}^{t,\pm}(-1)| \le 2$, $|\mathcal{I}^{t,\pm}(r)| \le \frac{4}{\gamma}\overset{r}{T}$, $|\mathcal{I}^-(\vec{r})| \le \frac{8}{\gamma^2}\overset{r_1}{T}\overset{r_1}{T}$ from [15, pages 32-33], where $t \in [0,T]$, $r \ge 0$, provided $\gamma \ge 4\Big(\frac{r_1}{T} + 2|\beta_2^0|\overset{r_2}{T}\Big) > 0$, the following inequalities are given

$$\max_{t\in[0,T]}|g_1^{(\alpha)}(t,\gamma)| \le \gamma^\alpha\big(\big|\mathcal{I}^{t-T,(-1)^{\alpha+1}}(-1)\big| + |\beta_2^0|\big|\mathcal{I}^{t,(-1)^{\alpha+1}}(r_2)\big|\big) \le 3\gamma^\alpha, \tag{3.2}$$

$$\max_{t\in[0,T]}|g_2^{(\alpha)}(t,\gamma)| \le 4\overset{r_1}{T}\gamma^{\alpha-1}, \tag{3.3}$$

$$|\Delta(\gamma)| \ge \frac{2}{\gamma}\overset{r_1}{T} - \frac{2}{\gamma}\big|\mathcal{I}_-(r_1-1)\big| - |\beta_2^0|\big|\mathcal{I}^-(\vec{r})\big| \ge \frac{1}{\gamma}\overset{r_1}{T} > 0. \tag{3.4}$$

The inequalities (3.4) imply that for a given $T \in [T_0,T_1]$ the possible elements of the null space of problem (1.1), (1.2) belong to the $\pi(\tau_1^*)$-dimentional space $\Pi_{\tau_1^*}\mathbf{C}^2([0,T];H_M)$, and the null space is trivial for those $T \in [T_0,T_1]$ which are not in the set of roots (finite, considering the multiplicities) of the finite number of equations $\Delta(\gamma_0\|\mu_k\|)\Delta(\gamma_1\|\mu_k\|) = 0$, where $\|\mu_k\| < \tau_1^*$.

Let us denote the set of such T points as $\mathcal{T}_0$. The solution of problem (1.1), (1.2) exists for all $T \in \mathcal{T}_0$ in the form $u = \Pi_{\tau_1}u + (I - \Pi_{\tau_1})u$ where $\Pi_{\tau_1}u$ is an almost periodic polynomial, and the estimate of the second term follows from the estimates (3.2), (3.4) and the formula (2.12), which induces the obvious inequality

$$\begin{aligned}
&\|(I-\Pi_\tau)u; H_M^{2,q}\|^2 \\
&\le 2\sum_{\|\mu_k\|<\tau}\sum_{j=1}^{2}\Big(\sum_{\alpha=0}^{2}\max_{l=0,1}\Big|\frac{g_j^{(\alpha)}(t,\gamma_l\|\mu_k\|)}{\Delta(\gamma_l\|\mu_k\|)}\Big|^2\big(1+\|\mu_k\|^2\big)^{q-\alpha}\Big)\|\varphi_{jk}^0\|^2,
\end{aligned} \tag{3.5}$$

for $\tau \ge 0$. This completes the proof. □

With the conditions of Theorem 3.1, problem (1.1), (1.2) is also solvable for $[T_0,T_1] \setminus \mathcal{T}_0$, if the functions φ_1 and φ_2 satisfy a finite number of orthogonality conditions (orthogonal to the elements of null space of the problem). In case when $r_1 = 0$, the solvability conditions of problem (1.1), (1.2) require more smooth functions φ_1 and φ_2 and the narrowing of the set $\mathcal{T}_0$. The null space of the problem can be infinite-dimensilonal, and the problem, affected by small denominators, can be unsolvable in the scale of spaces $\{H_M^\alpha\}_{\alpha\in\mathbb{R}}$ for T in set $[T_0,T_1] \setminus \mathcal{T}_0$.

Theorem 3.2. *Let $\alpha_1 = 0 = r_1$ and the conditions $\varphi_1 \in H_M^{q+1+2q_*}$, $\varphi_2 \in H_M^{q+2q_*}$, where $q_* > p/\theta_1$ are satisfied. Then for almost all $T \in [T_0,T_1]$, namely for all $T \in \mathcal{T}_0 \subset [T_0,T_1]$, $\operatorname{meas}\mathcal{T}_0 = T_1 - T_0$, there exists a unique solution of problem (1.1), (1.2) in the space $H_M^{2,q}$. Also for each $\varepsilon \in (0, 2c\zeta(q_*))$ there exists a set $\mathcal{T}_\varepsilon$*

with measure $\operatorname{meas}\mathcal{T}_\varepsilon \geq T_1 - T_0 - \varepsilon$ *such that for all* $T \in \mathcal{T}_\varepsilon$ *the following inequality is satisfied*

$$\begin{aligned} &\|(I-\Pi_{\tau_2})u; H_M^{2,q}\|^2 \\ &\leq \frac{C_2^2}{\varepsilon^4}\Big(\frac{9\gamma_1^2}{|\beta_1|^2}\|(I-\Pi_{\tau_2})\varphi_1; H_M^{q+1+2q_*}\|^2 + \frac{16}{|\alpha_2|^2}\|(I-\Pi_{\tau_2})\varphi_2; H_M^{q+2q_*}\|^2\Big), \end{aligned} \tag{3.6}$$

where $c = c(r_2, T_1 - T_0)$*,* $C_2 = 16c^2\gamma'\zeta^2(q_*)$*, and*

$$\gamma_0\tau_2 = \max\Big(4|\beta_2^0|\overset{r_2}{T} + \sqrt{\big(4|\beta_2^0|\overset{r_2}{T}\big)^2 + 4|\beta_2^0|\overset{r_2-1}{T}}, 1\Big).$$

Proof. Inequalities (3.2) can be interpreted with the formula $\max_{t\in[0,T]}|g_j^{(\alpha)}(t,\gamma)| \leq (2+j)\gamma^{\alpha+1-j}$, if $\gamma \geq 4|\beta_2^0|\overset{r_2}{T}$, and the quasipolynomial of variable T

$$\Delta(\gamma) = \frac{2i}{\gamma}(\cos\gamma T - 1) - 2i\beta_2^0\operatorname{Im}[\mathcal{I}_+(0)\mathcal{I}_-(r_2)] \tag{3.7}$$

is a small denominator in the formula (2.12), which can be estimated below (using the metrics lemma below) for $\gamma = \pm\gamma_l\|\mu_k\|$ with the measure of the the set of all such $T \in [T_0, T_1]$ for which this estimate holds.

Lemma 3.3 ([2]). *Let* f *be a quasipolynomial of the form*

$$f(y) = \sum_{j=1}^m p_j(y)e^{\lambda_j y}, \quad \lambda_j \neq \lambda_q \quad (j \neq q),$$

where $\lambda_j \in \mathbb{C}$*,* $|\lambda_1| \leq \dots \leq |\lambda_m|$*,* p_j *is a polynomial of degree* $n_j - 1$*,* $n_j \in \mathbb{N}$*,* $m \in \mathbb{N}$*. If, for* $\delta > 0$ *and certain complex numbers* $a_1, \dots, a_n$*, the condition*

$$\forall y \in [a,b] \subset \mathbb{R}\big|f^{(n)}(y) + \sum_{j=1}^n a_j f^{(n-j)}(y)\big| \geq \delta,$$

is satisfied, then for any ε *in the interval* $\big(0, \frac{\delta}{2n+2}\big(1 + \max_{1\leq j\leq n}|a_j|^{1/j}\big)^{-n}\big)$*, and for a certain* $c = c(b-a, n_1 + \dots + n_m, n)$*, we have the estimate*

$$\operatorname{meas}\{y \in [a,b] : |f(y) < \varepsilon|\} \leq c(1 + |\lambda_m|)\sqrt[n]{\varepsilon/\delta}.$$

Let us apply this Lemma to the quasipolynomial (3.7), where $[a,b] = [T_0, T_1]$, $m = 3$, $\lambda_1 = 0$, $\lambda_{2,3} = \pm i\gamma_l\|\mu_k\|$, $n_j = r_2$. For this purpose, we estimate below the function

$$\Gamma_\gamma\frac{d\Delta(\gamma)}{dT} = \Big(\frac{d^2}{dT^2} + i\gamma\frac{d}{dT}\Big)\Delta(\gamma),$$

where $\Gamma_\gamma = \frac{d}{dT} + i\gamma$ is a first order differential operation (according to Lemma, $n = 2$, $a_1 = i\gamma$, $a_2 = 0$). From the formula

$$\Gamma_\gamma\frac{d\Delta(\gamma)}{dT} = -2i\gamma e^{i\gamma T} + 2i\beta_2^0\big(\gamma e^{i\gamma T}\mathcal{I}_-(r_2) - \gamma\overset{r_2}{T}e^{-i\gamma T}\mathcal{I}_+(0) + \overset{r_2-1}{T}\operatorname{Im}\big[e^{-i\gamma T}\mathcal{I}_+(0)\big]\big),$$

we conclude the inequality

$$\big|\Gamma_\gamma\frac{d\Delta(\gamma)}{dT}\big| \geq 2\gamma - 4|\beta_2^0|\Big(2\overset{r_2}{T} + \frac{1}{\gamma}\overset{r_2-1}{T}\Big) \geq \gamma$$

in the case of $\gamma \ge 4|\beta_2^0|\Big(2\overset{r_2}{T} + \frac{1}{\gamma}\overset{r_2-1}{T}\Big)$ or

$$\gamma \ge 4|\beta_2^0|\overset{r_2}{T} + \sqrt{\big(4|\beta_2^0|\overset{r_2}{T}\big)^2 + 4|\beta_2^0|\overset{r_2-1}{T}} = \gamma_0\tau_2.$$

According to Lemma 3.3, the measure of the set of points $T \in [T_0, T_1]$ for which the inequality

$$|\Delta(\gamma_l\|\mu_k\|)| \ge \frac{\varepsilon^2\gamma_l\|\mu_k\|}{4c^2\zeta^2(q_*)}\frac{(1+\|\mu_k\|^2)^{-q_*}}{(1+\gamma_l\|\mu_k\|)^2} \tag{3.8}$$

is not satisfied, is less than

$$c(1+\gamma_l\|\mu_k\|)\sqrt{\frac{\varepsilon^2\gamma_l\|\mu_k\|}{4c^2\zeta^2(q_*)}\frac{(1+\|\mu_k\|^2)^{-q_*}}{(1+\gamma_l\|\mu_k\|)^2}\frac{1}{\gamma_l\|\mu_k\|}} = \frac{\varepsilon}{2}\frac{(1+\|\mu_k\|^2)^{-q_*/2}}{\zeta(q_*)}.$$

From the convergence of the series $\sum_{k\in\mathbb{Z}^p}(1+\|\mu_k\|^2)^{-q_*/2}$, that follows from (1.3), owing to the Borel-Cantelli lemma [15], for almost all $T \in [T_0, T_1]$ one can get the estimate (3.8) for large $\|\mu_k\|$. Let us exclude from this set of points T the roots of the equations $\Delta(\gamma_l\|\mu_k\|) = 0$, where $l = 0, 1$, $\|\mu_k\| < \tau_2$ and denote by $\mathcal{T}_0$ the obtained subset from the interval $[T_0, T_1]$. The measure of $\mathcal{T}_0$ is equal to $T_1 - T_0$ and for $T \in \mathcal{T}_0$ the first statement of the solvability theorem is established, since the solvability condition (2.11) is satisfied. In particular, for $\|\mu_k\| \ge \tau_2$ the function $\Delta(\gamma)$ can be estimated below by

$$|\Delta(\gamma_l\|\mu_k\|)| \ge \frac{1}{\gamma_l}\Big(\frac{\varepsilon}{4c\zeta(q_*)}\Big)^2(1+\|\mu_k\|^2)^{-q_*-1/2}. \tag{3.9}$$

Let us exclude from the interval $[T_0, T_1]$ the set of such T, for which the inequality (3.8) is not satisfied, and denote the obtained set by $\mathcal{T}_\varepsilon$. Note that the measure of the excluded set is less than $\frac{\varepsilon}{\zeta(q_*)}\sum_{\|\mu_k\|\ge\tau_2}(1+\|\mu_k\|^2)^{-q_*/2} = \varepsilon$. Then, $\operatorname{meas}\mathcal{T}_\varepsilon \ge T_1 - T_0 - \varepsilon$ and inequality (3.6) is satisfied for all $T \in \mathcal{T}_\varepsilon$, that follows from inequality (3.5) for $\tau = \tau_2$ and from the estimates (3.2) and (3.9). This completes our proof. □

4. Solvability conditions for case (2)

In this section, we investigate the second case of problem (1.1), (1.2), where the influence of small denominators on the solvability conditions is regulated by the non-negative integer parameter r_2 – the order of the measured moment.

Theorem 4.1. *Let $\alpha_2 = 0$ (therefore, $\alpha_1\beta_2 \ne 0$), $r_2 \ge 1$ and the conditions $\varphi_1 \in H_M^{q+q_*}$, $\varphi_2 \in H_M^{q+1+q_*}$, where $q_* > p/\theta_1$, be satisfied. Then, for almost all numbers $T \in [T_0, T_1]$, namely for all $T \in \mathcal{T}_0 \subset [T_0, T_1]$ with* $\operatorname{meas}\mathcal{T}_0 = T_1 - T_0$, *there exists a unique solution* (2.16) *of problem* (1.1), (1.2) *in the space $H_M^{2,q}$. Also for each $\varepsilon \in (0, 2c/\overset{r_2}{T_0})$ there exists a set $\mathcal{T}_\varepsilon$ with* $\operatorname{meas}\mathcal{T}_\varepsilon \ge T_1 - T_0 - \varepsilon$ *such that for all $T \in \mathcal{T}_\varepsilon$,*

$$\begin{aligned} &\|(I-\Pi_{\tau_3})u; H_M^{2,q}\|^2 \\ &\le \frac{C_3^2}{\varepsilon^2}\Big(\frac{16\big(\overset{r_2}{T}\big)^2}{\gamma_0^2|\alpha_1|^2}\|(I-\Pi_{\tau_2})\varphi_1; H_M^{q+q_*}\|^2 + \frac{9}{|\beta_2|^2}\|(I-\Pi_{\tau_2})\varphi_2; H_M^{q+1+q_*}\|^2\Big), \end{aligned} \tag{4.1}$$

where $c = c(\vec{r}, T_1 - T_0)$, $C_3 = 4c\gamma'\zeta(q_*)/\overset{r_2}{T_0}$, *and*

$$\tau_3 = \frac{4}{\gamma_0} \max_{T\in[T_0,T_1]} \Big(\frac{r_2}{T} + 4|\beta_1^0| \overset{r_1}{T}\Big).$$

Proof. The inequality (3.5) follows from formula (2.16). From (2.19) we derive the equalities

$$\begin{gathered} g_1^{(\alpha)}(t,\gamma) = (i\gamma)^\alpha \mathcal{I}^{t,(-1)^{\alpha+1}}(r_2), \\ g_2^{(\alpha)}(t,\gamma) = -(i\gamma)^\alpha \big(\mathcal{I}^{t,(-1)^{\alpha+1}}(-1) + \beta_1^0 \mathcal{I}^{t,(-1)^{\alpha+1}}(r_1)\big), \end{gathered}$$

where $\mathcal{I}^{t,(-1)^\alpha}(r) = \mathcal{I}^{t,+}(r)$ when α is even, and $\mathcal{I}^{t,(-1)^\alpha}(r) = \mathcal{I}^{t,-}(r)$ when α is odd; and, as in Theorem 3.1, for $\gamma \ge 4|\beta_1^0| \overset{r_1}{T}$ we derive the inequalities

$$\begin{gathered} \max_{t\in[0,T]} |g_1^{(\alpha)}(t,\gamma)| \le 4 \overset{r_2}{T} \gamma^{\alpha-1}, \\ \max_{t\in[0,T]} |g_2^{(\alpha)}(t,\gamma)| \le \gamma^\alpha \big(\big|\mathcal{I}^{t,(-1)^{\alpha+1}}(-1)\big| + |\beta_1^0| \big|\mathcal{I}^{t,(-1)^{\alpha+1}}(r_1)\big|\big) \le 3\gamma^\alpha. \end{gathered} \tag{4.2}$$

Let us estimate the expressions (2.17), which are the denominators in formula (3.5) and tend to zero when $\gamma \to \infty$. Assuming $\gamma \ge \gamma_0\tau_3$ in the equality

$$\Gamma_\gamma \Delta(\gamma) = -2 \overset{r_2}{T} e^{i\gamma T} + \mathcal{I}^{0,+}(r_2-1) + \beta_1^0 \Big(\overset{r_1}{T} \mathcal{I}^{T,-}(r_2) - \overset{r_2}{T} \mathcal{I}^{T,-}(r_1) + i\gamma \mathcal{I}^-(\vec{r}) \Big).$$

for $r_2 \ge 1$ one can get the below estimate:

$$\big|\Gamma_\gamma\Delta(\gamma)\big| \ge 2\overset{r_2}{T} - \big|\mathcal{I}^{0,+}(r_2-1)\big| - |\beta_1^0| \Big(\overset{r_1}{T} \big|\mathcal{I}^{T,-}(r_2)\big| + \overset{r_2}{T} \big|\mathcal{I}^{T,-}(r_1)\big| + \gamma \big|\mathcal{I}^-(\vec{r})\big| \Big) \ge \overset{r_2}{T_0}.$$

According to Lemma 3.3 ($m = 3$, $\lambda_1 = 0$, $\lambda_{2,3} = \pm i\lambda$, $n_1 = r_1 + r_2 - 1$, $n_2 = n_3 = \max(r_1, r_2)$, $n = 1$, $a_1 = i\gamma$), the measure of the set of points $T \in [T_0, T_1]$, for which the inequality

$$|\Delta(\gamma_l \|\mu_k\|)| \ge \frac{\varepsilon \overset{r_2}{T_0}}{2c\zeta(q_*)} \frac{(1 + \|\mu_k\|^2)^{-q_*/2}}{1 + \gamma_l\|\mu_k\|} \tag{4.3}$$

is not satisfied, where $\|\mu_k\| \ge \tau_3$, is not greater than

$$c(1 + \gamma_l\|\mu_k\|) \frac{\varepsilon \overset{r_2}{T_0}}{2c\zeta(q_*) \overset{r_2}{T_0}} \frac{(1 + \|\mu_k\|^2)^{-q_*/2}}{1 + \gamma_l\|\mu_k\|} \le \frac{\varepsilon}{2} \frac{(1 + \|\mu_k\|^2)^{-q_*/2}}{\zeta(q_*)}.$$

From the Borel-Cantelli lemma, we conclude the estimate (4.3) for almost all $T \in [T_0, T_1]$ (on the set $\mathcal{T}_0$) and for large $\|\mu_k\|$. Provided $\|\mu_k\| \ge \tau_3$ the inequality (4.3) gives the estimate

$$|\Delta(\gamma_l\|\mu_k\|)| \ge \frac{\varepsilon \overset{r_2}{T_0}}{2\sqrt{2}c\gamma_l\zeta(q_*)} (1 + \|\mu_k\|^2)^{-(1+q_*)/2},$$

i.e., the solvability condition (2.11) is satisfied, where $q_0 = 1 + q_*$ and $C_{0l} = C_{0l}(T)$ are some positive constants, and the first statement of the theorem (taking into account the estimates (4.2) for functions $g_j = g_j(t,\gamma)$) is satisfied as well.

Let $\mathcal{T}_\varepsilon$ be the set of such $T \in [T_0, T_1]$ that the inequality (4.3) is satisfied for all $k \in \mathbb{Z}^p$ provided $\|\mu_k\| \ge \tau_3$. Then $\operatorname{meas} \mathcal{T}_\varepsilon \ge T_1 - T_0 - \varepsilon$ and the solution satisfies condition (4.1). This completes our proof. $\square$

Theorem 4.2. *Let $\alpha_2 = 0 = r_2$, $\varphi_1 \in H_M^{q+2q_*}$, and $\varphi_2 \in H_M^{q+1+2q_*}$, where $q_* > p/\theta_1$. Then for almost all $T \in [T_0, T_1]$, namely for all $T \in \mathcal{T}_0 \subset [T_0, T_1]$ with* $\operatorname{meas} \mathcal{T}_0 = T_1 - T_0$, *there exists a unique solution (2.12) of problem (1.1), (1.2) in the space $H_M^{2,q}$. Also for each $\varepsilon \in (0, c\sqrt{2/3})$ there exists a set $\mathcal{T}_\varepsilon$ with* $\operatorname{meas} \mathcal{T}_\varepsilon \geq T_1 - T_0 - \varepsilon$ *such that for all $T \in \mathcal{T}_\varepsilon$,*

$$\begin{aligned} &\|(I - \Pi_{\tau_4})u; H_M^{2,q}\|^2 \\ &\leq \frac{C_4^2}{\varepsilon^4}\Big(\frac{16}{|\alpha_1|^2}\|(I - \Pi_{\tau_4})\varphi_1; H_M^{q+2q_*}\|^2 + \frac{9\gamma_1}{|\beta_2|^2}\|(I - \Pi_{\tau_4})\varphi_2; H_M^{q+1+2q_*}\|^2\Big), \end{aligned} \tag{4.4}$$

where $c = c(r_1, T_1 - T_0)$, $C_4 = 16c^2\gamma'\zeta^2(q_*)$, and

$$\gamma_0\tau_4 = \max\Big(4|\beta_1^0|\overset{r_1}{T} + \sqrt{\big(4|\beta_1^0|\overset{r_1}{T}\big)^2 + 4|\beta_1^0|\overset{r_1-1}{T}}, 1\Big).$$

Proof. Using estimates (4.2) for functions g_j, for $r_2 = 0$, the quasipolynomial $\Delta(\gamma)$ can be written as

$$\Delta(\gamma) = \frac{4i}{\gamma}\sin\frac{\gamma T}{2}\Big(\sin\frac{\gamma T}{2} - \beta_1^0 \operatorname{Im}\big[e^{i\gamma T/2}\mathcal{I}_-(r_1)\big]\Big)$$

and tends to zero when $\gamma \to \infty$. For estimating the speed of tending to zero of $\Delta(\gamma)$, we use the metric approach and Lemma 3.3 with $n = 2$, $m = 3$, $n_j = r_1$. Similar to Theorem 3.2, we write

$$\Gamma_\gamma \frac{d\Delta(\gamma)}{dT} = -2i\gamma e^{i\gamma T} + 2\beta_1^0\overset{r_1}{T}\big(e^{i\gamma T} - 1\big) + 2i\beta_1^0\frac{\overset{r_1-1}{T}}{\gamma}\big(1 - \cos\gamma T\big) - 2i\gamma\beta_1^0\mathcal{I}_-(r_1),$$

which yields the estimate

$$\big|\Gamma_\gamma\frac{d\Delta(\gamma)}{dT}\big| \geq 2\gamma - 4|\beta_1^0|\overset{r_1}{T} - 4|\beta_1^0|\frac{\overset{r_1-1}{T}}{\gamma} - 4|\beta_1^0|\overset{r_1}{T} = 2\gamma - 4|\beta_1^0|\overset{r_1}{T}\Big(\frac{r_1}{\gamma T} + 2\Big) \geq \gamma,$$

if $\gamma \geq 4|\beta_1^0|\overset{r_1}{T}\Big(\frac{r_1}{\gamma T} + 2\Big)$. According to Lemma 3.3, the measure of the set of such points T from $[T_0, T_1]$, that estimate (3.8) is not satisfied, is not greater than

$$c(1 + \gamma_l\|\mu_k\|)\sqrt{\frac{\varepsilon^2\gamma_l\|\mu_k\|}{4c^2\zeta^2(q_*)\gamma_l\|\mu_k\|}\frac{(1 + \|\mu_k\|^2)^{-q_*}}{(1 + \gamma_l\|\mu_k\|)^2}} = \frac{\varepsilon}{2}\frac{(1 + \|\mu_k\|^2)^{-q_*/2}}{\zeta(q_*)}.$$

By the Borel-Cantelli lemma and the above estimates, we can conclude the solvability of problem (1.1), (1.2) for almost all $T \in [T_0, T_1]$, namely for $T \in \mathcal{T}_0$, where $\operatorname{meas} \mathcal{T}_0 = T_1 - T_0$. For an arbitrary $\varepsilon \in (0, c\sqrt{2/3})$, the measure of such points $T \in [T_0, T_1]$ that inequality (3.8) is not satisfied at least for one $\|\mu_k\|$ from $\|\mu_k\| \geq \tau_4$, is not greater than

$$\frac{\varepsilon}{\zeta(q_*)}\sum_{\|\mu_k\| \geq \tau_4}(1 + \|\mu_k\|^2)^{-q_*/2} < \varepsilon,$$

so the second part of the theorem, including inequality (4.4), follows from estimates (3.8), (4.2) and formulas (2.16), (3.5). The proof is complete. □

Conclusions. We established conditions for the unique solvability of the problem with integral boundary conditions for the system of Lamé equations in spaces of almost periodic functions. Such problem is ill-posed in the sense of Hadamard and raises the problem of small denominators that are specific for such integral boundary conditions. To solve the problem of small denominators, we use the metric approach and the methodology of estimating the measures of exclusive sets on the real semiaxis.

We also determined the influence of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and the exponent θ_1, θ_2 on the solvability of the problem, in particular, the effectiveness of the combination of boundary and integral conditions and the corresponding spaces of almost periodic functions. We show that the impact of small denominators increases for zero and first order moments. We obtained the strengthening and the supplement for the results of the paper [12] (on the well-posed solvability of the problem (1.1), (1.2) in particular case $p = 3$ for real $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $r_1 < r_2$). Besides, the results of the paper [7] are also supplemented.

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