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# EXISTENCE OF HOMOCLINIC ORBITS FOR A CLASS OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. By using critical point theory, we prove the existence of a non-trivial homoclinic orbit for a class of nonlinear functional difference equations. Our conditions on the nonlinear term do not need to satisfy the well-known global Ambrosetti-Rabinowitz superquadratic condition.

## 1. Introduction

The article concerns the nonlinear functional difference equation

$$\Delta(p_n(\Delta u_{n-1})^\delta) - q_n u_n^\delta + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta$ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\delta$ is the ratio of odd positive integers, $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ are real sequences, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$, $T$ is a positive integer, $p_{n+T} = p_n$, $q_{n+T} = q_n$, and $f(n + T, v_1, v_2, v_3) = f(n, v_1, v_2, v_3)$.

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the natural numbers, integers and real numbers respectively. For $a$, $b \in \mathbb{Z}$, we define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \le b$. In this article we use the following assumptions:

- (A1) $p_n > 0$ for $n \in \mathbb{Z}$;
- (A2) $q_n > 0$ for $n \in \mathbb{Z}$;
- (A3) there exists a functional $F(n, v_1, v_2) \in C^1(\mathbb{Z}\times\mathbb{R}^2, \mathbb{R})$ with $F(n+T, v_1, v_2) = F(n, v_1, v_2)$ and satisfies
$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3);$$
- (A4) $\lim_{\varrho\to 0} f(n, v_1, v_2, v_3)/v_2^\delta = 0$ for $n \in \mathbb{Z}$, $\varrho = (v_1^{\delta+1} + v_2^{\delta+1} + v_3^{\delta+1})^{\frac{1}{\delta+1}}$;
- (A5) $\lim_{\sigma\to 0} F(n, v_1, v_2)/\sigma^{\delta+1} = 0$ for $n \in \mathbb{Z}$, $\sigma = (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}}$.

In general, (1.1) can be considered as a discrete analogue of the second order nonlinear functional differential equation

$$(p(t)\varphi(u'))' + q(t)u(t) + f(t, u(t+1), u(t), u(t-1)) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

This equation includes the equation

$$(p(t)\varphi(u'))' + f(t, u(t)) = 0, \quad t \in \mathbb{R},$$

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which has arose in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor [5, 12]. Equations similar in structure to (1.2) arise in the study of periodic solutions and homoclinic orbits of functional differential equations [13, 14].

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, queuing theory, ecology, cybernetics, biological systems, optimal control, population dynamics, etc. Since the past twenty years, there has been much progress on the qualitative properties of difference equations, which included results on homoclinic orbits, periodic solutions, boundary value problems, stability, attractivity, oscillation and other topics, see for example [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 27, 28, 29, 30, 31] and the references therein.

In 2004, Zhang, Wang and Yu [31] obtained necessary and sufficient conditions for the existence of strictly monotone increasing positive solutions of the following equation

$$\Delta(p_n(\Delta u_{n-1})^{\delta}) + q_n u_n^{\delta} = 0. \tag{1.3}$$

If $f(n, u_{n+1}, u_n, u_{n-1}) = f(n, u_n)$, Cai and Yu [4] considered the nonlinear difference equation of the type

$$\Delta(p_n(\Delta u_{n-1})^{\delta}) + q_n u_n^{\delta} = f(n, u_n), \quad n \in \mathbb{Z}, \tag{1.4}$$

using the critical point theory, and they obtained some new results on the existence of periodic solutions.

If $q_n \equiv 1$, Liu, Zhang and Shi in 2015 [18] and 2016 [27] respectively studied a class of nonlinear difference equation

$$\Delta(p_n(\Delta u_{n-1})^{\delta}) + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \quad n \in \mathbb{Z}, \tag{1.5}$$

has at least three $T$-periodic solutions.

By using the Symmetric Mountain Pass Theorem, Chen and Wang [7] established some existence criteria to guarantee a class of nonlinear difference equation

$$\Delta(p_n(\Delta u_{n-1})^{\delta}) - q_n u_n^{\delta} + f(n, u_n) = 0, \quad n \in \mathbb{Z}, \tag{1.6}$$

has infinitely many homoclinic orbits. Shi, Liu and Zhang [25] obtained the existence of a nontrivial homoclinic orbit for (1.1) based on the Mountain Pass Lemma in combination with periodic approximations.

In the superquadratic case, almost all the results in the literature (see e.g. [4, 9, 19, 22]) need the well-known global Ambrosetti-Rabinowitz superquadratic condition:

- There exists a constant $\beta > 2$ such that $0 < \beta F(n, u) \le u f(n, u)$ for all $n \in \mathbb{Z}$ and $u \in \mathbb{R} \setminus \{0\}$.

In this article, we introduce the following conditions that are weaker than the superquadratic condition

(A6) $\lim_{\sigma\to 0} F(n, v_1, v_2)/\sigma^{\delta+1} = \infty$ for $n \in \mathbb{Z}$, $\sigma = (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}}$;

(A7) for any $n \in \mathbb{Z}$, $F(n, 0, 0) = 0$, $F(n, v_1, v_2) \ge F(n, v_2) \ge 0$;

(A8) for any $\varrho > 0$, there exist $a = a_\varrho > 0$, $b = b_\varrho > 0$ and $\nu < \delta + 1$ such that for all $n \in \mathbb{Z}$, $(v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}} > \varrho$,

$$\Big[\delta + 1 + \frac{1}{a + b(v_1^{\delta+1} + v_2^{\delta+1})^{\frac{\nu}{\delta+1}}}\Big] F(n, v_1, v_2) \le \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2.$$

Our main results read as follows.

**Theorem 1.1.** *Suppose that* (A1)–(A8) *are satisfied. Then* (1.1) *possesses a nontrivial homoclinic orbit.*

**Theorem 1.2.** *Suppose that* (A1)–(A5) *and the following assumption are satisfied:*

(A9) $F(n, v_1, v_2) \geq 0$ *and there exists a constant* $\beta > 2$ *such that*
$$0 < \beta F(n, v_1, v_2) \leq \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2,$$
*for all* $(n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \setminus \{(0, 0)\}$.

*Then* (1.1) *possesses a nontrivial homoclinic orbit.*

For basic knowledge of variational methods, the reader is referred to [23, 24].

## 2. Variational structure and some lemmas

To apply critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notation.

Let $S$ be the set of sequences $u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty}$; that is,
$$S = \{\{u_n\} : u_n \in \mathbb{R},\, n \in \mathbb{Z}\}.$$
For $u, v \in S$, $a, b \in \mathbb{R}$, we define
$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$
Then $S$ is a vector space. Define
$$E = \big\{u \in S : \sum_{n=-\infty}^{+\infty} [p_n (\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}] < +\infty\big\},$$
and for $u \in E$,
$$\|u\| = \Big\{ \sum_{n=-\infty}^{+\infty} [p_n (\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}] \Big\}^{\frac{1}{\delta+1}}, \quad \forall u \in E. \tag{2.1}$$
Then $E$ is a uniform convex Banach space with this norm.

As usual, for $1 < s < +\infty$, we set
$$l^s = \big\{u \in S : \sum_{n=-\infty}^{+\infty} |u_n|^s < +\infty\big\}, \quad l^\infty = \{u \in S : \sup_{n \in \mathbb{Z}} |u_n| < +\infty\},$$
with their respective norms
$$\|u\|_s = \Big( \sum_{n=-\infty}^{+\infty} |u_n|^s \Big)^{1/s}, \quad \forall u \in l^s,$$
$$\|u\|_\infty = \sup_{n \in \mathbb{Z}} |u_n|, \quad \forall u \in l^\infty .$$

For $u \in E$, we define the functional
$$J(u) := \frac{1}{\delta+1} \sum_{n=-\infty}^{+\infty} [p_n (\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}] - \sum_{n=-\infty}^{+\infty} F(n, u_{n+1}, u_n). \tag{2.2}$$

If (A1)–(A3) hold, then $J \in C^1(E,\mathbb{R})$ and one can easily check that

$$\begin{aligned} \langle J'(u), v\rangle = &\sum_{n=-\infty}^{+\infty} [p_n(\Delta u_{n-1})^\delta \Delta v_{n-1} + q_n u_n^\delta v_n] \\ &- \sum_{n=-\infty}^{+\infty} f(n, u_{n+1}, u_n, u_{n-1}) v_n, \quad \forall u, v \in E. \end{aligned} \tag{2.3}$$

Thus, we can compute the partial derivative as

$$\frac{\partial J(u)}{\partial u_n} = -\Delta(p_n(\Delta u_{n-1})^\delta) + q_n u_n^\delta - f(n, u_{n+1}, u_n, u_{n-1}), \quad \forall n \in \mathbb{Z}. \tag{2.4}$$

So, the critical points of $J$ in $E$ are the solutions of (1.1) with $u_n \to 0$ as $|n| \to \infty$.

**Lemma 2.1** ([23])**.** *Let $E$ be a real Banach space with its dual space $E^*$ and assume that $J \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} J(u),$$

*for some $\eta_0 < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)),$$

*where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining $0$ to e; then there exists $\{u^{(k)}\}_{k\in\mathbb{N}} \subset E$ such that $J(u^{(k)}) \to c$ and $(1 + \|u^{(k)}\|)\|J'(u^{(k)})\|_{E^*} \to 0$ as $k \to \infty$.*

**Lemma 2.2.** *For $u \in E$ and $s > 1$,*

$$\underline{q}\|u\|_\infty^{\delta+1} \le \underline{q}\|u\|_s^{\delta+1} \le \|u\|^{\delta+1}, \tag{2.5}$$

*where $\underline{q} = \inf_{n\in\mathbb{Z}} q_n$.*

The proof of above lemma is routine; so we omit it.

**Lemma 2.3.** *Suppose that (A1)–(A8) are satisfied. Then there exists a constant $c > 0$ and a sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ satisfying*

$$J(u^{(k)}) \to c, \quad \|J'(u^{(k)})\|(1 + \|u^{(k)}\|) \to 0, \quad k \to \infty. \tag{2.6}$$

*Proof.* It follows from (A5) that there exists a constant $\rho > 0$ such that

$$F(n, v_1, v_2) \le \frac{\underline{q}}{4(\delta+1)}(v_1^{\delta+1} + v_2^{\delta+1}), \quad \forall n \in \mathbb{Z},\ (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}} \le \rho. \tag{2.7}$$

Let $\|u\| = \underline{q}^{\frac{1}{\delta+1}}\rho := \eta$, combining this and (2.5), we have $|u_n| \le \rho$ for all $n \in \mathbb{Z}$. Therefore, by (2.2) and (2.7), we have

$$\begin{aligned} J(u) &= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n=-\infty}^{+\infty} F(n, u_{n+1}, u_n) \\ &\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{\underline{q}}{4(\delta+1)} \sum_{n=-\infty}^{+\infty} (u_{n+1}^{\delta+1} + u_n^{\delta+1}) \\ &\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{\underline{q}}{2(\delta+1)}\|u\|_{\delta+1}^{\delta+1}, \\ &\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\eta^{\delta+1} \end{aligned}$$

$$= \frac{1}{2(\delta+1)}\eta^{\delta+1}, \quad \forall u \in E, \ \|u\| = \rho.$$

Choose $u^{(0)} \in E$ such that

$$u_0^{(0)} = 1, \quad u_n^{(0)} = 0, \quad \forall n \neq 0.$$

Then, for $\lambda > 0$ large enough, it follows from (A3)–(A6) and (2.3) that

$$\begin{aligned} J(\lambda u^{(0)}) &= \frac{\lambda^{\delta+1}}{\delta+1}\|u^{(0)}\|^{\delta+1} - \sum_{n=-\infty}^{+\infty} F(n, \lambda u_{n+1}^{(0)}, \lambda u_n^{(0)}) \\ &\le \frac{\lambda^{\delta+1}}{\delta+1}\|u^{(0)}\|^{\delta+1} - F(0, \lambda u_{n+1}^{(0)}, \lambda u_n^{(0)}) \\ &\le \lambda^{\delta+1}\Big[\frac{1}{\delta+1}\|u^{(0)}\|^{\delta+1} - \frac{F(0, \lambda u_{n+1}^{(0)}, \lambda u_n^{(0)})}{|\lambda u_0^{(0)}|^{\delta+1}}\Big] \le 0. \end{aligned}$$

Consequently, we can choose $\lambda_1 > 1$ such that $\lambda_1\|u^{(0)}\| > \eta$ and $J(\lambda_1 u^{(0)}) \le 0$. Define $e = \lambda_1 u^{(0)}$, then $e \in E$, $\|e\| > \eta$ and $J(e) \le 0$. From Lemma 2.1, one has that there exists a constant $c \ge \frac{1}{2(\delta+1)}\eta^{\delta+1}$ and a sequence $\{u^{(k)}\}_{k\in\mathbb{N}} \subset E$ such that (2.6) holds. $\square$

**Lemma 2.4.** *Suppose that* (A1)–(A8) *are satisfied. Then any sequence* $\{u^{(k)}\}_{k\in\mathbb{N}}$ *satisfying*

$$J(u^{(k)}) \to c > 0, \quad \|J'(u^{(k)})\|(1 + \|u^{(k)}\|) \to 0, \quad k \to \infty \tag{2.8}$$

*is bounded in* $E$.

*Proof.* By (A5), we know that there exists a constant $0 < \rho < 1$ such that

$$F(n, v_1, v_2) \le \frac{q}{4(\delta+1)}(v_1^{\delta+1} + v_2^{\delta+1}), \quad \forall n \in \mathbb{Z}, \ (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}} \le \rho. \tag{2.9}$$

It follows from (A3) and (A8) that

$$f(n, v_1, v_2, v_3)v_2 > (\delta+1)F(n, v_1, v_2) \ge 0, \ \forall n \in \mathbb{Z} \tag{2.10}$$

and

$$F(n, v_1, v_2) \le [a + b(v_1^{\delta+1} + v_2^{\delta+1})^{\frac{\nu}{\delta+1}}][f(v_1, v_2, v_3)v_2 - (\delta+1)F(n, v_1, v_2)], \tag{2.11}$$

for all $n \in \mathbb{Z}$, $(v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}} > \rho$.

From (2.2), (2.3) and (2.8), there exist two constants $M_1$ and $M_2$ such that

$$\begin{aligned} M_1 &\ge (\delta+1)J(u^{(k)}) - \langle J'(u^{(k)}), u^{(k)}\rangle \\ &= \sum_{n=-\infty}^{+\infty} [f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})u_n^{(k)} - (\delta+1)F(n, u_{n+1}^{(k)}, u_n^{(k)})] \end{aligned} \tag{2.12}$$

and

$$J(u^{(k)}) \le M_2. \tag{2.13}$$

By (2.3), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13), we have

$$
\begin{aligned}
&\frac{1}{\delta+1}\|u^{(k)}\|^{\delta+1}\\
&= J(u^{(k)}) + \sum_{n=-\infty}^{+\infty} F(n, u_{n+1}^{(k)}, u_n^{(k)})\\
&= J(u^{(k)}) + \sum_{n\in\mathbb{Z}([(u_{n+1}^{(k)})^{\delta+1}+(u_n^{(k)})^{\delta+1}]^{\frac{1}{\delta+1}}\le\rho)} F(n, u_{n+1}^{(k)}, u_n^{(k)})\\
&\quad + \sum_{n\in\mathbb{Z}([(u_{n+1}^{(k)})^{\delta+1}+(u_n^{(k)})^{\delta+1}]^{\frac{1}{\delta+1}}>\rho)} F(n, u_{n+1}^{(k)}, u_n^{(k)})\\
&\le J(u^{(k)}) + \frac{\underline{q}}{4(\delta+1)} \sum_{n\in\mathbb{Z}([(u_{n+1}^{(k)})^{\delta+1}+(u_n^{(k)})^{\delta+1}]^{\frac{1}{\delta+1}}>\rho)} \big[(u_{n+1}^{(k)})^{\delta+1} + (u_n^{(k)})^{\delta+1}\big]\\
&\quad + \sum_{n\in\mathbb{Z}([(u_{n+1}^{(k)})^{\delta+1}+(u_n^{(k)})^{\delta+1}]^{\frac{1}{\delta+1}}>\rho)} \big\{a + b[(u_{n+1}^{(k)})^{\delta+1} + (u_n^{(k)})^{\delta+1}]^{\frac{\nu}{\delta+1}}\big\}\\
&\quad \times [f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})u_n^{(k)} - (\delta+1)F(n, u_{n+1}^{(k)}, u_n^{(k)})]\\
&\le M_2 + \frac{1}{4}\|u^{(k)}\|^{\delta+1} + \sum_{n\in\mathbb{Z}} \big\{a + b[(u_{n+1}^{(k)})^{\delta+1} + (u_n^{(k)})^{\delta+1}]^{\frac{\nu}{\delta+1}}\big\}\\
&\quad \times \big[f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})u_n^{(k)} - (\delta+1)F(n, u_{n+1}^{(k)}, u_n^{(k)})\big]\\
&\le M_2 + \frac{1}{4}\|u^{(k)}\|^{\delta+1} + [a + 2b\|u^{(k)}\|_\infty^{\nu}]\\
&\times \big[f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})u_n^{(k)} - (\delta+1)F(n, u_{n+1}^{(k)}, u_n^{(k)})\big]\\
&\le M_2 + \frac{1}{4}\|u^{(k)}\|^{\delta+1} + M_1[a + 2b\|u^{(k)}\|_\infty^{\nu}]\\
&\le M_2 + \frac{1}{4}\|u^{(k)}\|^{\delta+1} + M_1[a + 2\underline{q}^{-\frac{\nu}{\delta+1}} b\|u^{(k)}\|^{\nu}],\ k\in\mathbb{N}.
\end{aligned}
\tag{2.14}
$$

Since $\nu < \delta+1$, by (2.14), we have that $\{u^{(k)}\}_{k\in\mathbb{N}}$ is bounded. Hence, the proof is complete. $\square$

## 3. Proof of main results

*Proof of Theorem 1.1.* From Lemma 2.3 there exists a sequence $\{u^{(k)}\}_{k\in\mathbb{N}} \subset E$ satisfying (2.6), and so (2.8). Hence, from Lemma 2.4, we have that $\{u^{(k)}\}_{k\in\mathbb{N}}$ is bounded in $E$. It follows from (2.5) that there exists a constant $M_3 > 0$ such that

$$
\underline{q}^{\frac{1}{\delta+1}}\|u^{(k)}\|_\infty \le \|u^{(k)}\| \le M_3, \forall n\in\mathbb{N}. \tag{3.1}
$$

From (A3)–(A5) we have

$$
\begin{gathered}
\big|\frac{1}{\delta+1} f(n, v_1, v_2, v_3)v_2 - F(n, v_1, v_2)\big| \le \frac{c\underline{q}}{4M_3^{\delta+1}} v_2^{\delta+1} + \frac{c\underline{q}}{8M_3^{\delta+1}}(v_1^{\delta+1} + v_2^{\delta+1}),\\
\text{for all } n\in\mathbb{Z},\ (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}} \le \frac{1}{\underline{q}^{\frac{1}{\delta+1}}} M_3.
\end{gathered}
\tag{3.2}
$$

If $\xi := \limsup_{k\to\infty} \|u^{(k)}\|_\infty = 0$. Then from (A4), (2.2), (2.3) and (3.2), one has

$$
\begin{aligned}
c &= J(u^{(k)}) - \frac{1}{\delta+1}\big\langle J'(u^{(k)}), u^{(k)}\big\rangle + o(1)\\
&= \frac{1}{\delta+1}\sum_{n=-\infty}^{+\infty} f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})u_n^{(k)} - \sum_{n=-\infty}^{+\infty} F\big(n, u_{n+1}^{(k)}, u_n^{(k)}\big) + o(1)\\
&\le \frac{c\underline{q}}{4M_3^{\delta+1}}\sum_{n=-\infty}^{+\infty}\big(u_n^{(k)}\big)^{\delta+1} + \frac{c\underline{q}}{8M_3^{\delta+1}}\sum_{n=-\infty}^{+\infty}[(u_{n+1}^{(k)})^{\delta+1} + (u_n^{(k)})^{\delta+1}]\\
&\le \frac{c\underline{q}}{4M_3^{\delta+1}}\|u^{(k)}\|_{\delta+1}^{\delta+1} + \frac{c\underline{q}}{4M_3^{\delta+1}}\|u^{(k)}\|_{\delta+1}^{\delta+1} + o(1)\\
&\le \frac{c}{2} + o(1), \quad k\to\infty.
\end{aligned}
$$

This contradiction shows that $\xi > 0$.

First, going to a subsequence if necessary, we can assume that the existence of $n^{(k)} \in \mathbb{Z}$ independent of $k$ such that

$$
|u_{n^{(k)}}^{(k)}| = \|u^{(k)}\|_\infty > \frac{\xi}{2}. \tag{3.3}
$$

Hence, making such shifts, we can assume that $n^{(k)} \in \mathbb{Z}(0, T-1)$ in (3.3). Moreover, passing to a subsequence of $k$s, we can even assume that $n^{(k)} = n^{(0)}$ is independent of $k$.

Next, we extract a subsequence, still denoted by $u^{(k)}$, such that

$$
u_n^{(k)} \to u_n, \quad k\to\infty,\ \forall n\in\mathbb{Z}.
$$

Inequality (3.3) implies that $|u_{n^{(0)}}| \ge \xi$ and, hence, $u = \{u_n\}$ is a nonzero sequence. Moreover,

$$
\begin{aligned}
&\Delta(p_n(\Delta u_{n-1})^\delta) - q_n u_n^\delta + f(n, u_{n+1}, u_n, u_{n-1})\\
&= \lim_{k\to\infty}\Big[\Delta\Big(p_n\big(\Delta(u_{n-1}^{(k)})\big)^\delta\Big) - q_n(u_n^{(k)})^\delta + f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)})\Big]\\
&= \lim_{k\to\infty} 0 = 0.
\end{aligned}
$$

So $u = \{u_n\}$ is a solution of (1.1).

Finally, for any fixed $D\in\mathbb{Z}$ and $k$ large enough, we have

$$
\sum_{n=-D}^{D} |u_n^{(k)}|^{\delta+1} \le \frac{1}{\underline{q}}\|u^{(k)}\|^{\delta+1} \le M_3^{\delta+1}.
$$

Since $M_3^{\delta+1}$ is a constant independent of $k$, passing to the limit, we have

$$
\sum_{n=-D}^{D} |u_n|^{\delta+1} \le M_3^{\delta+1}.
$$

Since $D$ is arbitrary and $u \in l^{\delta+1}$, the function $u$ satisfies $u_n \to 0$ as $|n|\to\infty$. The proof is complete. $\square$

Theorem 1.2 can be proved similarly as in the proof of Theorem 1.1 and using the process in [19]. For simplicity, we omit the proof. As an application of the main theorems, we give two examples to illustrate our results.

**Example 3.1.** In (1.1), let $p_n > 0$, $q_n > 0$, and

$$\begin{aligned}
&f(n, u_{n+1}, u_n, u_{n-1})\\
&= (\delta+1)u_n^{\delta}\ln[1+(u_{n+1}^{\delta+1}+u_n^{\delta+1})^{\frac{1}{\delta+1}}] + \frac{(u_{n+1}^{\delta+1}+u_n^{\delta+1})^{\frac{1}{\delta+1}}u_n^{\delta}}{1+(u_{n+1}^{\delta+1}+u_n^{\delta+1})^{\frac{1}{\delta+1}}}\\
&\quad + (\delta+1)u_n^{\delta}\ln\big[1+(u_n^{\delta+1}+u_{n-1}^{\delta+1})^{\frac{1}{\delta+1}}\big] + \frac{(u_n^{\delta+1}+u_{n-1}^{\delta+1})^{\frac{1}{\delta+1}}u_n^{\delta}}{1+(u_n^{\delta+1}+u_{n-1}^{\delta+1})^{\frac{1}{\delta+1}}}.
\end{aligned}$$

Since

$$F(n, v_1, v_2) = (v_1^{\delta+1}+v_2^{\delta+1})\ln[1+(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}],$$

we have

$$\begin{aligned}
&\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2}\\
&= (\delta+1)v_2^{\delta}\ln[1+(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}] + \frac{(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}v_2^{\delta}}{1+(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}}\\
&\quad + (\delta+1)v_2^{\delta}\ln[1+(v_2^{\delta+1}+v_3^{\delta+1})^{\frac{1}{\delta+1}}] + \frac{(v_2^{\delta+1}+v_3^{\delta+1})^{\frac{1}{\delta+1}}v_2^{\delta}}{1+(v_2^{\delta+1}+v_3^{\delta+1})^{\frac{1}{\delta+1}}}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial F(n, v_1, v_2)}{\partial v_1}v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2}v_2\\
&= (\delta+1)(v_1^{\delta+1}+v_2^{\delta+1})\ln\big[1+(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}\big] + \frac{(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{\delta+2}{\delta+1}}}{1+(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}}\\
&\ge \Big(\delta+1+\frac{1}{(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}}\Big)F(n, v_1, v_2)\\
&\ge 0, \quad \forall n \in \mathbb{Z}.
\end{aligned}$$

This shows that (A8) holds with $a = b = \nu = 1$. It is easy to verify all the conditions of Theorem 1.1 are satisfied. By Theorem 1.1, (1.1) possesses a nontrivial homoclinic orbit.

**Example 3.2.** In (1.1), let $p_n > 0$, $q_n > 0$, $\beta > 2$ and

$$f(n, u_{n+1}, u_n, u_{n-1}) = \beta\big[(u_{n+1}^{\delta+1}+u_n^{\delta+1})^{\frac{\beta-\delta-1}{\delta+1}} + (u_n^{\delta+1}+u_{n-1}^{\delta+1})^{\frac{\beta-\delta-1}{\delta+1}}\big]u_n^{\delta}.$$

Then we have

$$F(n, v_1, v_2) = (v_1^{\delta+1}+v_2^{\delta+1})^{\frac{\beta}{\delta+1}}.$$

By computations similar to those in [25], it is easy to verify all the assumptions of Theorem 1.2 are satisfied. Therefore (1.1) possesses a nontrivial homoclinic solution.

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## References

[1] R. P. Agarwal, P. J. Y. Wong; *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, Dordrecht, 1997.
[2] Z. AlSharawi, J. M. Cushing, S. Elaydi; *Theory and Applications of Difference Equations and Discrete Dynamical Systems*, Springer, New York, 2014.

[3] G. M. Bisci, D. Repovš; *Existence of solutions for p-Laplacian discrete equations*, Appl. Math. Comput., 242 (2014), 454-461.

[4] X. C. Cai, J. S. Yu; *Existence theorems of periodic solutions for second-order nonlinear difference equations*, Adv. Difference Equ., 2008 (2008), 1-11.

[5] A. Castro, R. Shivaji; *Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric*, Comm. Partial Differential Equations, 14(8-9) (1989), 1091-1100.

[6] P. Chen, X. H. Tang; *Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation*, Appl. Math. Comput., 217(9) (2011), 4408-4415.

[7] P. Chen, Z. M. Wang; *Infinitely many homoclinic solutions for a class of nonlinear difference equations*, Electron. J. Qual. Theory Differ. Equ., (47) (2012), 1-18.

[8] X. Q. Deng, G. Cheng, H. P. Shi; *Subharmonic solutions and homoclinic orbits of second order discrete Hamiltonian systems with potential changing sign*, Comput. Math. Appl., 58(6) (2009), 1198-1206.

[9] X. Q. Deng, X. Liu, H. P. Shi, T. Zhou; *Homoclinic orbits for second order nonlinear p-Laplacian difference equations*, J. Contemp. Math. Anal., 46(3) (2011), 172-181.

[10] X. Q. Deng, H. P. Shi, X. L. Xie; *Periodic solutions of second order discrete Hamiltonian systems with potential indefinite in sign*, Appl. Math. Comput., 218(1) (2011), 148-156.

[11] S. Elaydi; *An Introduction to Difference Equations*, Springer, New York, 2005.

[12] J. R. Esteban, J. L. Vázquez; *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal., 10(11) (1986), 1303-1325.

[13] C. J. Guo, D. O'Regan, Y. T. Xu, R. P. Agarwal; *Existence of homoclinic orbits for a singular second-order neutral differential equation*, J. Math. Anal. Appl., 366 (2011), 550-560.

[14] C. J. Guo, Y. T. Xu; *Existence of periodic solutions for a class of second order differential equation with deviating argument*, J. Appl. Math. Comput., 28(1-2) (2008), 425-433.

[15] Z. M. Guo, J. S. Yu; *The existence of periodic and subharmonic solutions of subquadratic second order difference equations*, J. London Math. Soc., 68(2) (2003), 419-430.

[16] J. H. Leng; *Existence of periodic solutions for a higher order nonlinear difference equation*, Electron. J. Differential Equations, 2016(134) (2016), 1-10.

[17] J. H. Leng; *Periodic and subharmonic solutions for 2nth-order $\phi_c$-Laplacian difference equations containing both advance and retardation*, Indag. Math. (N.S.), 27(4) (2016), 902-913.

[18] X. Liu, Y. B. Zhang, H. P. Shi; *Existence of periodic solutions for a class of nonlinear difference equations*, Qual. Theory Dyn. Syst., 14(1) (2015), 51-69.

[19] X. Liu, Y. B. Zhang, H. P. Shi; *Homoclinic orbits of second order nonlinear functional difference equations with Jacobi operators*, Indag. Math. (N.S.), 26(1) (2015), 75-87.

[20] X. Liu, Y. B. Zhang, H. P. Shi; *Homoclinic orbits and subharmonics for second order p-Laplacian difference equations*, J. Appl. Math. Comput., 43(1) (2013), 467-478.

[21] X. Liu, Y. B. Zhang, H. P. Shi, X. Q. Deng; *Periodic and subharmonic solutions for fourth-order nonlinear difference equations*, Appl. Math. Comput., 236(3) (2014), 613-620.

[22] M. J. Ma, Z. M. Guo; *Homoclinic orbits and subharmonics for nonlinear second order difference equations*, Nonlinear Anal., 67(6) (2007), 1737-1745.

[23] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer: New York, 1989.

[24] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Amer. Math. Soc., Providence, RI: New York, 1986.

[25] H. P. Shi, X. Liu, Y. B. Zhang; *Homoclinic orbits for a class of nonlinear difference equations*, Azerb. J. Math., 6(1) (2016), 87-102.

[26] H. P. Shi, X. Liu, Y. B. Zhang; *Homoclinic orbits of second-order nonlinear difference equations*, Electron. J. Differential Equations, 2015(150) (2015), 1-16.

[27] H. P. Shi, X. Liu, Y. B. Zhang; *Periodic solutions for a class of nonlinear difference equations*, Hokkaido Math. J., 45(1) (2016), 109-126.

[28] L. W. Yang; *Existence of homoclinic orbits for fourth-order p-Laplacian difference equations*, Indag. Math. (N.S.), 27(3) (2016), 879-892.

[29] L. W. Yang; *Existence theorems of periodic solutions for second-order difference equations containing both advance and retardation*, J. Contemp. Math. Anal., 51(2) (2016), 58-67.

[30] Q. Q. Zhang; *Homoclinic orbits for a class of discrete periodic Hamiltonian systems*, Proc. Amer. Math. Soc., 143(7) (2015), 3155-3163.

[31] R. Y. Zhang, Z. C. Wang, J. S. Yu; *Necessary and sufficient conditions for the existence of positive solutions of nonlinear difference equations*, Fields Inst. Commun.,42 (2004), 385-396.

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