Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 305, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NULL CONTROLLABILITY FOR LINEAR PARABOLIC CASCADE SYSTEMS WITH INTERIOR DEGENERACY

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ABSTRACT. We study the null controllability problem for linear degenerate parabolic systems with one control force through Carleman estimates for the associated adjoint problem. The novelty of this article is that for the first time it is considered a problem with an interior degeneracy and a control set that only requires to contain an interval lying on one side of the degeneracy points. The obtained result improves and complements a number of earlier works. As a consequence, observability inequalities are established.

1. INTRODUCTION

This article is devoted to the analysis of control properties for linear degenerate parabolic systems in one space dimension, governed in the bounded domain $(0,1)$ by means of one control force h, of the form

$$u_t - (a_1(x)u_x)_x + b_{11}(t,x)u = h(t,x)1_\omega, \quad (t,x) \in Q, \tag{1.1}$$

$$v_t - (a_2(x)v_x)_x + b_{22}(t,x)v + b_{21}(t,x)u = 0, \quad (t,x) \in Q, \tag{1.2}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (0,1). \tag{1.4}$$

where ω is an open subset of $(0,1)$, $T > 0$ fixed, $Q := (0,T) \times (0,1)$, the coefficients $b_{ij} \in L^\infty((0,T) \times (0,1))$, $i,j = 1,2$, $h \in L^2((0,T) \times (0,1))$, and every a_i, $i = 1,2$, degenerates at an interior point x_i of the spatial domain $(0,1)$ (for the precise assumptions we refer to section 2). 1_ω denotes the characteristic function of the set ω.

The study of degenerate parabolic equations is the subject of numerous articles and books (see e.g., [3, 10, 11, 12, 13, 14, 16, 23, 27]). As pointed out by several authors, many problems coming from physics (boundary layer models in [9], models of Kolmogorov type in [5], models of Grushin type in [4]), biology (Wright-Fisher models in [28] and Fleming-Viot models in [20]), and economics (Black-Merton-Scholes equations in [18]) are described by degenerate parabolic equations.

On the other hand, the fields of applications of Carleman estimates in studying controllability and inverse problems for degenerate parabolic coupled systems are

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2010 *Mathematics Subject Classification.* 35K65, 35K40, 35B45, 93B05, 93B07.
Key words and phrases. Degenerate parabolic systems; interior degeneracy; one control force; Carleman estimates, observability inequalities; null controllability.

Submitted October 14, 2016. Published November 30, 2016.

so wide that it is not surprising that also several papers are concerned with such a topic, see [1, 2, 7, 15, 26].

In most of the previous papers, the authors assume that the functions a_i degenerates at the boundary of the space domain, e.g., $a_i(x) = x_i^k(1-x_i)^{\kappa_i}$, $x \in [0,1]$, where k_i and κ_i are positive constants.

To our best knowledge, [22] is the first paper that studies Carleman estimates for degenerate operators when the degeneracy is in the interior of the domain. Recently, in [21], the authors analyzed control properties for interior degenerate non smooth parabolic equations and studied different situations in which the degeneracy point is inside or outside the control region in order to obtain an observability inequality for a degenerate parabolic single equation. For related systems of degenerate equations we refer to [6], where lipschitz stability for the source term from measurements of the component u is also treated, but using a locally distributed observation $\omega \subset (0,1)$, which contains the degeneracy points.

For this reason, in the present paper we focus on carleman estimates (and, consequently, null controllability) for parabolic system (1.1)–(1.4) with a control set ω such that, there exists a subinterval $\omega' \subset\subset \omega \subset (0,1)$ lying on one side of the degeneracy points x_i. We think that this latter situation is much more interesting. Indeed, our approach has an immediate application also in the case in which the control set ω is the union of two intervals ω_i, $i = 1,2$ each of them lying on one side of the degeneracy points and thus it permits to cover more involved situations.

Finally, let us emphasize the fact that the proof of Theorem 3.5 can be adapted to the case of a single equation in which way it unifies the results of [21, Lemma 5.1 and 5.2], and one does not need to distinguish between the different situations in which the degeneracy point is inside or outside the control region ω.

To study the controllability problem of linear degenerate parabolic system (1.1)–(1.4), we use the developed Carleman estimates for the internal degenerate parabolic equations in [22] to show a global Carleman inequality and deduce an observability estimate for the adjoint system. To prove our Carleman estimates, a crucial role is played by the following Hardy-Poincaré inequality.

Theorem 1.1 ([22, Proposition 2.1]). *Assume that p is any continuous function in $[0,1]$, with $p > 0$ on $[0,1] \setminus \{x_0\}$, $p(x_0) = 0$ and such that there exists $\vartheta \in (1,2)$ so that the function $x \longrightarrow \frac{p(x)}{|x-x_0|^\vartheta}$ is non-increasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$. Then there exists a constant $C_{HP} > 0$ such that for any function w locally absolutely continuous on $[0,x_0) \cup (x_0,1]$ and satisfying*

$$w(0) = w(1) = 0 \quad \textit{and} \quad \int_0^1 p(x)|w'(x)|^2\,dx < \infty,$$

the following inequality holds

$$\int_0^1 \frac{p(x)}{(x-x_0)^2} w^2(x)\,dx \le C_{HP} \int_0^1 p(x)|w'(x)|^2\,dx.$$

The rest of this article is organized as follows. In Section 2 we give the precise setting for the weak and the strong degenerate cases and discuss the well-posedness of the system (1.1)–(1.4). The Carleman estimate is proved in Section 3. Finally, in Section 4, by the Hilbert Uniqueness Method (HUM, [17, 25]), we deduce null controllability result by showing that the adjoint system is observable. In appendix, we give summarized proof of a Caccioppoli inequality corresponding to our context.

2. Assumptions and well-posedness

To study the well-posedness of the system (1.1)–(1.4), we consider two situations, namely the weakly degenerate (WD) and the strongly degenerate (SD) cases. Towards this end, as in [22], the associated weighted spaces and assumptions on diffusion coefficients are the following:

Case (WD)

$$\begin{aligned}\mathcal{H}^1_{a_i}(0,1) := \Big\{ & u \in L^2(0,1) : u \text{ is abs. cont. in } [0,1], \\ & \sqrt{a_i}u_x \in L^2(0,1),\ u(0)=u(1)=0\Big\},\end{aligned}$$

where the functions a_i, $i=1,2$, satisfy:

$$\begin{gathered}\exists x_i \in (0,1),\ i=1,2, \text{ s.t. } a_i(x_i)=0,\ a_i>0 \text{ in } [0,1]\setminus\{x_i\},\ a_i \in C^1([0,1]\setminus\{x_i\}), \\ \exists K_i \in (0,1) \text{ s.t. } (x-x_i)a_i' \le K_i a_i \quad \text{a.e. in } [0,1].\end{gathered} \tag{2.1}$$

Case (SD)

$$\begin{aligned}\mathcal{H}^1_{a_i}(0,1) := \Big\{ & u \in L^2(0,1) : u \text{ is l.a.c. in } [0,1]\setminus\{x_i\}, \\ & \sqrt{a_i}u_x \in L^2(0,1),\ u(0)=u(1)=0\Big\}\end{aligned}$$

and

$$\begin{aligned}&\exists x_i \in (0,1), i=1,2, \text{ s.t. } a_i(x_i)=0,\ a_i>0 \text{ in } [0,1]\setminus\{x_i\}, \\ &a_i \in C^1([0,1]\setminus\{x_i\}) \cap W^{1,\infty}(0,1), \\ &\exists K_i \in [1,2) \text{ s.t. } (x-x_i)a_i' \le K_i a_i \text{ a.e. in } [0,1], \text{and if } K_i > 4/3, \\ &\text{then exists } \mu_i \in (0,K_i] \text{ such that } \frac{a_i}{|x-x_i|^{\mu_i}} \text{ is non-increasing on} \\ &\text{the left of } x_i \text{ and non-decreasing on the right of } x_i.\end{aligned} \tag{2.2}$$

In both cases we consider the space

$$\mathcal{H}^2_{a_i}(0,1) := \big\{u \in \mathcal{H}^1_{a_i}(0,1) : a_i u_x \in H^1(0,1)\big\}$$

with the norms

$$\begin{gathered}\|u\|^2_{\mathcal{H}^1_{a_i}} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a_i}u_x\|^2_{L^2(0,1)}, \\ \|u\|^2_{\mathcal{H}^2_{a_i}} := \|u\|^2_{\mathcal{H}^1_{a_i}} + \|(a_i u_x)_x\|^2_{L^2(0,1)}.\end{gathered}$$

We recall from [22] that, for $i=1,2$, the operator $(A_i, D(A_i))$ defined by $A_i u := (a_i u_x)_x$, with $u \in D(A_i) = \mathcal{H}^2_{a_i}(0,1)$ is closed self-adjoint negative with dense domain in $L^2(0,1)$.

In the Hilbert space $\mathbb{H} = \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$, the system (1.1)–(1.4) can be transformed in the Cauchy problem (CP)

$$\begin{gathered}X'(t) = \mathcal{A}X(t) - BX(t) + f(t), \\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\end{gathered}$$

where

$$X=\begin{pmatrix}u(t)\\v(t)\end{pmatrix},\quad \mathcal{A}=\begin{pmatrix}A_1 & 0\\0 & A_2\end{pmatrix},\quad D(\mathcal{A})=D(A_1)\times D(A_2),$$
$$B=\begin{pmatrix}b_{11} & 0\\b_{21} & b_{22}\end{pmatrix},\quad f(t)=\begin{pmatrix}h(t,\cdot)1_\omega\\0\end{pmatrix}$$

As the operator $\mathcal{A}$ is diagonal and since B is a bounded perturbation, the following well-posedness and regularity results hold.

Proposition 2.1. *(i) The operator $\mathcal{A}$ generates a contraction strongly continuous semigroup $(T(t))_{t\geq 0}$.*

(ii) for all $h\in L^2(Q)$ and $u_0,v_0\in L^2(0,1)$, there exists a unique weak solution $(u,v)\in C([0,T];\mathbb{H})\cap L^2(0,T;\mathcal{H}^1_{a_1}(0,1)\times\mathcal{H}^1_{a_2}(0,1))$ of (1.1)–(1.4) and

$$\begin{aligned}&\sup_{t\in[0,T]}\|(u,v)(t)\|^2_{\mathbb{H}}+\int_0^T\Big(\|\sqrt{a_1}u_x\|^2_{L^2(0,1)}+\|\sqrt{a_2}v_x\|^2_{L^2(0,1)}\Big)dt\\&\leq C_T(\|(u_0,v_0)\|^2_{\mathbb{H}}+\|h\|^2_{L^2(Q)}),\end{aligned}\tag{2.3}$$

for a positive constant C_T.

(iii) Moreover, if $(u_0,v_0)\in\mathcal{H}^1_{a_1}\times\mathcal{H}^1_{a_2}$, then (u,v) is in the space

$$H^1(0,T;\mathbb{H})\cap L^2(0,T;\mathcal{H}^2_{a_1}(0,1)\times\mathcal{H}^2_{a_2}(0,1))\cap C([0,T];\mathcal{H}^1_{a_1}(0,1)\times\mathcal{H}^1_{a_2}(0,1)),\tag{2.4}$$

and there exists a positive constant C such that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big(\|(u,v)(t)\|^2_{\mathcal{H}^1_{a_1}(0,1)\times\mathcal{H}^1_{a_2}(0,1)}\Big)\\&+\int_0^T\Big(\big\|(u_t,v_t)\big\|^2_{\mathbb{H}}+\big\|\big((a_1u_x)_x,(a_2v_x)_x\big)\big\|^2_{\mathbb{H}}\Big)dt\\&\leq C\Big(\|(u_0,v_0)\|^2_{\mathcal{H}^1_{a_1}(0,1)\times\mathcal{H}^1_{a_2}(0,1)}+\|h\|^2_{L^2(Q)}\Big).\end{aligned}\tag{2.5}$$

3. Carleman estimates for the adjoint cascade system

The goal of this section is to establish a Carleman estimate for the homogeneous adjoint system of (1.1)–(1.4). Thus, let us consider the problem

$$U_t-(a_1(x)U_x)_x+b_{11}(t,x)U+b_{21}(t,x)V=0,\quad (t,x)\in Q,\tag{3.1}$$
$$V_t-(a_2(x)V_x)_x+b_{22}(t,x)V=0,\quad (t,x)\in Q,\tag{3.2}$$
$$U(t,1)=U(t,0)=V(t,1)=V(t,0)=0,\quad t\in(0,T),\tag{3.3}$$
$$U(0,x)=U_0(x),\quad V(0,x)=V_0(x),\quad x\in(0,1).\tag{3.4}$$

Towards this end, we define the following time and space weight functions. For $x\in[0,1]$,

$$\varphi_i(t,x)=\theta(t)\psi_i(x),\tag{3.5}$$

where

$$\theta(t):=\frac{1}{[t(T-t)]^4},\quad \psi_i(x):=c_i\Big[\int_{x_i}^x\frac{y-x_i}{a_i(y)}dy-d_i\Big].$$

For $x\in[A,B]$:

$$\Phi_i(t,x)=\theta(t)\Psi_i(x),\quad \Psi_i(x)=e^{2\rho_i}-e^{r_i\zeta_i(x)},\tag{3.6}$$

where

$$\zeta_i(x) = \int_x^B \frac{dy}{\sqrt{a_i(y)}}, \quad \rho_i = r_i \zeta_i(A).$$

Here the functions a_i, $i = 1, 2$, satisfy (2.1) or (2.2) and the positive constants c_i, d_i, r_i and ρ_i are chosen such that

$$d_1 > d_1^\star, \quad d_2 > 16 d_2^\star, \quad \rho_2 > 2\ln(2), \tag{3.7}$$

$$e^{2\rho_1} - e^{\rho_1} \geq e^{2\rho_2} - 1, \tag{3.8}$$

$$\frac{e^{2\rho_2} - 1}{d_2 - d_2^\star} \leq c_2 < \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2}), \tag{3.9}$$

$$\text{and } c_1 \geq \max\Big\{\frac{e^{2\rho_1} - 1}{d_1 - d_1^\star}, \frac{c_2 d_2}{d_1 - d_1^\star}\Big\}, \tag{3.10}$$

where

$$d_i^\star := \sup_{[0,1]} \int_{x_i}^x \frac{y - x_i}{a_i(y)}\, dy.$$

Remark 3.1. The interval

$$\Big[\frac{e^{2\rho_2} - 1}{d_2 - d_2^\star}, \frac{4(e^{2\rho_2} - e^{\rho_2})}{3d_2}\Big)$$

is not empty. In fact, from $\rho_2 > 2\ln 2$, and $d_2 > 16 d_2^\star$, we have

$$\begin{aligned}
\frac{d_2^\star}{d_2} < \frac{1}{16} &\Leftrightarrow \frac{1}{4} < \frac{1}{3} - \frac{4d_2^\star}{3d_2} \\
&\Leftrightarrow e^{-\rho_2} < \frac{1}{3} - \frac{4d_2^\star}{3d_2} \\
&\Leftrightarrow \frac{e^{2\rho_2} - 1}{e^{2\rho_2} - e^{\rho_2}} < \frac{4(d_2 - d_2^\star)}{3d_2} \\
&\Leftrightarrow \frac{e^{2\rho_2} - 1}{d_2 - d_2^\star} < \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2}).
\end{aligned}$$

From (3.7)-(3.10), we have the following results.

Lemma 3.2. (i) *For* $(t, x) \in [0, T] \times [0, 1]$,

$$\varphi_1 \leq \varphi_2, \quad -\Phi_1 \leq -\Phi_2, \quad \varphi_i \leq -\Phi_i. \tag{3.11}$$

(ii) *For* $(t, x) \in [0, T] \times [0, 1]$,

$$4\Phi_2(t, x) + 3\varphi_2(t, x) > 0. \tag{3.12}$$

Proof. (i)

(1) $\varphi_1 \leq \varphi_2$: since $\theta \geq 0$ it is sufficient to prove $\psi_1 \leq \psi_2$. By the choice of c_1 we have $c_1 \geq \frac{c_2 d_2}{d_1 - d_1^\star}$. Then, $\max\{\psi_1(0), \psi_1(1)\} \leq -c_2 d_2$. Hence, $\psi_1(x) \leq \psi_2(x)$.
(2) $-\Phi_1 \leq -\Phi_2$:
since Ψ_i is increasing, it is sufficient to prove that $\min \Psi_1(x) \geq \max \Psi_2(x)$. Indeed $\Psi_1(0) = e^{2\rho_1} - e^{r_1\zeta_1(0)} \geq e^{2\rho_2} - 1 = \Psi_2(1)$.
(3) $\varphi_i \leq -\Phi_i$: since $c_i \geq \frac{e^{2\rho_i} - 1}{d_i - d_i^\star}$, it follows that $\max\{\psi_i(0), \psi_i(1)\} \leq -\Psi_i(1)$ and the conclusion follows immediately.

(ii) $4\Phi_2(t,x) + 3\varphi_2(t,x) > 0$: This follows easily by the assumption $3c_2d_2 < 4\Psi_2(0)$. □

We show now an intermediate Carleman-type estimate which could be used to show the null controllability for parabolic systems with two control forces. As a first step, consider the adjoint problem

$$\begin{gathered} U_t - (a_1(x)U_x)_x + b_{11}(t,x)U + b_{21}(t,x)V = 0, \quad (t,x) \in Q, \\ V_t - (a_2(x)V_x)_x + b_{22}(t,x)V = 0, \quad (t,x) \in Q, \\ U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \quad t \in (0,T), \\ U(0,x) = U_0(x) \in D(A_1^2), \quad V(0,x) = V_0(x) \in D(A_2^2), \end{gathered} \tag{3.13}$$

where $D(A_i^2) = \Big\{u \in D(A_i) \Big| A_i u \in D(A_i)\Big\}$, for $i = 1, 2$. Observe that for $i = 1, 2$, $D(A_i^2)$ is densely defined in $D(A_i)$ for the graph norm (see, e.g., [8, Lemma 7.2]) and hence in $L^2(0,1)$. As in [21] or [22], letting (U_0, V_0) vary in $\big(D(A_1^2), D(A_2^2)\big)$, we define the class of functions

$$\mathcal{W} := \big\{(U,V) \text{ is a solution of (3.13)}\big\}.$$

Obviously (see, e.g., [8, Theorem 7.5]) $\mathcal{W} \subset C^1\big([0,T]; D(\mathcal{A})\big) \subset \mathcal{V} \subset \mathcal{U}$, where

$$\begin{gathered} \mathcal{V} := \mathrm{L}^2\big(0,T; D(\mathcal{A})\big) \cap \mathrm{H}^1(0,T; \mathcal{H}^1_{a_1}(0,1) \times \mathcal{H}^1_{a_2}(0,1)), \\ \mathcal{U} := C\big([0,T]; \mathbb{H}\big) \cap L^2(0,T; \mathcal{H}^1_{a_1}(0,1) \times \mathcal{H}^1_{a_2}(0,1)). \end{gathered}$$

To prove the forthcoming theorems we use the following Carleman estimate proved in [22, Corollary 5.1].

Theorem 3.3. *Let $w \in L^2\big(0,T; \mathcal{H}^2_a(0,1)\big) \cap H^1\big(0,T; \mathcal{H}^1_a(0,1)\big)$ solution of*

$$\begin{gathered} w_t - (aw_x)_x + cw = H, \quad (t,x) \in Q, \\ w(t,0) = w(t,1) = 0, \quad t \in (0,T), \\ w(0,x) = w_0(x), \quad x \in (0,1), \end{gathered}$$

where $c \in L^\infty(Q)$ and a satisfies Hypothesis 2.1 or 2.2 and $H \in L^2(Q)$. Then, there exist two positive constants C and s_0, such that, for all $s \ge s_0$,

$$\begin{aligned} &\int_0^T\int_0^1 \Big(s\theta a w_x^2 + s^3\theta^3 \frac{(x-x_0)^2}{a} w^2\Big) e^{2s\varphi}\,dx\,dt \\ &\le C\Big(\int_0^T\int_0^1 H^2 e^{2s\varphi}\,dx\,dt + sc_1\int_0^T [a\theta e^{2s\varphi}(x-x_0)w_x^2 dt]_{x=0}^{x=1}\Big). \end{aligned}$$

Remark 3.4. In this article, we are interested in the case where the control subdomain ω is such that, it contains a subinterval lying on one side of the degeneracy points x_i, more precisely: $\omega' = (\alpha, \beta) \subset\subset \omega \subset (0,1)$, such that $0 < x_1 < x_2 < \alpha < \beta < 1$.

Now we are ready to state Carleman estimates related to (3.13).

Theorem 3.5. *Let $T > 0$ be given. There exist two positive constants C and s_0 such that, every solution $(U,V) \in \mathcal{W}$ of (3.13) satisfies*

$$\int_0^T\int_0^1 [s\theta(t)a_1(x)U_x^2(t,x) + s^3\theta^3(t)\frac{(x-x_1)^2}{a_1(x)}U^2(t,x)]e^{2s\varphi_1(t,x)}\,dx\,dt$$

$$
\begin{aligned}
&+\int_0^T\int_0^1[s\theta(t)a_2(x)V_x^2(t,x)+s^3\theta^3(t)\frac{(x-x_2)^2}{a_2(x)}V^2(t,x)]e^{2s\varphi_2(t,x)}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'}s^3\theta^3[U^2(t,x)+V^2(t,x)]e^{-2s\Phi_2(t,x)}\,dx\,dt
\end{aligned}
$$

for all $s\ge s_0$.

For the proof of the above theorem, we shall use the following classical Carleman estimate in suitable interval (A,B) [22].

Proposition 3.6. *Let* z *be the solution of*

$$
\begin{gathered}
z_t-(az_x)_x+cz=h,\quad x\in(A,B),\ t\in(0,T),\\
z(t,A)=z(t,B)=0,\quad t\in(0,T),
\end{gathered}
$$

where $a\in C^1([A,B])$ *is a strictly positive function and* $c\in L^\infty$. *Then there exist two positive constants* r *and* s_0, *such that for any* $s>s_0$

$$
\begin{aligned}
&\int_0^T\int_A^B s\theta e^{r\zeta}z_x^2e^{-2s\Phi}\,dx\,dt+\int_0^T\int_A^B s^3\theta^3e^{3r\zeta}z^2e^{-2s\Phi}\,dx\,dt\\
&\le c\Big(\int_0^T\int_A^B h^2e^{-2s\Phi}\,dx\,dt-\int_0^T\Big[\sigma(t,\cdot)z_x^2(t,\cdot)e^{-2s\Phi(t,\cdot)}\Big]_{x=A}^{x=B}dt\Big),
\end{aligned}
\tag{3.14}
$$

for some positive constant c. *Here the functions* θ, Φ *and* ζ *are defined, as in* (3.5)–(3.6), *with* $\sigma(t,x):=rs\theta(t)e^{r\zeta(x)}$, *for* $r,s>0$.

Proof of Theorem 3.5. Let us suppose that $0<x_1<x_2<\alpha<\beta<1$ (the proof is analogous wehn we assume that $0<\alpha<\beta<x_1<x_2<1$ with obvious adaptation). Also set $\lambda:=\frac{2\alpha+\beta}{3}$ and $\gamma:=\frac{\alpha+2\beta}{3}$, so that $\alpha<\lambda<\gamma<\beta$. Now, we consider a smooth function $\eta:[0,1]\to[0,1]$ such that

$$
\begin{gathered}
\eta(x)=1,\quad x\in[\gamma,1]\\
\eta(x)=0,\quad x\in[0,\lambda].
\end{gathered}
$$

Then, define $\hat p=\eta U$ and $\hat q=\eta V$, where (U,V) is the solution of (3.13).

Hence, $\hat p$ and $\hat q$ satisfy the system

$$
\begin{gathered}
\hat p_t-(a_1\hat p_x)_x+b_{11}\hat p=-b_{21}\hat q-(a_1\eta_xU)_x-\eta_xa_1U_x,\quad (t,x)\in Q,\\
\hat q_t-(a_2\hat q_x)_x+b_{22}\hat q=-(a_2\eta_xV)_x-\eta_xa_2V_x,\quad (t,x)\in Q,\\
\hat p(t,\alpha)=\hat p(t,1)=\hat q(t,\alpha)=\hat q(t,1)=0,\quad t\in(0,T).
\end{gathered}
$$

Applying the classical Carleman estimate stated in Proposition 3.6, with $A=\alpha$ and $B=1$, one has

$$
\begin{aligned}
&\int_0^T\int_\alpha^1(s\theta e^{r_1\zeta_1}\hat p_x^2+s^3\theta^3e^{3r_1\zeta_1}\hat p^2)e^{-2s\Phi_1}\,dx\,dt\\
&\le\tilde C\int_0^T\int_\alpha^1\hat q^2e^{-2s\Phi_1}\,dx\,dt+C\int_0^T\int_{\hat\omega}(U^2+U_x^2)e^{-2s\Phi_1}\,dx\,dt,
\end{aligned}
$$

for all $s\ge s_0$ with $\hat\omega=[\lambda,\gamma]$. Let us remark that the boundary term in $x=1$ is nonpositive, while the one in $x=\alpha$ is 0, so that they can be neglected in the classical Carleman estimate.

Analogously, one can prove that $\hat q$ satisfies

$$
\int_0^T\int_\alpha^1(s\theta e^{r_2\zeta_2}\hat q_x^2+s^3\theta^3e^{3r_2\zeta_2}\hat q^2)e^{-2s\Phi_2}\,dx\,dt
$$

$$\le C\int_0^T\int_{\hat\omega}(V^2+V_x^2)e^{-2s\Phi_2}\,dx\,dt.$$

Thus combining the last two inequalities, it follows that

$$\begin{aligned}
&\int_0^T\int_\alpha^1(s\theta e^{r_1\zeta_1}\hat p_x^2+s^3\theta^3e^{3r_1\zeta_1}\hat p^2)e^{-2s\Phi_1}\,dx\,dt\\
&+\int_0^T\int_\alpha^1(s\theta e^{r_2\zeta_2}\hat q_x^2+s^3\theta^3e^{3r_2\zeta_2}\hat q^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le \tilde C\int_0^T\int_\alpha^1\hat q^2e^{-2s\Phi_1}\,dx\,dt+C\int_0^T\int_{\hat\omega}(U^2+U_x^2)e^{-2s\Phi_1}\,dx\,dt\\
&\quad+C\int_0^T\int_{\hat\omega}(V^2+V_x^2)e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

Taking s such that $\tilde C\le\frac12 s^3\theta^3e^{3r_2\zeta_2}$, using $-\Phi_1\le-\Phi_2$ and Caccioppoli's inequality (5.1), we obtain

$$\begin{aligned}
&\int_0^T\int_\alpha^1(s\theta e^{r_1\zeta_1}\hat p_x^2+s^3\theta^3e^{3r_1\zeta_1}\hat p^2)e^{-2s\Phi_1}\,dx\,dt\\
&+\int_0^T\int_\alpha^1(s\theta e^{r_2\zeta_2}\hat q_x^2+s^3\theta^3e^{3r_2\zeta_2}\hat q^2)e^{-2s\Phi_2}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'}s^2\theta^2[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

Then, by Lemma 3.2 one can prove that there exists a positive constant C such that for every $(t,x)\in[0,T]\times[\alpha,1]$

$$a_i(x)e^{2s\varphi_i(t,x)}\le Ce^{r_i\zeta_i}e^{-2s\Phi_i},\quad \frac{(x-x_i)^2}{a_i(x)}e^{2s\varphi_i(t,x)}\le Ce^{3r_i\zeta_i}e^{-2s\Phi_i}. \tag{3.15}$$

Consequently,

$$\begin{aligned}
&\int_0^T\int_\alpha^1\Big(s\theta a_1\hat p_x^2+s^3\theta^3\frac{(x-x_1)^2}{a_1}\hat p^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+\int_0^T\int_\alpha^1\Big(s\theta a_2\hat q_x^2+s^3\theta^3\frac{(x-x_2)^2}{a_2}\hat q^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'}s^2\theta^2[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

By the definition of $\hat p$ and $\hat q$, we obtain

$$\begin{aligned}
&\int_0^T\int_\gamma^1\Big(s\theta a_1U_x^2+s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+\int_0^T\int_\gamma^1\Big(s\theta a_2V_x^2+s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'}s^2\theta^2[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt,
\end{aligned}\tag{3.16}$$

for a positive constant C and for s large enough.

On the other hand, by the properties of the weight functions, calculations show that

$$s^3\theta^3\frac{(x-x_i)^2}{a_i}e^{2s\varphi_i} \le Cs^2\theta^2 e^{-2s\Phi_2}, \quad \forall (t,x) \in (0,T)\times(\lambda,\gamma) \tag{3.17}$$

for a positive constant C. In addition, arguing as in the proof of Caccioppoli's inequality 5.1, one can easily show that

$$\int_0^T\int_\lambda^\gamma s\theta[U_x^2+V_x^2]e^{-2s\Phi_2}\,dx\,dt \le C\int_0^T\int_{\omega'} s^3\theta^3[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt, \tag{3.18}$$

for some constant $C>0$.

By (3.17) and (3.18) we can find a positive constant C such that

$$\begin{aligned}
&\int_0^T\int_\lambda^\gamma \Big(s\theta a_1 U_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+\int_0^T\int_\lambda^\gamma \Big(s\theta a_2 V_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'} s^3\theta^3[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}\tag{3.19}$$

Thus (3.16) and (3.19) imply

$$\begin{aligned}
&\int_0^T\int_\lambda^1 \Big(s\theta a_1 U_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2\Big)e^{2s\varphi_1}\,dx\,dt\\
&+\int_0^T\int_\lambda^1 \Big(s\theta a_2 V_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2\Big)e^{2s\varphi_2}\,dx\,dt\\
&\le C\int_0^T\int_{\omega'} s^3\theta^3[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt,
\end{aligned}\tag{3.20}$$

for a positive constant C and for s large enough.

To complete the proof, it is sufficient to prove a similar inequality on the interval $[0,\lambda]$. To this aim, we follow a reflection procedure. Consider the functions

$$W(t,x) := \begin{cases} U(t,x), & x\in[0,1],\\ U(t,-x), & x\in[-1,0],\end{cases} \qquad Z(t,x) := \begin{cases} V(t,x), & x\in[0,1],\\ V(t,-x), & x\in[-1,0],\end{cases}$$

where (U,V) solves (3.13), and

$$\tilde\psi_i(x) := \begin{cases} \psi_i(x), & x\in[0,1],\\ c_i[\int_{-x_i}^x \frac{y+x_i}{\tilde a_i(y)}\,dy - d_i], & x\in[-1,0],\end{cases}$$

$$\tilde a_i(x) = \begin{cases} a_i(x), & x\in[0,1],\\ a_i(-x), & x\in[-1,0],\end{cases} \qquad \tilde b_{ij}(x) := \begin{cases} b_{ij}(x), & x\in[0,1],\\ b_{ij}(-x), & x\in[-1,0].\end{cases}$$

Therefore, (W,Z) solves the system

$$\begin{gathered}
W_t - (\tilde a_1 W_x)_x + \tilde b_{11}W = -\tilde b_{21}Z, \quad x\in(-1,1),\ t\in(0,T),\\
Z_t - (\tilde a_2 Z_x)_x + \tilde b_{22}Z = 0, \quad x\in(-1,1),\ t\in(0,T),\\
W(t,-1) = W(t,1) = Z(t,-1) = Z(t,1) = 0,\ t\in(0,T).
\end{gathered}\tag{3.21}$$

Now, we consider a smooth function $\tau : [-1,1] \to [0,1]$ such that

$$\tau(x) = \begin{cases} 1, & x \in [-x_1/3, \lambda], \\ 0, & x \in [-1, -x_1/2] \cup [\gamma, 1], \end{cases}$$

and define the functions $\bar{p} = \tau W$ and $\bar{q} = \tau Z$, where (W, Z) is the solution of (3.21). Then $(\bar{p}, \bar{q})$ satisfies

$$\begin{gathered} \bar{p}_t - (\tilde{a}_1 \bar{p}_x)_x + \tilde{b}_{11}\bar{p} = -\tilde{b}_{21}\bar{q} - (\tilde{a}_1 \tau_x W)_x - \tau_x \tilde{a}_1 W_x, \quad x \in (-1,1),\ t \in (0,T), \\ \bar{q}_t - (\tilde{a}_2 \bar{q}_x)_x + \tilde{b}_{22}\bar{q} = -(\tilde{a}_2 \tau_x Z)_x - \tau_x \tilde{a}_2 Z_x, \quad x \in (-1,1),\ t \in (0,T), \\ \bar{p}(t, -\frac{2x_1}{3}) = \bar{p}(t,1) = \bar{q}(t, -\frac{2x_1}{3}) = \bar{q}(t,1) = 0, \quad t \in (0,T). \end{gathered}$$

Now, define $\tilde{\varphi}_i := \theta(t)\tilde{\psi}_i(x)$, where $\tilde{\psi}_i$ is defined as above. Using the analogue of Theorem 3.3 for the first component $\bar{p}$ on $(-\frac{2x_1}{3}, 1)$ in place of $(0,1)$ and with φ_i replaced by $\tilde{\varphi}_i$, by the equalities $\bar{p}_x(t, -\frac{2x_1}{3}) = \bar{p}_x(t,1) = 0$ and the definition of W, we obtain

$$\begin{aligned} &\int_0^T \int_{-2x_1/3}^1 (s\theta \tilde{a}_1 \bar{p}_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{\tilde{a}_1} \bar{p}^2) e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\le C \int_0^T \int_{-2x_1/3}^1 \tilde{b}_{21}^2 \bar{q}^2 e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\quad + C \int_0^T \int_{-\frac{x_1}{2}}^{-\frac{x_1}{3}} (W^2 + W_x^2) e^{2s\tilde{\varphi}_1}\,dx\,dt + C \int_0^T \int_\lambda^\gamma (W^2 + W_x^2) e^{2s\varphi_1}\,dx\,dt \\ &\le C \int_0^T \int_{-2x_1/3}^1 \tilde{b}_{21}^2 \bar{q}^2 e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\quad + C \underbrace{\int_0^T \int_{\frac{x_1}{3}}^{\frac{x_1}{2}} (U^2 + U_x^2) e^{2s\varphi_1}\,dx\,dt}_{J} + C \int_0^T \int_\lambda^\gamma (U^2 + U_x^2) e^{2s\varphi_1}\,dx\,dt. \end{aligned}$$

To absorb J, let $\epsilon_1 > 0$ be small enough. Since $\inf_{t\in[0,T]} \theta(t) > 0$, $\inf_{x\in[\frac{x_1}{3},\frac{x_1}{2}]} a_1(x) > 0$ and $\inf_{x\in[\frac{x_1}{3},\frac{x_1}{2}]} \frac{(x-x_1)^2}{a_1(x)} > 0$, for s large enough, it follows that

$$\begin{aligned} &\int_0^T \int_{\frac{x_1}{3}}^{\frac{x_1}{2}} (U^2 + U_x^2) e^{2s\varphi_1}\,dx\,dt \\ &\le \epsilon_1 \int_0^T \int_{\frac{x_1}{3}}^{\frac{x_1}{2}} (s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2) e^{2s\varphi_1}\,dx\,dt \\ &\le \epsilon_1 \int_0^T \int_0^\lambda (s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2) e^{2s\varphi_1}\,dx\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^T \int_{-2x_1/3}^1 (s\theta \tilde{a}_1 \bar{p}_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{\tilde{a}_1} \bar{p}^2) e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\le C \int_0^T \int_{-2x_1/3}^1 \tilde{b}_{21}^2 \bar{q}^2 e^{2s\tilde{\varphi}_1}\,dx\,dt + C \int_0^T \int_\lambda^\gamma (U^2 + U_x^2) e^{2s\varphi_1}\,dx\,dt \end{aligned}$$

$$+ \epsilon_1 \int_0^T\int_0^\lambda (s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2) e^{2s\varphi_1}\,dx\,dt.$$

Using similar arguments, for the second component $\bar{q}$, we obtain

$$\begin{aligned}
&\int_0^T\int_{-2x_1/3}^1 (s\theta \tilde{a}_2 \bar{q}_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{\tilde{a}_2}\bar{q}^2) e^{2s\tilde{\varphi}_2}\,dx\,dt \\
&\le C\int_0^T\int_\lambda^\gamma (V^2 + V_x^2) e^{2s\varphi_2}\,dx\,dt \\
&\quad + \epsilon_2 \int_0^T\int_0^\lambda (s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2) e^{2s\varphi_2}\,dx\,dt,
\end{aligned}$$

where $\epsilon_2 > 0$ is taken small.

Combining the last two inequalities and using Lemma 3.2, by Caccioppoli's inequality 5.1, we obtain

$$\begin{aligned}
&\int_0^T\int_{-2x_1/3}^1 (s\theta \tilde{a}_1 \bar{p}_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{\tilde{a}_1}\bar{p}^2) e^{2s\tilde{\varphi}_1}\,dx\,dt \\
&+ \int_0^T\int_{-2x_1/3}^1 (s\theta \tilde{a}_2 \bar{q}_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{\tilde{a}_2}\bar{q}^2) e^{2s\tilde{\varphi}_2}\,dx\,dt \\
&\le C\int_0^T\int_{\omega'} s^2\theta^2 (U^2+V^2) e^{-2s\Phi_2}\,dx\,dt + C\int_0^T\int_{-2x_1/3}^1 \tilde{b}_{21}^2 \bar{q}^2 e^{2s\tilde{\varphi}_1}\,dx\,dt \qquad (3.22)\\
&\quad + \epsilon_1 \int_0^T\int_0^\lambda (s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2) e^{2s\varphi_1}\,dx\,dt \\
&\quad + \epsilon_2 \int_0^T\int_0^\lambda (s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2) e^{2s\varphi_2}\,dx\,dt.
\end{aligned}$$

On the other hand, using that $\tilde{\varphi}_1 < \tilde{\varphi}_2$, we have

$$\begin{aligned}
&\int_0^T\int_{-2x_1/3}^1 \tilde{b}_{21}^2 \bar{q}^2 e^{2s\tilde{\varphi}_1}\,dx\,dt \\
&\le \|\tilde{b}_{21}\|_\infty^2 \int_0^T\int_{-2x_1/3}^1 (\bar{q} e^{s\tilde{\varphi}_2})^2\,dx\,dt \\
&= \|\tilde{b}_{21}\|_\infty^2 \int_0^T\int_{-2x_1/3}^1 \Big(\frac{|x-x_2|^2}{\tilde{a}_2(x)}\bar{q}^2 e^{2s\tilde{\varphi}_2}\Big)^{1/4} \Big(\frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\bar{q}^2 e^{2s\tilde{\varphi}_2}\Big)^{3/4}\,dx\,dt \qquad (3.23)\\
&\le \frac{\|\tilde{b}_{21}\|_\infty^2}{4} \int_0^T\int_{-2x_1/3}^1 \frac{|x-x_2|^2}{\tilde{a}_2(x)}\bar{q}^2 e^{2s\tilde{\varphi}_2}\,dx\,dt \\
&\quad + \frac{3\|\tilde{b}_{21}\|_\infty^2}{4} \int_0^T\int_{-2x_1/3}^1 \frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\bar{q}^2 e^{2s\tilde{\varphi}_2}\,dx\,dt.
\end{aligned}$$

Now, define $w(t,x) := e^{s\tilde{\varphi}_2(t,x)}\bar{q}(t,x)$, one has

$$\int_0^T\int_{-2x_1/3}^1 \frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\bar{q}^2 e^{2s\tilde{\varphi}_2}\,dx\,dt = \int_0^T\int_{-2x_1/3}^1 \frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}} w^2\,dx\,dt.$$

First observe that, since $w(-2x_1/3) = w(0) = 0$, by the classical Poincaré inequality, one has

$$
\begin{aligned}
&\int_0^T\int_{-2x_1/3}^0 \frac{\tilde{a}_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt \\
&\le \max_{x\in[-\frac{2x_1}{3},0]} \Big\{\frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\Big\} \int_0^T\int_{-2x_1/3}^0 w^2\,dx\,dt \\
&\le C_P \max_{x\in[-\frac{2x_1}{3},0]} \Big\{\frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\Big\} \int_0^T\int_{-2x_1/3}^0 w_x^2\,dx\,dt \\
&\le CC_P \max_{x\in[-\frac{2x_1}{3},0]} \Big\{\frac{\tilde{a}_2^{1/3}(x)}{|x-x_2|^{2/3}}\Big\} \int_0^T\int_{-2x_1/3}^0 \tilde{a}_2 w_x^2\,dx\,dt,
\end{aligned}
\tag{3.24}
$$

where C_P is the Poincaré constant.

Now, to obtain an estimate on $(0,1)$, we distinguish between two cases. First, if $K_2 \le \frac{4}{3}$, we consider the function $p_2(x) = |x-x_2|^{\frac{4}{3}}$. Obviously, there exists $q \in \big(1, \frac{4}{3}\big)$ such that the function $\frac{p_2(x)}{|x-x_2|^q}$ is non-increasing on the left of $x = x_2$ and nondecreasing on the right of $x = x_2$. Then, we can apply the Theorem 1.1, obtaining

$$
\begin{aligned}
\int_0^T\int_0^1 \frac{a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt &\le \max_{x\in[0,1]} a_2^{1/3}(x) \int_0^T\int_0^1 \frac{1}{|x-x_2|^{2/3}} w^2\,dx\,dt \\
&= \max_{x\in[0,1]} a_2^{1/3}(x) \int_0^T\int_0^1 \frac{p_2(x)}{|x-x_2|^{2}} w^2\,dx\,dt \\
&\le \max_{x\in[0,1]} a_2^{1/3}(x) C_{HP} \int_0^T\int_0^1 p_2(x) w_x^2\,dx\,dt \\
&= \max_{x\in[0,1]} a_2^{1/3}(x) C_{HP} \int_0^T\int_0^1 a_2 \frac{|x-x_2|^{\frac{4}{3}}}{a_2} w_x^2\,dx\,dt \\
&\le \max_{x\in[0,1]} a_2^{1/3}(x) C_{HP} C_1 \int_0^T\int_0^1 a_2 w_x^2\,dx\,dt,
\end{aligned}
$$

where C_{HP} is the Hardy-Poincaré constant and $C_1 = \max\big(\frac{x_2^{\frac{4}{3}}}{a_2(0)}, \frac{(1-x_2)^{\frac{4}{3}}}{a_2(1)}\big)$. In the previous inequality we have used the property that the map $x \mapsto \frac{|x-x_2|^\mu}{a_2(x)}$ is non-increasing on the left of $x = x_2$ and nondecreasing on the right of $x = x_2$ for all $\mu \ge K_2$, see [22, Lemma 2.1].

If $K_2 > 4/3$, we can consider the function $p_2(x) = (a_2(x)|x-x_2|^4)^{1/3}$. Then

$$
p_2(x) = a_2(x)\Big(\frac{(x-x_2)^2}{a_2(x)}\Big)^{2/3} \le C_1 a_2(x),
$$

where

$$
C_1 := \max\Big\{\big(\frac{x_2^2}{a_2(0)}\big)^{2/3}, \big(\frac{(1-x_2)^2}{a_2(1)}\big)^{2/3}\Big\}, \quad \frac{a_2^{1/3}}{|x-x_2|^{2/3}} = \frac{p_2(x)}{(x-x_2)^2}.
$$

Moreover, using Hypothesis 2.2, one has that the function $\frac{p_2(x)}{|x-x_2|^q}$, where $q := \frac{4+\mu}{3} \in (1,2)$, is non-increasing on the left of $x = x_2$ and nondecreasing on the right of $x = x_2$. The Hardy-Poincaré inequality implies

$$\begin{aligned}\int_0^T\int_0^1 \frac{a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt &= \int_0^T\int_0^1 \frac{p_2(x)}{(x-x_2)^2} w^2 dx\,dt\\ &\le C_{HP}\int_0^T\int_0^1 p_2(x) w_x^2\,dx\,dt\\ &\le C_{HP}C_1\int_0^T\int_0^1 a_2 w_x^2\,dx\,dt,\end{aligned}$$

where C_{HP} and C_1 are the Hardy-Poincaré constant and the constant introduced before, respectively. Thus, in every case,

$$\int_0^T\int_0^1 \frac{a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt \le C\int_0^T\int_0^1 a_2 w_x^2\,dx\,dt, \tag{3.25}$$

for a positive constant C.

Combining (3.24) and (3.25), we obtain

$$\begin{aligned}&\int_0^T\int_{-2x_1/3}^1 \frac{\tilde a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt\\ &= \int_0^T\int_{-2x_1/3}^0 \frac{\tilde a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt + \int_0^T\int_0^1 \frac{a_2^{1/3}}{|x-x_2|^{2/3}} w^2\,dx\,dt\\ &\le CC_P \max_{x\in[-\frac{2x_1}{3},0]}\{\frac{\tilde a_2^{1/3}(x)}{|x-x_2|^{2/3}}\}\int_0^T\int_{-2x_1/3}^0 \tilde a_2 w_x^2\,dx\,dt + C\int_0^T\int_0^1 a_2 w_x^2\,dx\,dt\\ &\le C\int_0^T\int_{-2x_1/3}^1 \tilde a_2 w_x^2\,dx\,dt\\ &\le C\int_0^T\int_{-2x_1/3}^1 \tilde a_2 \bar q_x^2 e^{2s\tilde\varphi_2}\,dx\,dt + C\int_0^T\int_{-2x_1/3}^1 s^2\theta^2\frac{|x-x_2|^2}{\tilde a_2(x)}\bar q^2 e^{2s\tilde\varphi_2}\,dx\,dt.\end{aligned} \tag{3.26}$$

From (3.23) and (3.26), it results that

$$\begin{aligned}&\int_0^T\int_{-2x_1/3}^1 \tilde b_{21}^2\bar q^2 e^{2s\tilde\varphi_1}\,dx\,dt\\ &\le C\int_0^T\int_{-2x_1/3}^1 \Big(\tilde a_2\bar q_x^2 + s^2\theta^2\frac{|x-x_2|^2}{\tilde a_2(x)}\bar q^2\Big)e^{2s\tilde\varphi_2}\,dx\,dt,\end{aligned} \tag{3.27}$$

for a positive constant C.

Taking s large enough, by (3.27), one can estimate (3.22) in the following way

$$\begin{aligned} &\int_0^T\int_{-2x_1/3}^1 (s\theta\tilde{a}_1\bar{p}_x^2 + s^3\theta^3\frac{(x-x_1)^2}{\tilde{a}_1}\bar{p}^2)e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &+ \int_0^T\int_{-2x_1/3}^1 (s\theta\tilde{a}_2\bar{q}_x^2 + s^3\theta^3\frac{(x-x_2)^2}{\tilde{a}_2}\bar{q}^2)e^{2s\tilde{\varphi}_2}\,dx\,dt \\ &\le C\int_0^T\int_{\omega'} s^2\theta^2(U^2+V^2)e^{-2s\Phi_2}\,dx\,dt \\ &\quad + \epsilon_1\int_0^T\int_0^\lambda (s\theta a_1U_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2)e^{2s\varphi_1}\,dx\,dt \\ &\quad + \epsilon_2\int_0^T\int_0^\lambda (s\theta a_2V_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2)e^{2s\varphi_2}\,dx\,dt. \end{aligned} \tag{3.28}$$

Hence, by (3.28), the definition of W, Z, $\bar{p}$ and $\bar{q}$, we obtain

$$\begin{aligned} &\int_0^T\int_0^\lambda (s\theta a_1U_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2)e^{2s\varphi_1}\,dx\,dt \\ &+ \int_0^T\int_0^\lambda (s\theta a_2V_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2)e^{2s\varphi_2}\,dx\,dt \\ &= \int_0^T\int_0^\lambda (s\theta a_1W_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}W^2)e^{2s\varphi_1}\,dx\,dt \\ &\quad + \int_0^T\int_0^\lambda (s\theta a_2Z_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}Z^2)e^{2s\varphi_2}\,dx\,dt \\ &\le \int_0^T\int_{-\frac{x_1}{3}}^\lambda (s\theta\tilde{a}_1W_x^2 + s^3\theta^3\frac{(x-x_1)^2}{\tilde{a}_1}W^2)e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\quad + \int_0^T\int_{-\frac{x_1}{3}}^\lambda (s\theta\tilde{a}_2Z_x^2 + s^3\theta^3\frac{(x-x_2)^2}{\tilde{a}_2}Z^2)e^{2s\tilde{\varphi}_2}\,dx\,dt \\ &= \int_0^T\int_{-\frac{x_1}{3}}^\lambda (s\theta\tilde{a}_1\bar{p}_x^2 + s^3\theta^3\frac{(x-x_1)^2}{\tilde{a}_1}\bar{p}^2)e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\quad + \int_0^T\int_{-\frac{x_1}{3}}^\lambda (s\theta\tilde{a}_2\bar{q}_x^2 + s^3\theta^3\frac{(x-x_2)^2}{\tilde{a}_2}\bar{q}^2)e^{2s\tilde{\varphi}_2}\,dx\,dt \\ &\le \int_0^T\int_{-2x_1/3}^1 (s\theta\tilde{a}_1\bar{p}_x^2 + s^3\theta^3\frac{(x-x_1)^2}{\tilde{a}_1}\bar{p}^2)e^{2s\tilde{\varphi}_1}\,dx\,dt \\ &\quad + \int_0^T\int_{-2x_1/3}^1 (s\theta\tilde{a}_2\bar{q}_x^2 + s^3\theta^3\frac{(x-x_2)^2}{\tilde{a}_2}\bar{q}^2)e^{2s\tilde{\varphi}_2}\,dx\,dt \\ &\le C\int_0^T\int_{\omega'} s^2\theta^2(U^2+V^2)e^{-2s\Phi_2}\,dx\,dt \\ &\quad + \epsilon_1\int_0^T\int_0^\lambda (s\theta a_1U_x^2 + s^3\theta^3\frac{(x-x_1)^2}{a_1}U^2)e^{2s\varphi_1}\,dx\,dt \\ &\quad + \epsilon_2\int_0^T\int_0^\lambda (s\theta a_2V_x^2 + s^3\theta^3\frac{(x-x_2)^2}{a_2}V^2)e^{2s\varphi_2}\,dx\,dt. \end{aligned} \tag{3.29}$$

Adding up (3.20) and (3.29), we finally obtain Theorem 3.5. □

To study the null-controllability of the parabolic system (1.1)–(1.4) with one control force, we need to show the following Carleman estimate.

Theorem 3.7. *Let $T > 0$. Moreover, assume that*

$$b_{21} \geq \mu > 0 \quad \text{on } [0,T] \times \omega_1', \tag{3.30}$$

for some $\omega_1' \subset\subset \omega'$. Then there exist two positive constants C and s_0 such that, for all $s \geq s_0$, the solution $(U,V) \in \mathcal{W}$ of (3.13) satisfies

$$\begin{aligned} &\int_0^T\int_0^1 \Big(s\theta a_1 U_x^2 + s^3\theta^3 \frac{(x-x_1)^2}{a_1} U^2\Big) e^{2s\varphi_1}\,dx\,dt \\ &+ \int_0^T\int_0^1 \Big(s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2\Big) e^{2s\varphi_2}\,dx\,dt \\ &\leq C\int_0^T\int_{\omega'} U^2\,dx\,dt. \end{aligned} \tag{3.31}$$

Theorem 3.7 is a consequence of Theorem 3.5 applied to ω_1' and of the following Lemma.

Lemma 3.8. *For each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$\int_0^T\int_{\omega_1'} s^3\theta^3 V^2 e^{-2s\Phi_2(t,x)}\,dx\,dt \leq \varepsilon J(v) + C_\varepsilon \int_0^T\int_{\omega'} U^2\,dx\,dt,$$

where $\varepsilon > 0$ is small enough, s is large enough and

$$J(V) = \int_0^T\int_0^1 (s\theta a_2 V_x^2 + s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2) e^{2s\varphi_2}\,dx\,dt.$$

Proof. The choice of the weight functions given in Lemma 3.2 will play a crucial role. We will adapt the technique used in [2]. Let $\chi \in C^\infty(0,1)$, such that $\operatorname{supp}(\chi) \subset \omega'$ and $\chi \equiv 1$ on ω_1'. Multiplying the first equation of system (3.13) by $s^3\theta^3\chi e^{-2s\Phi_2(x)}V$ and integrating over Q, we obtain

$$\begin{aligned} \int_Q s^3\theta^3 b_{21}\chi e^{-2s\Phi_2(x)} V^2\,dx\,dt = &-\int_Q s^3\theta^3\chi e^{-2s\Phi_2(x)} U_t V\,dx\,dt \\ &+ \int_Q s^3\theta^3\chi e^{-2s\Phi_2(x)} (a_1U_x)_x V\,dx\,dt \\ &- \int_Q s^3\theta^3 b_{11}\chi e^{-2s\Phi_2(x)} UV\,dx\,dt. \end{aligned} \tag{3.32}$$

Integrating by parts and using the second equation in (3.13), we obtain

$$\begin{aligned} &\int_Q s^3\theta^3\chi e^{-2s\Phi_2(x)} VU_t\,dx\,dt \\ &= \int_Q s^3\theta^3 a_2\chi e^{-2s\Phi_2(x)} U_xV_x\,dx\,dt \\ &\quad + \int_Q s^3\theta^3 a_2(\chi e^{-2s\Phi_2(x)})_x UV_x\,dx\,dt \\ &\quad + \int_Q \big[s^3\theta^3 b_{22} + 2s^4\theta^3\dot\theta\Psi_2(x) - 3s^3\theta^2\dot\theta\big]\chi e^{-2s\Phi_2(x)} VU\,dx\,dt, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\int_Q s^3\theta^3\chi e^{-2s\Phi_2(x)}(a_1U_x)_xV\,dx\,dt = &-\int_Q s^3\theta^3 a_1\chi e^{-2s\Phi_2(x)}V_xU_x\,dx\,dt\\ &+\int_Q s^3\theta^3 a_1(\chi e^{-2s\Phi_2(x)})_xV_xU\,dx\,dt\\ &+\int_Q s^3\theta^3(a_1(\chi e^{-2s\Phi_2(x)})_x)_xVU\,dx\,dt.\end{aligned}\tag{3.34}$$

So, combining (3.32)-(3.34), we obtain

$$\int_Q s^3\theta^3 b_{21}\chi e^{-2s\Phi_2(x)}V^2\,dx\,dt = I_1+I_2+I_3,$$

where

$$I_1 = -\int_Q s^3\theta^3(a_1+a_2)\chi e^{-2s\Phi_2(x)}U_xV_x\,dx\,dt,$$

$$\begin{aligned}I_2 &= \int_Q s^3\theta^3(a_1-a_2)(\chi e^{-2s\Phi_2(x)})_xUV_x\,dx\,dt\\ &= \int_Q \big(s^3\theta^3\chi' - 2s^4\theta^4\Psi_{2,x}(x)\chi\big)(a_1-a_2)e^{-2s\Phi_2(x)}UV_x\,dx\,dt,\end{aligned}$$

$$\begin{aligned}I_3 = &\int_Q \big[3s^3\theta^2\dot\theta - s^3\theta^3(b_{11}+b_{22}) - 2s^4\theta^3\dot\theta\Psi_2(x)\big]\chi e^{-2s\Phi_2(x)}UV\,dx\,dt\\ &+\int_Q s^3\theta^3(a_1(\chi e^{-2s\Phi_2(x)})_x)_xUV\,dx\,dt.\end{aligned}$$

For $\varepsilon>0$, we have

$$\begin{aligned}|I_1| &= \int_Q (\sqrt{s\theta a_2}e^{s\varphi_2}V_x)((s\theta)^{5/2}(a_2)^{-\frac12}(a_1+a_2)\chi e^{-s(2\Phi_2(x)+\varphi_2)}U_x)\,dx\,dt\\ &\le \varepsilon\int_Q s\theta a_2 e^{2s\varphi_2}V_x^2\,dx\,dt + \frac{1}{2\varepsilon}\underbrace{\int_Q s^5\theta^5\frac{(a_1^2+a_2^2)}{a_2}\chi^2 e^{-2s(2\Phi_2(x)+\varphi_2)}U_x^2\,dx\,dt}_{L}.\end{aligned}$$

The integral L should be estimated by an integral in U^2. For this, we multiply the first equation in (3.13) by $s^5\theta^5\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2e^{-2s(2\Phi_2(x)+\varphi_2)}U$ and we integrate by parts, obtaining

$$L = L_1+L_2+L_3+L_4,$$

where

$$\begin{aligned}L_1 = &\frac12\int_Q s^5(5\theta^4-2s\theta^5(2\Psi_2(x)+\psi_2))\dot\theta\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2\\ &\times e^{-2s(2\Phi_2(x)+\varphi_2)}U^2\,dx\,dt,\end{aligned}$$

$$L_2 = \frac12\int_Q s^5\theta^5\Big(a_1\Big(\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2e^{-2s(2\Phi_2(x)+\varphi_2)}\Big)_x\Big)_xU^2\,dx\,dt,$$

$$L_3 = -\int_Q s^5\theta^5\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2 b_{11}e^{-2s(2\Phi_2(x)+\varphi_2)}U^2\,dx\,dt,$$

$$L_4 = -\int_Q s^5\theta^5\frac{(a_1^2+a_2^2)}{a_1a_2}\chi^2 b_{21}e^{-2s(2\Phi_2(x)+\varphi_2)}UV\,dx\,dt.$$

Since $|\dot{\theta}| \leq C\theta^2$ and $\operatorname{supp}(\chi) \subset \omega'$, the functions a_i, $\frac{1}{a_i}$, χ, ψ_i, Ψ_i and their derivatives are bounded on ω' and also b_{11}, b_{21} and b_{22}. We deduce that, for $i \in \{1,2,3\}$

$$|L_i| \leq C \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(2\Phi_2(x)+\varphi_2)} U^2\,dx\,dt.$$

For $i = 4$ we have

$$\begin{aligned}
|L_4| &= \int_Q \big[(s\theta)^{\frac{3}{2}} \frac{(x-x_2)}{\sqrt{a_2}} e^{s\varphi_2} V\big]\big[(s\theta)^{\frac{7}{2}} b_{21}\chi^2 \frac{(a_1^2+a_2^2)}{a_1\sqrt{a_2}(x-x_2)} e^{-s(4\Phi_2+3\varphi_2)} U\big]\,dx\,dt\\
&\leq \varepsilon^2 \int_Q s^3\theta^3 \frac{(x-x_2)^2}{a_2} e^{2s\varphi_2} V^2\,dx\,dt\\
&\quad + \frac{1}{4\varepsilon^2}\int_Q s^7\theta^7 b_{21}^2\chi^4 \frac{(a_1^2+a_2^2)^2}{a_1^2 a_2 (x-x_2)^2} e^{-2s(4\Phi_2+3\varphi_2)} U^2\,dx\,dt\\
&\leq \varepsilon^2 \int_Q s^3\theta^3 \frac{(x-x_2)^2}{a_2} e^{2s\varphi_2} V^2\,dx\,dt + C_\varepsilon \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(4\Phi_2+3\varphi_2)} U^2\,dx\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
|L| &\leq C_\varepsilon \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(4\Phi_2+3\varphi_2)} U^2\,dx\,dt\\
&\quad + \varepsilon^2 \int_Q s^3\theta^3 \frac{(x-x_2)^2}{a_2} e^{2s\varphi_2} V^2\,dx\,dt.
\end{aligned}$$

Furthermore

$$|I_1| \leq C_\varepsilon \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(4\Phi_2+3\varphi_2)} U^2\,dx\,dt + \varepsilon J(V).$$

Using the fact that χ' and χ are supported in ω' and $x_2 \notin \omega'$, proceeding as before, one has

$$\begin{aligned}
|I_2| &\leq C \int_Q s^4\theta^4 (\chi'+\chi) e^{-2s\Phi_2} U V_x\,dx\,dt\\
&\leq C \int_Q (\sqrt{s\theta a_2} e^{s\varphi_2} V_x)((s\theta)^{7/2} (a_2)^{-\frac{1}{2}} (\chi'+\chi) e^{-s(2\Phi_2+\varphi_2)} U)\,dx\,dt\\
&\leq \varepsilon \int_Q s\theta a_2 V_x^2 e^{2s\varphi_2}\,dx\,dt + C_\varepsilon \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(2\Phi_2+\varphi_2)} U^2\,dx\,dt.\\
|I_3| &\leq C \int_Q s^5\theta^5 (\chi''+\chi'+\chi) e^{-2s\Phi_2} UV\,dx\,dt\\
&\leq C \int_Q (s^{\frac{3}{2}}\theta^{\frac{3}{2}} \frac{x-x_2}{\sqrt{a_2}} e^{s\varphi_2} V)((s\theta)^{\frac{7}{2}} \frac{\sqrt{a_2}}{x-x_2} (\chi''+\chi'+\chi) e^{-s(2\Phi_2+\varphi_2)} U)\,dx\,dt\\
&\leq \varepsilon \int_Q s^3\theta^3 \frac{(x-x_2)^2}{a_2} V^2 e^{2s\varphi_2}\,dx\,dt + C_\varepsilon \int_0^T\int_{\omega'} s^7\theta^7 e^{-2s(2\Phi_2+\varphi_2)} U^2\,dx\,dt.
\end{aligned}$$

So, thanks to Lemma 3.2, we have

$$\begin{gathered}
e^{-2s(2\Phi_2+\varphi_2)} \leq e^{-2s(4\Phi_2+3\varphi_2)} \leq 1,\\
\sup_{(t,x)\in Q} s^r\theta^r(t) e^{-2s(4\Phi_2+3\varphi_2)} < \infty, \quad r \in \mathbb{R}.
\end{gathered}$$

Then, for ε small enough and s large enough, we have

$$|\int_Q s^3\theta^3 b_{21}\chi e^{-2s\Phi_2} V^2\,dx\,dt| \le C_\varepsilon \int_0^T\int_{\omega'} U^2\,dx\,dt + 3\varepsilon J(V).$$

Finally, by the definition of χ and the previous inequality, it follows that

$$\begin{aligned}
\mu\int_0^T\int_{\omega_1'} s^3\theta^3 e^{-2s\Phi_2}V^2\,dx\,dt &\le |\int_0^T\int_{\omega_1'} s^3\theta^3 b_{21} e^{-2s\Phi_2}V^2\,dx\,dt| \\
&\le |\int_Q s^3\theta^3 b_{21}\chi e^{-2s\Phi_2}V^2\,dx\,dt| \\
&\le C_\varepsilon\int_0^T\int_{\omega'} U^2\,dx\,dt + \varepsilon J(V).
\end{aligned}$$

This completes the proof. □

4. Observability and null controllability

In this section we prove, as a consequence of the Carleman estimates established in the above section, observability inequalities for the adjoint problem (3.1)-(3.4).

Theorem 4.1. *Let $T > 0$ be given. Then there exists a positive constant C_T such that every (U, V) solution of* (3.1)-(3.4) *satisfies*

$$\int_0^1 [U^2(T,x) + V^2(T,x)]dx \le C_T\int_0^T\int_{\omega'} U^2(t,x)\,dx\,dt.$$

To prove the above theorem we need the following result.

Lemma 4.2. *Let $T > 0$ be given. Then there exists a positive constant C_T such that every $(U, V) \in \mathcal{W}$ solution of* (3.13) *satisfies*

$$\int_0^1 [U^2(T,x) + V^2(T,x)]dx \le C_T\int_0^T\int_{\omega'} U^2(t,x)\,dx\,dt.$$

Proof. Multiplying the first and the second equations in the system (3.13) respectively by U_t and V_t. Integrating over $(0, 1)$ the sum of the new equations, we obtain

$$\begin{aligned}
0 = \int_0^1 [U_t^2 + V_t^2]\,dx - [a_1(x)U_xU_t]_0^1 - [a_2(x)V_xV_t]_0^1 + \int_0^1 b_{11}UU_t\,dx \\
+ \int_0^1 b_{22}VV_t\,dx + \int_0^1 b_{21}VU_t\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx.
\end{aligned}$$

Using the Young's inequality we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx &\le \int_0^1 b_{11}^2U^2\,dx + \int_0^1 (b_{22}^2 + b_{21}^2)V^2\,dx, \\
&\le C\int_0^1 [U^2(t,x) + V^2(t,x)]\,dx.
\end{aligned}$$

By [22, Lemma 2.1], the map $x \mapsto \frac{(x-x_i)^2}{a_i(x)}$ is non-increasing on $[0, x_i)$ and non-decreasing on $(x_i, 1]$, then

$$\Big(\frac{(x-x_i)^2}{a_i(x)}\Big)^{1/3} \le \max\Big\{\Big(\frac{x_i^2}{a_i(0)}\Big)^{1/3}, \Big(\frac{(1-x_i)^2}{a_i(1)}\Big)^{1/3}\Big\}.$$

Hence

$$\frac{d}{dt}\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx \le CC_0\int_0^1 \big[\frac{a_1^{1/3}(x)}{(x-x_1)^{2/3}}U^2(t,x) + \frac{a_2^{1/3}(x)}{(x-x_2)^{2/3}}V^2(t,x)\big]\,dx,$$

where

$$C_0 := \max\big\{\big(\frac{x_1^2}{a_1(0)}\big)^{1/3}, \big(\frac{(1-x_1)^2}{a_1(1)}\big)^{1/3}, \big(\frac{x_2^2}{a_2(0)}\big)^{1/3}, \big(\frac{(1-x_2)^2}{a_2(1)}\big)^{1/3}\big\}.$$

Moreover, by the Hardy-Poincaré inequality, and proceeding as in (3.25), one has

$$\frac{d}{dt}[a_1U_x^2 + a_2V_x^2]dx \le C_1\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx.$$

Hence

$$\frac{d}{dt}\big\{e^{-C_1t}\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx\big\} \le 0.$$

Consequently, the function $t \mapsto e^{-C_1t}\int_0^1 [a_1U_x^2 + a_2V_x^2]\,dx$ is not increasing. Thus,

$$\int_0^1 [a_1(x)U_x^2(T,x) + a_2(x)V_x^2(T,x)]\,dx \le e^{C_1T}\int_0^1 [a_1(x)U_x^2(t,x) + a_2(x)V_x^2(t,x)]\,dx.$$

Moreover, by the fact that

$$\inf_{(\frac{T}{4},\frac{3T}{4})\times(0,1)} s\theta e^{2s\varphi_i} > 0,$$

integrating over $[T/4, 3T/4]$, and using the Carleman estimate (3.31), we obtain

$$\begin{aligned}
&\int_0^1 [a_1(x)U_x^2(T,x) + a_2(x)V_x^2(T,x)]\,dx \\
&\le \frac{2e^{C_1T}}{T}\int_{T/4}^{3T/4}\int_0^1 [a_1(x)U_x^2(t,x) + a_2(x)V_x^2(t,x)]\,dx\,dt \\
&\le C_T\int_{T/4}^{3T/4}\int_0^1 s\theta[a_1(x)U_x^2e^{2s\varphi_1} + a_2(x)V_x^2e^{2s\varphi_2}]\,dx\,dt \\
&\le C_T\int_0^T\int_{\omega'} U^2(t,x)\,dx\,dt.
\end{aligned} \tag{4.1}$$

On the other hand, applying the Hardy-Poincaré inequality one gets

$$\begin{aligned}
&\int_0^1 [U^2(T,x) + V^2(T,x)]\,dx \\
&\le C\int_0^1 \big[\frac{a_1^{1/3}(x)}{(x-x_1)^{2/3}}U^2(T,x) + \frac{a_2^{1/3}(x)}{(x-x_2)^{2/3}}V^2(T,x)\big]\,dx \\
&\le C\int_0^1 [a_1(x)U_x^2(T,x) + a_2(x)V_x^2(T,x)]\,dx,
\end{aligned} \tag{4.2}$$

for a positive constant C.
Combining (4.1) and (4.2) the conclusion follows. □

The proof of Theorem 4.1 is now standard using Lemma 4.2 and proceeding as in [22, Proposition 4.1], but we give it for the reader's convenience.

Proof of Proposition 4.1. Let $(U_0, V_0) \in L^2(0,1) \times L^2(0,1)$ and let (U, V) be the solution of (3.13) associated to (U_0, V_0). Since $D(A_i^2)$ is densely defined in $L^2(0,1)$, there exists a sequence $(U_0^n, V_0^n)_n \subset D(A_1^2) \times D(A_2^2)$ which converge to (U_0, V_0) in $L^2(0,1) \times L^2(0,1)$. Now, consider the solution (U_n, V_n) associated to (U_0^n, V_0^n). Since the semigroup generated by $\mathcal{A}$ is analytic, hence $\mathcal{A}$ is closed (e.g., see [19, Theorem I.1.4]), thus, by [19, Theorem II.6.7], we obtain that $(U_n, V_n)_n$ converges to a certain (U, V) in $C([0,T];\mathbb{H})$, so that

$$\begin{aligned} \lim_{n\to+\infty} \int_0^1 U_n^2(T,x)\,dx &= \int_0^1 U^2(T,x)\,dx \\ \lim_{n\to+\infty} \int_0^1 V_n^2(T,x)\,dx &= \int_0^1 V^2(T,x)\,dx \\ \lim_{n\to+\infty} \int_0^T\int_{\omega'} U_n^2(t,x)\,dx\,dt &= \int_0^T\int_{\omega'} U^2(t,x)\,dx\,dt. \end{aligned}$$

But by Lemma 4.2 we know that

$$\int_0^1 [U_n^2(T,x) + V_n^2(T,x)]dx \le C_T \int_0^T\int_{\omega'} U_n^2(t,x)\,dx\,dt.$$

Thus Theorem 4.1 is now proved. □

By Theorem 4.1, using a standard technique (e.g., see [24, Section 7.4]), one can deduce the following controllability result.

Theorem 4.3. *If the assumption 3.30 is satisfied, then the cascade degenerate parabolic system* (1.1)–(1.4) *with one control force is null controllable.*

5. Appendix

As in [1, 2], we give the proof of the Caccioppoli's inequality for linear cascade systems with two interior degeneracies.

Lemma 5.1 (Caccioppoli's inequality)**.** *Let ω'' and ω' two open subintervals of $(0,1)$ such that $\omega'' \subset \omega' \subset\subset \omega \subset (0,1)$ and $x_i \notin \overline{\omega'}$. Then, there exist two positive constants C and s_0 such that every solution $(U,V) \in \mathcal{W}$ of the adjoint problem* (3.13) *satisfies*

$$\begin{aligned} &\int_0^T\int_{\omega''} [U_x^2(t,x) + V_x^2(t,x)]e^{-2s\Phi_2}\,dx\,dt \\ &\le C\int_0^T\int_{\omega'} s^2\theta^2[U^2(t,x) + V^2(t,x)]e^{-2s\Phi_2}\,dx\,dt, \end{aligned} \tag{5.1}$$

for all $s \ge s_0$.

Proof. Define a smooth cut-off function $\xi \in C^\infty(0,1)$ such that $\operatorname{supp}(\xi) \subset \omega'$ and $\xi \equiv 1$ on ω''. Since (U,V) solves (3.13), we have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}[\int_0^1 \xi^2 e^{-2s\Phi_2}(U^2+V^2)\,dx]dt \\ &= -2\int_0^T\int_0^1 s\dot{\Phi}_2\xi^2 e^{-2s\Phi_2}(U^2+V^2)\,dx\,dt - 2\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} a_1(x)U_x^2\,dx\,dt \\ &\quad - 2\int_0^T\int_0^1 (\xi^2 e^{-2s\Phi_2})_x a_1(x)UU_x\,dx\,dt - 2\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} b_{22}(x)V^2\,dx\,dt \end{aligned}$$

$$
\begin{aligned}
&-2\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} b_{11}U^2 dx\,dt - 2\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} b_{21}UV\,dx\,dt\\
&-2\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} a_2(x)V_x^2\,dx\,dt - 2\int_0^T\int_0^1 (\xi^2 e^{-2s\Phi_2})_x a_2(x)VV_x\,dx\,dt.
\end{aligned}
$$

Then, integration by parts yields

$$
\begin{aligned}
&\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2}\big[a_1U_x^2 + a_2V_x^2\big]dxdt\\
&= -\int_0^T\int_0^1 s\dot{\Phi}_2\xi^2 e^{-2s\Phi_2}(U^2+V^2)\,dx\,dt - \int_0^T\int_0^1 (\xi^2 e^{-2s\Phi_2})_x(a_1UU_x + a_2VV_x)dx\,dt\\
&\quad -\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2}(b_{11}U^2 + b_{22}V^2)\,dx\,dt - \int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} b_{21}UV\,dx\,dt\\
&= -\int_0^T\int_0^1 s\dot{\Phi}_2\xi^2 e^{-2s\Phi_2}(U^2+V^2)\,dx\,dt + \frac{1}{2}\int_0^T\int_0^1 \Big((\xi^2 e^{-2s\Phi_2})_x a_1\Big)_x U^2 dx\,dt\\
&\quad +\frac{1}{2}\int_0^T\int_0^1 \Big((\xi^2 e^{-2s\Phi_2})_x a_2\Big)_x V^2 dx\,dt - \int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2}(b_{11}U^2 + b_{22}V^2)\,dx\,dt\\
&\quad -\int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2} b_{21}UV\,dx\,dt.
\end{aligned}
$$

Since $x_i \notin \overline{\omega'}$, $\operatorname{supp}(\xi) \subset \omega'$, $\xi \equiv 1$ on ω'' and $|\dot{\theta}| \leq c\theta^2$ then, using the Young inequality, we obtain

$$
\begin{aligned}
&\min_{x\in\omega''}\{a_1(x), a_2(x)\}\int_0^T\int_{\omega''} e^{-2s\Phi_2}[U_x^2 + V_x^2]\,dx\,dt\\
&\leq \int_0^T\int_0^1 \xi^2 e^{-2s\Phi_2}[a_1U_x^2 + a_2V_x^2]\,dx\,dt\\
&\leq C\int_0^T\int_{\omega'} (1 + s^2\theta^2 + s|\dot{\theta}|)[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt\\
&\leq C\int_0^T\int_{\omega'} s^2\theta^2[U^2+V^2]e^{-2s\Phi_2}\,dx\,dt,
\end{aligned}
$$

and the proof is complete. □

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