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COMPLEX OSCILLATIONS OF NON-DEFINITE STURM-LIOUVILLE PROBLEMS

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Abstract. We expand upon the basic oscillation theory for general boundary problems of the form

$$-y'' + q(t)y = \lambda r(t)y, \quad t \in I = [a, b]$$

where q and r are real-valued piecewise continuous functions and y is required to satisfy a pair of homogeneous separated boundary conditions at the endpoints. The *non-definite case* is characterized by the indefiniteness of each of the quadratic forms

$$B + \int_a^b (|y'|^2 + q|y|^2) \quad \text{and} \quad \int_a^b r|y|^2,$$

over a suitable space where B is a boundary term. In 1918 Richardson proved that, in the case of the Dirichlet problem, if $r(t)$ changes its sign exactly once and the boundary problem is non-definite then the zeros of the real and imaginary parts of any non-real eigenfunction interlace. We show that, unfortunately, this result is false in the case of two turning points, thus removing any hope for a general separation theorem for the zeros of the non-real eigenfunctions. Furthermore, we show that when a non-real eigenfunction vanishes inside I, the absolute value of the difference between the total number of zeros of its real and imaginary parts is exactly 2.

1. Introduction

We are concerned here with Sturm-Liouville problems of the form

$$-y'' + q(t)y = \lambda r(t)y \tag{1.1}$$

where $-\infty < a \le t \le b < \infty$ and y satisfies the boundary conditions

$$y(a)\cos\alpha - y'(a)\sin\alpha = 0, \tag{1.2}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \tag{1.3}$$

$0 \le \alpha, \beta < \pi$, the potential function $q(t)$ and the weight function $r(t)$ are real-valued in general. The value of the parameter $\lambda \in \mathbb{C}$ for which there exists a solution $y(t, \lambda)$ which is non-identically zero on $[a, b]$ is called an *eigenvalue* of problem (1.1)-(1.2)-(1.3), and the corresponding function $y(t, \lambda)$ is called an *eigenfunction* of the problem. The set consisting of all the eigenvalues of the problem is called

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the *spectrum* of (1.1)-(1.2)-(1.3). For the sake of simplicity we assume occasionally that q, r are both continuous or both piecewise continuous on $[a, b]$.

As alluded to in [3], the classical Sturm-Liouville oscillation theory of problems of the form (1.1)-(1.3) is concerned with the position and the number of zeros of solutions. The position of such zeros varies when the functions q, r, and/or the parameter λ are changed. In particular, the weight function r plays a critical role in the form and nature of these results. For example, in [7] the authors show that the oscillation of the weight function can drive away the real eigenvalues from the real line into the complex plane. It is also known [2] that for a fixed weight function, an increase in the number of negative squares of

$$B + \int_a^b (|y'|^2 + q|y|^2)$$

by varying q, can lead to an increase in the number of non-real eigenvalues. For an historical overview of this subject until 1986, see [6].

It is clear that, in the non-definite case, the weight function r must take on both signs in the interval (a, b), [8]. A point about which the weight function $r(t)$ actually changes its sign in the interval (a, b) is called a **turning point** of r.

We now focus on the the Dirichlet problem (i.e., $\alpha = \beta = 0$ in (1.2)-(1.3)).

Theorem 1.1 ([5, Theorem 3]). *Let λ and $y(t, \lambda)$ be a non-real eigenvalue and associated non-real eigenfunction of the problem* (1.1). *If $r(t)$ has precisely n turning points in (a, b) then $y(t, \lambda)$ may vanish at most $(n - 1)$ times in (a, b).*

Corollary 1.2 ([5, Corollary 1]). *Let λ and $y(t, \lambda)$ be a non-real eigenvalue and associated non-real eigenfunction of the problem* (1.1)-(1.3). *If $r(t)$ has exactly one turning point in (a, b) then $y(t, \lambda) \neq 0$ in (a, b).*

Below we show that the previous conclusion fails in the case of more than one turning point.

Theorem 1.3 (See [8, 5, 6]). *Let r be continuous and not vanish identically in any right neighborhood of $t = a$. If $r(t)$ changes its sign precisely once in (a, b) then the roots of the real and imaginary parts ψ and φ of any non-real eigenfunction $y = \psi + i\varphi$ corresponding to a non-real eigenvalue, separate one another (or interlace).*

Thus, Theorem 1.1 implies that if the weight function has one turning point (i.e., $n = 1$) then no non-real eigenfunction can have a zero in (a, b). Of course, Richardson's separation theorem, Theorem 1.3 above, also gives the same conclusion. In the case where the weight function $r(t)$ has exactly two turning points, numerical results in [4] indicate that some non-real eigenfunctions can vanish once in (a, b), in agreement with Theorem 1.1.

In the sequel we present basic results in the non-definite case of Sturm-Liouville problems and give necessary examples in some cases. In Section 2 we present a non-definite Sturm-Liouville problem in which the weight function has more than one turning point in (a, b) which then violates Richardson's separation theorem, Theorem 1.3. This shows that we cannot easily generalize the separation theorem to the case of more than one turning point. In Section 3, we consider the case in which the weight function $r(t)$ has two turning points in (a, b) with the assumption that $r(t)$ does not vanish identically on a subinterval of (a, b). We prove that the absolute value of the difference between the total number of zeros of the real and imaginary parts of a given non-real eigenfunction (corresponding to a non-real eigenvalue) of a problem of the form (1.1)-(1.3) is equal to 2. The main stimulus for the work covered in Section 3 arose out of the numerical results presented in the paper [4].

2. Failure of the interlacing property

In this section we show that Richardson's separation theorem, Theorem 1.3, fails for a weight function having more than one turning point. We do this by exhibiting a non-definite Sturm-Liouville problem whose weight function has more than one turning point in (a, b) having a non-real eigenfunction that vanishes there.

Let $\lambda = \sigma + i\tau$, $y(t)$ be some non-real eigenvalue-eigenfunction pair of the *complex* coefficient Sturm-Liouville equation

$$-y'' + \exp(it)y = \lambda y, \tag{2.1}$$

satisfying the boundary conditions

$$y(a) = y(b) = 0. \tag{2.2}$$

The existence of such eigenvalues is due to Hilb, see [6].

Next, let $\kappa = \mu + i\eta$, z, be a non-real eigenvalue-eigenfunction pair of the problem

$$-z'' + \exp(it)z = \kappa z, \tag{2.3}$$

satisfying the new set of boundary conditions, namely,

$$z(b) = z(2b) = 0, \quad z'(b) = y'(b), \tag{2.4}$$

where y already satisfies (2.1)-(2.2) (and, of course, $y'(b) \neq 0$). Separating real and imaginary parts in (2.1) and in (2.3) we get

$$-y'' + (\cos t - \sigma)y = i\,(\tau - \sin t)y, \quad y(a) = y(b) = 0, \tag{2.5}$$

$$-z'' + (\cos t - \mu)z = i\,(\eta - \sin t)z, \quad z(b) = z(2b) = 0, \tag{2.6}$$

with z being normalized by setting $z'(b) = y'(b)$.

Now, on the interval $[a, 2b]$, consider the equation

$$-W'' + \Big(\cos t - \rho(t)\Big)W = i\Big(r(t) - \sin t\Big)W, \tag{2.7}$$

where,

$$\rho(t) = \begin{cases} \sigma, & \text{if } t \in (a, b) \\ \mu, & \text{if } t \in (b, 2b), \end{cases}$$

$$r(t) = \begin{cases} \tau, & \text{if } t \in (a, b) \\ \eta, & \text{if } t \in (b, 2b). \end{cases}$$

Then ρ, r are real piecewise continuous functions on $[a, 2b]$. In addition, we know that $\tau - \sin t$ must change its sign at least once in (a, b) since i is a non-real eigenvalue in (2.5). So, the function $r(t)$ changes its sign on (a, b) on account of (2.5) and then again on $(b, 2b)$ on account of (2.6). Now the function

$$W(t) = \begin{cases} y(t), & \text{if } t \in [a, b] \\ z(t), & \text{if } t \in [b, 2b] \end{cases}$$

satisfies the boundary conditions

$$W(a) = W(2b) = 0. \tag{2.8}$$

Claim: W is an eigenfunction of the Sturm-Liouville problem (2.7)-(2.8) having the complex eigenvalue, i. In addition, $W(b) = 0$.

Proof. That $W(b) = 0$ is clear from the definition. Clearly, $W'(t)$ exists in the two intervals (a, b) and $(b, 2b)$. We show that $W'(b)$ exists. Let $W'_{\pm}(b)$ be the right/left derivatives of W at $t = b$. By the definition of $W(t)$, we know that $W'_{+}(b) = z'(b) = y'(b) = W'_{-}(b)$, so we conclude that $W'(b)$ exists, and that the function $W(t)$ is thus continuously differentiable on the interval $(a, 2b)$. Finally, W' is an absolutely continuous function on each of (a, b) and $(b, 2b)$ since y', z' have this property. It follows that, in fact, W' is itself absolutely continuous on $[a, 2b]$, and therefore W is an eigenfunction of (2.7)-(2.8), corresponding to the non-real eigenvalue i that vanishes at an interior point (i.e, $t = b$). □

We have therefore proved the following theorem.

Theorem 2.1. *There exists a regular non-definite Sturm-Liouville problem on a finite interval I having a non-real eigenfunction $y(t, \lambda)$, corresponding to a non-real eigenvalue λ, such that $y(t, \lambda) = 0$ for some t in the interior of I.*

Of course, this shows that the expected interlacing property of the zeros of the real and imaginary parts of a non-real eigenfunction cannot hold, in general.

3. Zeros of real and imaginary parts of non-real eigenfunctions in the two-turning point case

When the weight function $r(t)$ has two turning points, a non-real eigenfunction may vanish at most once in the interval (a, b), by Theorem 1.1. If $r(t)$ has two turning points, then the two turning points divide the interval (a, b) into three subintervals and if a non-real eigenfunction vanishes once in the interval (a, b), it will vanish in the middle interval. Moreover, we establish the difference between the number of zeros of the real and imaginary parts of a non-real eigenfunction corresponding to a non-real eigenvalue of a problem of the form (1.1)-(1.3).

Theorem 3.1. *Let $q, r \in C[a, b]$ and assume that the weight function r has precisely two turning points in the interval (a, b), and that it does not vanish identically in any subinterval of (a, b). Let λ be a non-real eigenvalue of problem*

$$-y'' + q(t)y = \lambda r(t)y, \tag{3.1}$$

$$y(a) = y(b) = 0 \tag{3.2}$$

and let $y(t, \lambda)$ be a corresponding, necessarily non-real, eigenfunction having exactly one zero in (a, b). Then the absolute value of the difference between the total number of zeros of the real and imaginary part of y in (a, b) is two.

Proof. Since $\lambda \notin \mathbb{R}$, classical arguments imply that

$$\int_a^b r|y|^2 dt = 0. \tag{3.3}$$

We define a function f by $f(t) = \int_a^t r\,|y|^2\,dx$, (see [5]). It then follows that

$$f(a) = f(b) = 0. \tag{3.4}$$

By hypothesis there exists $k \in (a, b)$ such that $y(k) = 0$. Hence, y is a non-real eigenfunction of (3.1) on the interval $[a, k]$ satisfying $y(a) = 0 = y(k)$ and, as a result, $f(a) = 0 = f(k)$ must hold. Similarly, $y(k) = 0 = y(b)$ forces $f(k) = f(b) = 0$. Let c_1 and c_2 be the two turning points of r with $a < c_1 < c_2 < b$.

We now claim that $k \in (c_1, c_2)$. Without loss of generality, let us assume that $k \in (a, c_1)$. Then there is a number $t_1 \in (a, c_1)$ such that $0 = f'(t_1) = r(t_1)|y(t_1)|^2$. This means that $y(t_1) = 0$, since $r \neq 0$ in (a, c_1). Hence t_1 is another zero of y, contradicting the assumption that k is the only zero of y in (a, b). The same argument holds if $k \in (c_2, b)$. So, this case is impossible.

In the second case, without loss of generality, we assume that $k = c_1$. Then the open interval (a, c_1) is turning point free, by hypothesis. This means that $r(t) \neq 0$ for any $t \in (a, c_1)$. But then the Dirichlet problem for (3.1) must be definite on $[a, c_1]$. Classical Sturm-Liouville Theory now implies that all its Dirichlet eigenvalues on $[a, c_1]$ must be real. But this is impossible as we started with a non-real eigenvalue! A similar argument applies in the case where $k = c_2$. Thus, this case cannot occur. Since the first two cases are impossible, it must be the case that $k \in (c_1, c_2)$, as stated.

Now, k divides the interval (a, b) into two intervals (a, k) and (k, b) on each of which $r(t)$ has one turning point. So, in particular, our non-real eigenfunction y satisfies a Dirichlet problem for (3.1) on the interval $[a, k]$, where r has one turning point (namely, c_1) in (a, k). By Richardson's Theorem 1.3, the zeros of the real and imaginary parts of y must interlace in the interval (a, k). Similarly, the same argument applied to $[k, b]$ yields that the zeros of the real and imaginary parts of y must interlace in the interval (k, b). This means that the zeros of the real and imaginary parts of y interlace on *almost* the whole interval (a, b) except near, and at, the only zero of $y(t)$, i.e., where $t = k$.

We write $y(t) = \varphi(t) + i\,\psi(t)$. If $\varphi(t)$ has n zeros in (a, k), then $\psi(t)$ has $n-1$ zeros in (a, k), since the zeros interlace in (a, k). Similarly, if $\varphi(t)$ has m zeros in (k, b), then $\psi(t)$ has $m-1$ zeros in (k, b). Recall that both $\varphi(k) = \psi(k) = 0$ by hypothesis. Adding the total number of zeros we find that φ has $n+m+1$ zeros in (a, b) while $\psi(t)$ must have $n+m-1$ zeros in (a, b). The difference in the number of zeros being equal to two, the proof is complete. □

4. Conclusion

We have proved two main results that further develop the work that Richardson started some 100 years ago. First, we show that Richardson's separation theorem (1918) for the zeros of the real and imaginary part of a non-real eigenfunction (corresponding to a non-real eigenvalue) of a non-definite Sturm-Liouville Dirichlet problem in the case of one turning point is false, in general, by exhibiting a counterexample in the case of two turning points. The counterexample shows that a complex eigenfunction can actually vanish in the interior of the interval of definition!

Then we show that if a non-real eigenfunction (corresponding to a non-real eigenvalue) of a non-definite Sturm-Liouville Dirichlet problem in the case of two turning points vanishes in the interior of the interval under consideration then the absolute value of the difference between the total number of zeros of the real and imaginary parts of this eigenfunction must be equal to 2.

Many questions in this area remain unanswered. For instance, one observation on the spectrum of a non-definite Sturm-Liouville problem is that if the problem

$$-y'' + q(t)y = \lambda r(t)y, \tag{4.1}$$

$$y(a) = y(b) = 0 \tag{4.2}$$

has a non-real eigenvalue, $c + id$, $d \neq 0$ and a real eigenvalue, say γ, then $c \neq \gamma$. In other words we claim that, in the non-definite case, there cannot exist a non-real eigenvalue whose real part is also an eigenvalue. Whether this is an accident or a result of a more general yet unproven theorem, is unknown, but we conjecture that it is so and leave this for future research.

Furthermore, there is a need to prove general results on the behaviour of the real and imaginary parts of non-real eigenfunctions in the case where the weight function has a finite number of turning points. For further open questions on the non-real spectrum of non-definite problems see the monograph, [1].

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