

EIGENVALUES HOMOGENIZATION FOR THE FRACTIONAL p -LAPLACIAN

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ABSTRACT. In this work we study the homogenization for eigenvalues of the fractional p -Laplace operator in a bounded domain both with Dirichlet and Neumann conditions. We obtain the convergence of eigenvalues and the explicit order of the convergence rates when periodic weights are considered.

1. INTRODUCTION

The purpose of this article is to study the asymptotic behavior as $\varepsilon \rightarrow 0$ of the eigenvalues of the non-local problem

$$\begin{aligned} (-\Delta_p)^s u &= \lambda_{p,\varepsilon} \rho_\varepsilon(x) |u|^{p-2} u \quad \text{in } \Omega \subset \mathbb{R}^n \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{aligned} \tag{1.1}$$

where for $\varepsilon > 0$, the parameter $\lambda_{p,\varepsilon}$ is the eigenvalue and $1 < p < \infty$. The domain Ω is assumed to be a bounded and open set in \mathbb{R}^n , $n \geq 1$. The weight functions ρ_ε are positive and bounded away from zero and infinity, i.e., for some constants ρ_- and ρ_+ it holds that

$$0 < \rho_- \leq \rho_\varepsilon(x) \leq \rho_+ < \infty \quad x \in \Omega. \tag{1.2}$$

Here, for $s \in (0, 1)$ we denote by $(-\Delta_p)^s$ the fractional p -Laplace operator, which is defined as

$$(-\Delta_p)^s u(x) = 2p \cdot \text{v.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{n+sp}} dy.$$

As $\varepsilon \rightarrow 0$ in (1.1), the following limit problem is obtained

$$\begin{aligned} (-\Delta_p)^s u &= \lambda_p \rho_0(x) |u|^{p-2} u \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{aligned} \tag{1.3}$$

where $\rho_0(x)$ is the weak* limit in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0$ of the sequence $\{\rho_\varepsilon\}_\varepsilon$.

For each fixed value of ε it is known that there exists a sequence of variational eigenvalues $\{\lambda_{k,p}^\varepsilon\}_{k \geq 1}$ of (1.1) such that $\lambda_{k,p}^\varepsilon \rightarrow \infty$ as $k \rightarrow \infty$. Analogously, for the limit problem (1.3), there exists a sequence of variational eigenvalues $\{\lambda_{k,p}^0\}_{k \geq 1}$ such that $\lambda_{k,p}^0 \rightarrow \infty$ as $k \rightarrow \infty$ (see Section 2). We are interested in studying the behavior of the sequence $\{\lambda_{k,p}^\varepsilon\}_{k \geq 1}$ as $\varepsilon \rightarrow 0$.

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When $s = 1$ and $p = 2$, (1.1) becomes the eigenvalue problem for the Laplacian operator with Dirichlet boundary conditions. This problem has been extensively studied and a complete description of the asymptotic behavior of its spectrum was obtained in the 70's. Boccardo and Marcellini [3], and Kesavan [12] proved that for each fixed k ,

$$\lim_{\varepsilon \rightarrow 0} \lambda_{k,2}^\varepsilon = \lambda_{k,2}^0.$$

Later on, in [4] and [9] this result was extended to p -Laplacian type operators.

One of the purposes of our paper is to extend this results to non-local eigenvalue problems. Our first result states the convergence of the k -th eigenvalue of problem (1.1) to the k -th eigenvalue of the limit problem (1.3) when a general family of weight functions is considered.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $s \in (0, 1)$ and let ρ_ε satisfying (1.2) such that $\rho_\varepsilon \xrightarrow{*} \rho_0$ weakly* in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0$. Let $\lambda_{k,p}^\varepsilon$ and $\lambda_{k,p}^0$ be the k -th (variational) eigenvalues of (1.1) and (1.3), respectively. Then*

$$\lim_{\varepsilon \rightarrow 0} \lambda_{k,p}^\varepsilon = \lambda_{k,p}^0 \quad (1.4)$$

for each fixed $k \geq 1$.

A slight modification in the arguments of the proof in the previous result allow us to deal with the following non-local Neumann eigenvalue problem considered recently in [7]

$$\begin{aligned} \mathcal{L}_{s,p}u + |u|^{p-2}u &= \Lambda_{p,\varepsilon}\rho_\varepsilon(x)|u|^{p-2}u \quad \text{in } \Omega \\ u &\in W^{s,p}(\Omega) \end{aligned} \quad (1.5)$$

where $\mathcal{L}_{s,p}$ denotes the regional fractional p -Laplacian defined as

$$\mathcal{L}_{s,p} := 2 \text{p. v.} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy \quad (1.6)$$

for which, again, the min-max theory provides a sequence of variational eigenvalues tending to $+\infty$, denoted by $\{\Lambda_{k,p}^\varepsilon\}_{k \geq 1}$. Analogously to the Dirichlet case, as $\varepsilon \rightarrow 0$, a limit problem is obtained in terms of ρ_0 , the weak* limit of ρ_ε in $L^\infty(\Omega)$,

$$\begin{aligned} \mathcal{L}_{s,p}u + |u|^{p-2}u &= \Lambda_p\rho_0(x)|u|^{p-2}u \quad \text{in } \Omega \\ u &\in W^{s,p}(\Omega) \end{aligned} \quad (1.7)$$

which has a sequence of variational eigenvalues denoted by $\{\Lambda_{k,p}^0\}_{k \geq 1}$. Here $W^{s,p}(\Omega)$ stands for a fractional order Sobolev space, which is defined in Sections 2. The corresponding convergence result is stated as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $s \in (0, 1)$ and let ρ_ε satisfying (1.2) such that $\rho_\varepsilon \xrightarrow{*} \rho_0$ weakly* in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0$. Let $\Lambda_{k,p}^\varepsilon$ and $\Lambda_{k,p}^0$ be the k -th (variational) eigenvalues of (1.5) and (1.7), respectively. Then*

$$\lim_{\varepsilon \rightarrow 0} \Lambda_{k,p}^\varepsilon = \Lambda_{k,p}^0 \quad (1.8)$$

for each fixed $k \geq 1$.

Homogenization theory dates back to the late sixties with the works of Spagnolo and de Giorgi and it developed very rapidly during the last two decades. Homogenization theory tries to get a good approximation of a macroscopic behavior of the

heterogeneous material by letting the parameter $\varepsilon \rightarrow 0$. A case of relevant importance is the study of periodic homogenization problems due to the many applications to physics and engineering. The main references for the homogenization theory of (local) periodic structures are the books by Bensoussan-Lions-Papanicolaou [1], Sanchez-Palencia [17], Oleřnik-Shamaev-Yosifian [16] among others.

An interesting issue in the homogenization theory is to estimate the rates of convergence of the eigenvalues in (1.4) and (1.8); that is, to find bounds for the differences $|\lambda_{k,p}^\varepsilon - \lambda_{k,p}^0|$ and $|\Lambda_{k,p}^\varepsilon - \Lambda_{k,p}^0|$. Since it is desirable to obtain the explicit dependence on ε , we restrict our study to periodic weights, i.e., we consider a family of weight functions ρ_ε given in terms of a single-bounded Q -periodic function ρ in the form

$$\rho_\varepsilon(x) := \rho(x/\varepsilon), \quad \varepsilon > 0,$$

with Q being the unit cube of \mathbb{R}^n . The function ρ is assumed to satisfy the bounds (1.2). Under these assumptions it is well-known that

$$\rho_\varepsilon \xrightarrow{*} \bar{\rho} \quad \text{in } L^\infty(\Omega) \text{ as } \varepsilon \rightarrow 0,$$

where $\bar{\rho}$ is the average of ρ on Q .

In the local case, the rates of convergence for the eigenvalues of the p -Laplace operator were studied in several papers. The authors in [16] proved some estimates for the Dirichlet and the Neumann case when $p = 2$ by using tools from functional analysis in Hilbert spaces. Assuming that Ω is a Lipschitz domain they showed that there exists a constant C depending on k and Ω such that

$$|\lambda_{k,2}^\varepsilon - \lambda_{k,2}^0| \leq C\varepsilon^{\frac{1}{2}}.$$

Later on, under the same assumptions on Ω it was proved in [13] the following bounds for both Dirichlet and Neumann boundary conditions,

$$|\lambda_{k,2}^\varepsilon - \lambda_{k,2}^0| \leq C\varepsilon |\log \varepsilon|^{\frac{1}{2}+\gamma}$$

for any $\gamma > 0$, C depending on k and γ . When the domain is more regular ($C^{1,1}$ is enough) in [11] explicit dependence of the constant C on k was obtained. It was proved that

$$|\lambda_{k,2}^\varepsilon - \lambda_{k,2}^0| \leq C\varepsilon k^{\frac{3}{n}} \varepsilon |\log \varepsilon|^{\frac{1}{2}+\gamma}$$

for any $\gamma > 0$, C depending on γ . In both cases, when the domain Ω is smooth, the logarithmic term can be removed.

In [9] the results were extended to the local p -Laplace operator via non-linear techniques and the dependence on the constant was improved. The authors in [9] proved that

$$|\lambda_{k,p}^\varepsilon - \lambda_{k,p}^0| \leq Ck^{\frac{p+1}{n}} \varepsilon, \quad |\Lambda_{k,p}^\varepsilon - \Lambda_{k,p}^0| \leq Ck^{\frac{2p}{n}} \varepsilon \quad (1.9)$$

where C is a constant independent on k and ε which can be explicitly computed.

To the best of our knowledge, no investigation was made on the homogenization and convergence rates for the weighted fractional p -Laplacian eigenvalue problem. In contrast with the p -Laplacian operator, the non-local nature of (1.1) makes more difficult to deal with the convergence rates. The main obstacle is how to manage the boundedness of fractional norms in order to obtain relations between the variational characterization of eigenvalues.

In the two next results we obtain the rates of the convergence of the eigenvalues of problems (1.1) and (1.5) when periodicity assumptions are made on the weight family.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $\rho \in L^\infty(\mathbb{R}^n)$ be a Q -periodic function satisfying (1.2), Q being the unit cube of \mathbb{R}^n . Let $\lambda_{k,p}^\varepsilon$ and $\lambda_{k,p}^0$ be the k -th variational eigenvalues of (1.1) and (1.3), respectively. Then*

$$|\lambda_{k,p}^\varepsilon - \lambda_{k,p}^0| \leq C\varepsilon^s (\lambda_{k,p})^{1+\frac{1}{p}}$$

for every $k \in \mathbb{N}$ and $s \in (0, 1)$, $\lambda_{k,p}$ being the k -th variational eigenvalue of the Dirichlet fractional p -Laplacian of order s . The constant C depends only on Ω , s , n , p and the bounds of ρ . In the case $p = 2$ the previous inequality becomes

$$|\lambda_{k,2}^\varepsilon - \lambda_{k,2}^0| \leq C\varepsilon^s k^{3s/n}$$

for every $k \in \mathbb{N}$ and $s \in (0, 1)$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary and $\rho \in L^\infty(\mathbb{R}^n)$ be a Q -periodic function satisfying (1.2), Q being the unit cube of \mathbb{R}^n . Let $\Lambda_{k,p}^\varepsilon$ and $\Lambda_{k,p}^0$ be the k -th variational eigenvalues of (1.5) and (1.7), respectively. Then*

$$|\Lambda_{k,p}^\varepsilon - \Lambda_{k,p}^0| \leq C\varepsilon^s (\Lambda_{k,p})^2$$

for every $k \in \mathbb{N}$ and $s \in (\frac{1}{p}, 1)$, $\Lambda_{k,p}$ being the k -th variational eigenvalue of the regional Neumann fractional p -Laplacian (1.7) with $\rho_0 \equiv 1$. The constant C depends only on Ω , s , n and the bounds of ρ . In the case $p = 2$ the previous inequality becomes

$$|\Lambda_{k,2}^\varepsilon - \Lambda_{k,2}^0| \leq C\varepsilon^s k^{4s/n}$$

for every $k \in \mathbb{N}$ and $s \in (\frac{1}{p}, 1)$.

Although the rates obtained in the two previous results are similar, in the Neumann case the range of allowed values for s is smaller, and more assumptions on the boundary of Ω have to be made. Such restrictions arise from the use of trace arguments in the proof. Observe that the rates obtained in Theorems 1.3 and 1.4 are the natural generalization of the results for the local case stated in (1.9).

This article is organized as follows: in Section 2 we introduce some definitions and properties of the eigenvalues of non-local problems meanwhile that in Section 3 we prove the results before stated.

2. FRACTIONAL SOBOLEV SPACES AND FRACTIONAL EIGENVALUES

In this section we present some well-known results about fractional Sobolev spaces and the eigenvalues of non-local problems. For more detailed information we refer to the reader, for instance to [8].

Let Ω be a subset of \mathbb{R}^n , $n \geq 1$. For any $s \in (0, 1)$ and $p \geq 1$ we denote $W^{s,p}(\Omega)$ the fractional Sobolev space defined as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{u(x) - u(y)}{|x - y|^{\frac{n}{p} + s}} \in L^p(\Omega \times \Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)}^p := \|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p,$$

where $[u]_{W^{s,p}(\Omega)}$ is the so-called Gagliardo semi-norm of u defined as

$$[u]_{W^{s,p}(\Omega)}^p = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy.$$

Hereafter we denote $\mathcal{X}_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$.

It is well-known that the space $W^{s',p}$ is continuously embedded in $W^{s,p}$ when $s \leq s'$, (see for instance [8]).

Lemma 2.1. *Let $p \in [1, \infty)$ and $0 < s < s' < 1$. Let Ω be an open set in \mathbb{R}^n and u a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s',p}(\Omega)}$$

for some positive constant $C = C(n, s, p)$.

Here we denote by $(v)_U$ the average of the function v on the set U . This is useful tool to be used is the following fractional Poincaré inequality on cubes of side ε .

Lemma 2.2. *Let Q be the unit cube in \mathbb{R}^n , $n \geq 1$. Then, for every $u \in W^{s,p}(Q_\varepsilon)$, $1 < p < \infty$ we have*

$$\|u - (u)_{Q_\varepsilon}\|_{L^p(Q_\varepsilon)} \leq c\varepsilon^s [u]_{W^{s,p}(Q_\varepsilon)},$$

where $Q_\varepsilon = \varepsilon Q$ and c is a constant depending only on n .

Proof. Given $u \in W^{s,p}(Q_\varepsilon)$, by using Jensen’s inequality it follows that

$$\begin{aligned} \int_{Q_\varepsilon} |u - (u)_{Q_\varepsilon}|^p dx &= \int_{Q_\varepsilon} \left| \int_{Q_\varepsilon} (u(x) - u(y)) dy \right|^p dx \\ &\leq \int_{Q_\varepsilon} \int_{Q_\varepsilon} |u(x) - u(y)|^p dy dx \\ &\leq c\varepsilon^{sp} \int_{Q_\varepsilon} \int_{Q_\varepsilon} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx, \end{aligned}$$

from where the result follows. □

It is readily seen that the following Poincaré-type inequality holds:

$$\|u\|_{L^p(Q_\varepsilon)} \leq c\varepsilon^s [u]_{W^{s,p}(Q_\varepsilon)}$$

for all $u \in \mathcal{X}_0^{s,p}(Q_\varepsilon)$, from where it follows that $[\cdot]_{W^{s,p}(\Omega)}$ is an equivalent norm in the space $\mathcal{X}_0^{s,p}$.

Another result that we will use is the trace’s inequality for fractional spaces proved in [18], which it is necessary for our auxiliary computations.

Proposition 2.3. *Let Ω be a bounded C^1 domain and $\frac{1}{p} < s < 1$. Then*

$$\|u\|_{W^{s-\frac{1}{p},p}(\partial\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)},$$

where C is a constant depending on s, p and Ω .

2.1. Fractional eigenvalues. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Given a weight function ρ bounded away from zero and infinity, we consider the Dirichlet eigenvalue problem

$$(-\Delta_p)^s u = \lambda_D \rho |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{2.1}$$

Because of the non-local nature of the problem it is needed to consider the boundary condition not only on $\partial\Omega$ but in $\mathbb{R}^n \setminus \Omega$. This problem has a variational structure. We say that $u \in \mathcal{X}_0^{s,p}(\Omega)$ is a weak solution of (2.1) if

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} = \lambda_D \int_{\Omega} \rho(x) |u|^{p-2} uv$$

for every $v \in \mathcal{X}_0^{s,p}(\Omega)$.

The following non-local Neumann eigenvalue problem for the regional p -Laplacian defined in (1.6) was considered recently in [7]

$$\mathcal{L}_{s,p}u + |u|^{p-2}u = \lambda_N \rho(x)|u|^{p-2}u \quad \text{in } \Omega, \quad u \in W^{s,p}(\Omega). \quad (2.2)$$

In this case, we say that a function $u \in W^{s,p}(\Omega)$ is a weak solution of (2.2) if

$$\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} + \int_{\Omega} |u|^{p-2}uv = \lambda_N \int_{\Omega} \rho |u|^{p-2}uv$$

for every $v \in W^{s,p}(\Omega)$.

As in [10], a non-decreasing sequence of eigenvalues for (2.1) and (2.2) can be defined by means of cohomological index. We will denote by $\{\lambda_{Dk,p}\}_{k \geq 1}$ and $\{\Lambda_{Nk,p}\}_{k \geq 1}$ such sequences, respectively. They can be written by using the following inf-sup characterization:

$$\lambda_{Dk,p} = \inf_{\mathcal{C} \in \mathcal{D}_k} \sup_{u \in \mathcal{C}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\|\rho^{\frac{1}{p}}u\|_{L^p(\Omega)}^p}, \quad \lambda_{Nk,p} = \inf_{\mathcal{C} \in \tilde{\mathcal{D}}_k} \sup_{u \in \mathcal{C}} \frac{\|u\|_{W^{s,p}(\Omega)}^p}{\|\rho^{\frac{1}{p}}u\|_{L^p(\Omega)}^p}, \quad (2.3)$$

where $\mathcal{D}_k = \{\mathcal{W} \subset \mathcal{X}_0^{s,p}(\Omega) : i(\mathcal{W}) \geq k\}$ and $\tilde{\mathcal{D}}_k = \{\mathcal{W} \subset W^{s,p}(\Omega) : i(\mathcal{W}) \geq k\}$. Here i denotes the cohomological index, see for instance [15] for the definition and further properties. These formulas differ from the classical ones by the use of the index instead of the genus, but they coincide in the lineal case. In fact, when $p = 2$ the sequence defined in (2.3) coincides with the sequence of variational eigenvalues which uses *dimension* instead of *index* (see for instance [19])

$$\lambda_{Dk,2} = \min_{\mathcal{C} \in \mathcal{D}_k} \max_{u \in \mathcal{C}} \frac{[u]_{W^{s,2}(\mathbb{R}^n)}^2}{\|\rho^{\frac{1}{2}}u\|_{L^2(\Omega)}^2}, \quad \lambda_{Nk,2} = \min_{\mathcal{C} \in \tilde{\mathcal{D}}_k} \max_{u \in \mathcal{C}} \frac{\|u\|_{W^{s,2}(\Omega)}^2}{\|\rho^{\frac{1}{2}}u\|_{L^2(\Omega)}^2}, \quad (2.4)$$

where $\mathcal{D}_k = \{\mathcal{W} \subset \mathcal{X}_0^{s,2}(\Omega) : \dim \mathcal{W} = k\}$ and $\tilde{\mathcal{D}}_k = \{\mathcal{W} \subset W^{s,2}(\Omega) : \dim \mathcal{W} = k\}$. The variational characterization of eigenvalues plays a fundamental role in our analysis and the proof of our results since it allows to reduce the eigenvalues convergence to the study of oscillating integrals.

Bounds for the eigenvalues (2.3) are also necessary for our arguments. When the weight function ρ satisfy (1.2), it is easy to see that

$$(\rho_+)^{-1}\lambda_{k,p} \leq \lambda_{Dk,p} \leq (\rho_-)^{-1}\lambda_{k,p}, \quad (2.5)$$

where $\lambda_{k,p}$ is the k -th eigenvalue of the Dirichlet fractional Laplacian, i.e., it satisfies the following equation

$$(-\Delta_p)^s u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \quad (2.6)$$

Moreover, observe that for any $u \in W^{s,p}(\mathbb{R}^n)$, $u \neq 0$, (1.2) gives

$$\frac{\|u\|_{W^{s,p}(\Omega)}^p}{\|\rho^{1/p}u\|_{L^p(\Omega)}^p} \leq \frac{1}{\rho_-} \left(1 + \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\|u\|_{L^p(\Omega)}^p} \right),$$

and using that $\mathcal{X}_0^{s,p}(\Omega) \subset W^{s,p}(\Omega)$ it is straightforward to see that Neumann eigenvalues can be bounded with the Dirichlet ones, i.e.,

$$\Lambda_{Nk,p} \leq (\rho_-)^{-1}(1 + \lambda_{k,p}). \quad (2.7)$$

Since the sequences (2.3) are bounded in terms of the eigenvalues of (2.6), it is desirable to estimate such eigenvalues.

For $p > 1$, $s \in (0, 1)$ and $sp > n$, the authors in [10] proved that for k large the following bounds hold

$$c_1|\Omega|^{-\frac{sp}{n}}k^{\frac{sp-n}{n}} \leq \lambda_{k,p} \leq c_2|\Omega|^{-\frac{sp}{n}}k^{\frac{np-n+sp}{n}} \tag{2.8}$$

for some positive constants c_1 and c_2 depending on s , p and n . Nevertheless, they suspect that these estimates are not optimal. However when $p = 2$, precise estimates for this sequences are known. In 1959, Blumenthal and Gettoor [2] proved a Weyl’s formula for $\lambda_{k,2}$ in the context of s –stable symmetric processes, whose generators are the fractional Laplacians, more precisely, they proved the asymptotic formula

$$\lambda_{k,2} \sim (4\pi)^s \left(k|\Omega|^{-1}\Gamma\left(1 + \frac{n}{2}\right) \right)^{2s/n}, \quad k \rightarrow +\infty.$$

Moreover, in [6] it was proved that there exists some constant c independent on k such that $c(\tilde{\mu}_{k,2})^s \leq \lambda_{k,2} \leq (\tilde{\mu}_{k,2})^s$, where $\tilde{\mu}_{k,2}$ is the k -th eigenvalue of the Laplacian with Dirichlet boundary conditions on $\partial\Omega$. Since it is well-known that there exist constants c_1 and c_2 independent on k such that $c_1k^{\frac{2}{n}} \leq \tilde{\mu}_{k,2} \leq c_2k^{\frac{2}{n}}$ (see for instance [5]), for the case $p = 2$, inequality (2.5) reads as

$$C_1k^{2s/n} \leq \lambda_{k,2} \leq C_2k^{2s/n} \tag{2.9}$$

where C_1 and C_2 are two constant independent on k and s .

3. PROOF OF MAIN RESULTS

The convergence of the sequence of Dirichlet and Neumann eigenvalues is a consequence of the following simple lemma concerning oscillating integrals. When periodicity is not assumed on the weight functions, the result does not provide any information about the order of the convergence.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\{g_\varepsilon\}_{\varepsilon>0}$ be a set functions such that $0 < g_- \leq g_\varepsilon \leq g_+ < +\infty$ for g_\pm constants and $g_\varepsilon \rightarrow g$ weakly* in $L^\infty(\Omega)$. Then*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (g_\varepsilon - g)|u|^p = 0$$

for every $u \in W^{s,p}(\Omega)$, $0 < s < 1$.

Proof. The weak* convergence of g_ε in $L^\infty(\Omega)$ says that $\int_{\Omega} g_\varepsilon \varphi \rightarrow \int_{\Omega} g \varphi$ for all $\varphi \in L^1(\Omega)$. In particular, since $u \in W^{s,p}(\Omega)$, we have that $|u|^p \in L^1(\Omega)$ and the result is proved. \square

Proof of Theorem 1.1. Let $\delta > 0$ and $\mathcal{C}_{k,\delta} \subset \mathcal{X}_0^{s,p}(\Omega)$ be a set of index greater or equal than k such that

$$\lambda_{k,p}^0 = \inf_{C \in \mathcal{D}_k} \sup_{u \in C} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_0 |u|^p} = \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_0 |u|^p} + O(\delta)$$

where $\mathcal{D}_k = \{\mathcal{W} \in \mathcal{X}_0^{s,p}(\Omega) : i(\mathcal{W}) \geq k\}$.

We use now the set $\mathcal{C}_{k,\delta}$, which is admissible in the variational characterization of the k th–eigenvalue of (1.1), to find a bound for it as follows,

$$\lambda_{k,p}^\varepsilon \leq \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_\varepsilon |u|^p} = \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_0 |u|^p} \frac{\int_{\Omega} \rho_0 |u|^p}{\int_{\Omega} \rho_\varepsilon |u|^p}. \tag{3.1}$$

To bound $\lambda_{k,p}^\varepsilon$ we look for bounds of the two quotients in (3.1). For every function $u \in \mathcal{C}_{k,\delta}$ we have

$$\frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_0 |u|^p} \leq \sup_{v \in \mathcal{C}_{k,\delta}} \frac{[v]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_0 |v|^p} = \lambda_{k,p}^0 + O(\delta). \quad (3.2)$$

Since $u \in \mathcal{C}_{k,\delta} \subset \mathcal{X}_0^{s,p}(\Omega)$, by Lemma 3.1 we obtain

$$\frac{\int_{\Omega} \rho_0 |u|^p}{\int_{\Omega} \rho_\varepsilon |u|^p} \leq 1 + O(\varepsilon). \quad (3.3)$$

Then, combining (3.1), (3.2) and (3.3) we find that $\lambda_{k,p}^\varepsilon \leq (\lambda_{k,p}^0 + O(\delta))(1 + O(\varepsilon))$, from where it follows that

$$\lambda_{k,p}^\varepsilon - \lambda_{k,p}^0 \leq O(\varepsilon, \delta).$$

In a similar way, interchanging the roles of $\lambda_{k,p}^0$ and $\lambda_{k,p}^\varepsilon$, we obtain that $\lambda_{k,p}^0 - \lambda_{k,p}^\varepsilon \leq O(\varepsilon, \delta)$. Gathering both inequalities and letting $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ it is obtained the desired result. \square

Proof of Theorem 1.2. The proof of the Neumann case follows with an analogous argument to that of Theorem 1.1 by considering the Rayleigh quotients related to $\Lambda_{k,p}^0$ and $\Lambda_{k,p}^\varepsilon$ and by applying Lemma 3.1. \square

When periodicity assumptions are made on the weight functions, besides the convergence of the eigenvalues, estimates on the rates of the convergence are obtained. The proofs of Theorems 1.3 and 1.4 follow the ideas introduced by Oleinik et al. in [16], where the problem of obtaining rates on the eigenvalues is reduced to the study of the convergence rates of oscillating integrals. First we prove the Dirichlet case. Later, since the Neumann case involves estimates on the boundary of the domain, it will be necessary to assume some additional hypothesis; nevertheless the main idea in the proof still being the same.

The following inequality will be useful to prove our next lemma. We refer to [14] for a proof.

Lemma 3.2. For $p > 1$ and $x, y \in \mathbb{R}^n$, $x \neq y$,

$$|x|^p - |y|^p \leq p|x|^{p-2}x \cdot (x - y).$$

Lemma 3.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and denote by Q the unit cube in \mathbb{R}^n . Let $g \in L^\infty(\mathbb{R}^n)$ be a Q -periodic function such that $\bar{g} = 0$. Then the inequality

$$\left| \int_{\Omega} g\left(\frac{x}{\varepsilon}\right) |v|^p \right| \leq c\varepsilon^s [v]_{W^{s,p}(\mathbb{R}^n)} \|v\|_{L^p(\Omega)}^{p-1}$$

holds for every $v \in \mathcal{X}_0^{s,p}(\Omega)$ with $s \in (0, 1)$. The constant c depends only on Ω , n , p and the bounds of g .

Proof. Denote by I^ε the set of all $z \in \mathbb{Z}^n$ such that $Q_{z,\varepsilon} \cap \Omega \neq \emptyset$, $Q_{z,\varepsilon} := \varepsilon(z + Q)$. Given $v \in \mathcal{X}_0^{s,p}(\Omega)$ we consider the function \bar{v}_ε given by

$$\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{Q_{z,\varepsilon}} v(y) dy$$

for $x \in Q_{z,\varepsilon}$. We denote by $\Omega_1 = \cup_{z \in I^\varepsilon} Q_{z,\varepsilon} \supset \Omega$. Thus, we can write

$$\int_{\Omega} g_\varepsilon |v|^p = \int_{\Omega_1} g_\varepsilon (|v|^p - |\bar{v}_\varepsilon|^p) + \int_{\Omega_1} g_\varepsilon |\bar{v}_\varepsilon|^p,$$

and we can bound the previous expression as

$$\int_{\Omega} g_{\varepsilon} |v|^p \leq g_+ \int_{\Omega_1} |v|^p - |\bar{v}_{\varepsilon}|^p + \left| \int_{\Omega_1} g_{\varepsilon} |\bar{v}_{\varepsilon}|^p \right|. \tag{3.4}$$

The first integral in (3.4) can be split as

$$\int_{\Omega_1} |v|^p - |\bar{v}_{\varepsilon}|^p = \int_{I_1} |v|^p - |\bar{v}_{\varepsilon}|^p + \int_{I_2} |\bar{v}_{\varepsilon}|^p - |v|^p \tag{3.5}$$

where $I_1 = \{x \in \Omega : |v|^p - |\bar{v}_{\varepsilon}|^p \geq 0\}$ and $I_2 = \{x \in \Omega : |v|^p - |\bar{v}_{\varepsilon}|^p < 0\}$. Then, by using Lemma 3.2 we can bound (3.5) as

$$p \int_{\Omega_1} |v|^{p-1} |v - \bar{v}_{\varepsilon}| + p \int_{\Omega_1} |\bar{v}_{\varepsilon}|^{p-1} |v - \bar{v}_{\varepsilon}|. \tag{3.6}$$

To bound this expression, first we observe that by using Lemma 2.2 we have

$$\begin{aligned} \int_{\Omega_1} |v - \bar{v}_{\varepsilon}|^p &= \sum_{z \in I^{\varepsilon}} \int_{Q_{z,\varepsilon}} |v - \bar{v}_{\varepsilon}|^p dx \\ &\leq c^p \varepsilon^{sp} \sum_{z \in I^{\varepsilon}} [v]_{W^{s,p}(Q_{z,\varepsilon})}^p \\ &= c^p \varepsilon^{sp} \sum_{z \in I^{\varepsilon}} \int_{Q_{z,\varepsilon} \times Q_{z,\varepsilon}} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\leq c^p \varepsilon^{sp} \sum_{z \in I^{\varepsilon}} \sum_{\tilde{z} \in I^{\varepsilon}} \int_{Q_{z,\varepsilon} \times Q_{\tilde{z},\varepsilon}} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dx dy \\ &= c^p \varepsilon^{sp} [v]_{W^{s,p}(\Omega_1)}^p \\ &\leq c^p \varepsilon^{sp} [v]_{W^{s,p}(\mathbb{R}^n)}^p. \end{aligned} \tag{3.7}$$

Secondly, if $\frac{1}{p} + \frac{1}{p'} = 1$ we have

$$\int_{\Omega_1} |v|^{(p-1)p'} = \int_{\Omega_1} |v|^p. \tag{3.8}$$

Moreover, since $|\bar{v}_{\varepsilon}| \leq \int_Q |v| \leq C \left(\int_{\Omega_1} |v|^p \right)^{1/p}$ we obtain

$$\int_{\Omega_1} |\bar{v}_{\varepsilon}|^{(p-1)p'} \leq C \int_{\Omega_1} |v|^p. \tag{3.9}$$

Then, combining (3.7), (3.8) and (3.9) we can bound (3.6) as

$$\begin{aligned} &\left(\int_{\Omega_1} |v - \bar{v}_{\varepsilon}|^p \right)^{1/p} \left(\left(\int_{\Omega_1} |v|^{(p-1)p'} \right)^{1/p'} + \left(\int_{\Omega_1} |\bar{v}_{\varepsilon}|^{(p-1)p'} \right)^{1/p'} \right) \\ &\leq c \varepsilon^s [v]_{W^{s,p}(\mathbb{R}^n)} \|v\|_{L^p(\Omega)}^{p-1}. \end{aligned} \tag{3.10}$$

Finally, since $\bar{g} = 0$ and since g is Q -periodic, we obtain

$$\int_{\Omega_1} g_{\varepsilon} |\bar{v}_{\varepsilon}|^p = \sum_{z \in I^{\varepsilon}} |\bar{v}_{\varepsilon}|^p \int_{Q_{z,\varepsilon}} g_{\varepsilon} = 0. \tag{3.11}$$

Now, combining (3.10) and (3.11) we can bound (3.4) by

$$\left| \int_{\Omega} g_{\varepsilon} |v|^p \right| \leq C \varepsilon^s [v]_{W^{s,p}(\mathbb{R}^n)} \|v\|_{L^p(\Omega)}^{p-1},$$

and the proof is complete. □

Now we are ready to prove our main result.

Proof of Theorem 1.3. Let $\delta > 0$ and let $\mathcal{C}_{k,\delta} \subset \mathcal{X}_0^{s,p}(\Omega)$ be a set of index greater or equal then k such that

$$\lambda_{k,p}^0 = \inf_{C \in \mathcal{D}_k} \sup_{u \in C} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\bar{\rho} \int_{\Omega} |u|^p} = \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\bar{\rho} \int_{\Omega} |u|^p} + O(\delta)$$

where $\mathcal{D}_k = \{\mathcal{W} \in \mathcal{X}_0^{s,p}(\Omega) : i(\mathcal{W}) \geq k\}$. We use now the set $\mathcal{C}_{k,\delta}$, which is admissible in the variational characterization of the k -th eigenvalue of (1.1), in order to find a bound for it as follows,

$$\lambda_{k,p}^{\varepsilon} \leq \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} \rho_{\varepsilon} |u|^p} = \sup_{u \in \mathcal{C}_{k,\delta}} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\bar{\rho} \int_{\Omega} |u|^p} \frac{\bar{\rho} \int_{\Omega} |u|^p}{\int_{\Omega} \rho_{\varepsilon} |u|^p}. \quad (3.12)$$

To bound $\lambda_{k,p}^{\varepsilon}$ we look for bounds of the two quotients in (3.12). For every function $u \in \mathcal{C}_{k,\delta}$ we have that

$$\frac{[u]_{W^{s,p}(\mathbb{R}^n)}^p}{\bar{\rho} \int_{\Omega} |u|^p} \leq \sup_{v \in \mathcal{C}_{k,\delta}} \frac{[v]_{W^{s,p}(\mathbb{R}^n)}^p}{\bar{\rho} \int_{\Omega} |v|^p} = \lambda_{k,p}^0 + O(\delta). \quad (3.13)$$

Since $u \in \mathcal{C}_{k,\delta} \subset \mathcal{X}_0^{s,p}(\Omega)$, by Lemma 3.3 we obtain

$$\begin{aligned} \frac{\bar{\rho} \int_{\Omega} |u|^p}{\int_{\Omega} \rho_{\varepsilon} |u|^p} &\leq 1 + c\varepsilon^s \frac{[u]_{W^{s,p}(\mathbb{R}^n)} \|u\|_{L^p(\Omega)}^{p-1}}{\int_{\Omega} \rho_{\varepsilon} |u|^p} \\ &\leq 1 + c\varepsilon^s \frac{\rho_+}{\rho_-} \frac{[u]_{W^{s,p}(\mathbb{R}^n)} \|u\|_{L^p(\Omega)}^{p-1}}{\bar{\rho} \int_{\Omega} |u|^p} \\ &\leq 1 + c\varepsilon^s \frac{\rho_+}{\rho_-} \frac{[u]_{W^{s,p}(\mathbb{R}^n)}}{\bar{\rho} (\int_{\Omega} |u|^p)^{1/p}} \\ &\leq 1 + C\varepsilon^s (\lambda_{k,p}^0 + O(\delta))^{1/p}. \end{aligned} \quad (3.14)$$

Then, combining (3.12), (3.13) and (3.14) we find that

$$\lambda_{k,p}^{\varepsilon} \leq (\lambda_{k,p}^0 + O(\delta))(1 + C\varepsilon^s (\lambda_{k,p}^0)^{1/p}).$$

Letting $\delta \rightarrow 0$ we obtain

$$\lambda_{k,p}^{\varepsilon} - \lambda_{k,p}^0 \leq C\varepsilon^s (\lambda_{k,p}^0)^{1+\frac{1}{p}}. \quad (3.15)$$

In a similar way, interchanging the roles of $\lambda_{k,p}^0$ and $\lambda_{k,p}^{\varepsilon}$, we obtain

$$\lambda_{k,p}^0 - \lambda_{k,p}^{\varepsilon} \leq C\varepsilon^s (\lambda_{k,p}^{\varepsilon})^{1+\frac{1}{p}}. \quad (3.16)$$

Hence, from (3.15) and (3.16), we arrive at

$$|\lambda_{k,p}^{\varepsilon} - \lambda_{k,p}^0| \leq C\varepsilon^s \max\{\lambda_{k,p}^0, \lambda_{k,p}^{\varepsilon}\}^{1+\frac{1}{p}},$$

and by using the bounds given (2.5) and (2.9) the result follows. \square

The following Lemma is necessary to deal with the convergence rates of functions in $W^{s,p}(\Omega)$.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and, for $\delta > 0$, let G_δ be a tubular neighborhood of $\partial\Omega$, i.e. $G_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and every $v \in W^{s,p}(\Omega)$ we have*

$$\|u\|_{L^p(G_\delta)} \leq c\delta^{1/p} \|u\|_{W^{s,p}(\Omega)}.$$

whenever $1/p < s < 1$.

Proof. Given G_δ , the sets ∂G_δ are uniformly smooth surfaces. By the trace's inequality stated in Proposition 2.3 and the continuous inclusion given in Lemma 2.1 we have

$$\begin{aligned} \|u\|_{L^p(\partial G_\delta)} &\leq \|u\|_{W^{s-\frac{1}{p},p}(\partial G_\delta)} \\ &\leq c \|u\|_{W^{s,p}(G_\delta)} \\ &\leq c \|u\|_{W^{s,p}(\Omega)}, \quad \delta \in (0, \delta_0) \end{aligned}$$

provided that $1/p < s < 1$, where c is a constant independent on δ and u . Integrating this inequality with respect to δ we obtain

$$\|u\|_{L^p(G_\delta)}^p = \int_0^\delta \left(\int_{\partial G_\tau} |u|^p dS \right) d\tau \leq c\delta \|u\|_{W^{s,p}(\Omega)}^p$$

and the result is proved. □

The proof of the next Lemma follows with a slight modification to that of Lemma 3.3, and it is essential for handling the convergence rates of eigenvalues of the Neumann problem (1.5).

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and denote by Q the unit cube in \mathbb{R}^n . Let $g \in L^\infty(\mathbb{R}^n)$ be a Q -periodic function such that $\bar{g} = 0$. Then the inequality*

$$\left| \int_{\Omega} g\left(\frac{x}{\varepsilon}\right) |v|^p \right| \leq c\varepsilon^s \|v\|_{W^{s,p}(\Omega)}^p$$

holds for every $v \in W^{s,p}(\Omega)$ with $1/p < s < 1$. The constant c depends only on Ω , p , n and the bounds of g .

Proof. The proof is quite similar to that of Lemma 3.3, however there are some details to have into account. Denote by I^ε the set of all $z \in \mathbb{Z}^n$ such that $Q_{z,\varepsilon} \cap \Omega \neq \emptyset$ and $Q_{z,\varepsilon}$ is completely contained in Ω , being $Q_{z,\varepsilon} := \varepsilon(z + Q)$. Given $v \in W^{s,p}(\Omega)$ we consider the function \bar{v}_ε given by

$$\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{Q_{z,\varepsilon}} v(y) dy$$

for $x \in Q_{z,\varepsilon}$. We denote by $\Omega_1 = \cup_{z \in I^\varepsilon} Q_{z,\varepsilon}$. Thus, we can write

$$\int_{\Omega} g_\varepsilon |v|^p = \int_G g_\varepsilon |v|^p + \int_{\Omega_1} g_\varepsilon (|v|^p - |\bar{v}_\varepsilon|^p) + \int_{\Omega_1} g_\varepsilon |\bar{v}_\varepsilon|^p. \tag{3.17}$$

where $G = \Omega \setminus \bar{\Omega}_1$.

As in the Dirichlet Lemma we have

$$\int_{\Omega_1} g_\varepsilon (|v|^p - |\bar{v}_\varepsilon|^p) + \int_{\Omega_1} g_\varepsilon |\bar{v}_\varepsilon|^p \leq c\varepsilon^s [v]_{W^{s,p}(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}. \tag{3.18}$$

The set G is contained in a δ -neighborhood of $\partial\Omega$ with $\delta = c\varepsilon$ for some constant c , and therefore, according to Lemma 3.4 we have

$$\int_G g_\varepsilon |v|^p \leq c\varepsilon \|v\|_{W^{s,p}(\Omega)}^p. \quad (3.19)$$

Since ε and s are less than 1, gathering (3.17), (3.18) and (3.19) we obtain

$$\begin{aligned} \left| \int_\Omega g_\varepsilon |v|^p \right| &\leq C\varepsilon^s [v]_{W^{s,p}(\Omega)} \|u\|_{L^p(\Omega)}^{p-1} + C\varepsilon \|v\|_{W^{s,p}(\Omega)}^p \\ &\leq C\varepsilon^s \|v\|_{W^{s,p}(\Omega)}^p \end{aligned}$$

and the proof is complete. \square

Since Lemma 3.5 has been proved, the proof of Theorem 1.4 is analogous to that of Theorem 1.3, by using Lemma 3.5 instead of Lemma 3.3 with the bound in (2.7).

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