Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 316, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

CONNECTION OF TWO NONLINEAR DIFFERENTIAL EQUATIONS WITH A TWO-DIMENSIONAL HARDY OPERATOR IN WEIGHTED LEBESGUE SPACES WITH MIXED NORMS

ROVSHAN A. BANDALIYEV

Abstract. In this article we study a connection between two nonlinear differential equations with a two-dimensional Hardy operator in weighted Lebesgue spaces with mixed norm. In particular, it is proved that the weight functions which are the coefficients of nonlinear differential equations are included in the estimate of the two-dimensional Hardy operator in this space.

1. Introduction

In the modern theory of the equations of mathematical physics the functional analysis methods, founded in the classical works of Hilbert, are widely applied. In the study of elliptic equations, embedding theorems play an important role, and been studied by several mathematicians; see for example [12]. The study of the embedding theorems in arbitrary open sets in $\mathbb{R}^n$ requires investigation of the multidimensional Hardy operator, and there is a need an estimates of this operator in various function spaces, among which the weighted Lebesgue spaces are very important. On the other hand, the multidimensional Hardy operator has numerous applications in the spectral theory of operators, in the theory of partial differential equations, in the theory of integral equations, in the theory of function spaces and etc. see [12, 10, 7] and the references therein). In the one-dimensional case, we refer to the well-known monograph [9]. A systematic study of the Hardy inequality was carried out in [4, 14, 15, 13, 6, 10] and references therein. Note that in [4, 14, 15] the Hardy inequality in its equivalent differential form

$$\begin{gathered}\Big(\int_0^\infty |f(x)|^q\,w(x)\,dx\Big)^{1/q}\le C\Big(\int_0^\infty |f'(x)|^p\,u(x)\,dx\Big)^{1/p}\\ f(0)=f(+0)=0\end{gathered}\tag{1.1}$$

was connected with the Euler-Lagrange differential equation. In particular, in [14] and [15] a necessary and sufficient condition on weight functions for validity (1.1) was obtained in the case $p=q$. The study of the case with different parameters p and q was started in [6] and developed in [10], [12] and etc. In the case $p\neq q$ the other type criterion on weight functions for validity (1.1) was obtained in [8].

2010 *Mathematics Subject Classification.* 46B50, 47B38, 34L30.
Key words and phrases. Two-dimensional Hardy operator; weight functions;
Lebesgue spaces with mixed norm.

Submitted February 5, 2016. Published December 12, 2016.

Namely, in [8] the inequality (1.1) was connected with nonlinear ordinary differential equation in weighted Lebesgue spaces.

In this paper we prove a connection between two nonlinear differential equations with a two-dimensional Hardy operator in weighted Lebesgue spaces with mixed norm. In other words, we prove that the weight function involved in the definition of the weighted Lebesgue spaces with mixed norm connects these differential equations with a two-dimensional Hardy operator in this space.

This article is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3.

2. Preliminaries

Let $1 < p_1, p_2 < \infty$ and $\rho_1(t)$, $\rho_2(t)$ be weight functions, that are, Lebesgue measurable, positive and a.e. finite functions on $(0,\infty)$. Let $p_i' = \frac{p_i}{p_i-1}$ and $q_i' = \frac{q_i}{q_i-1}$, where $i = 1,2$. Throughout this article we assume that the functions are Lebesgue measurable. Now we define the weighted Lebesgue spaces with mixed norm. Let $(0,\infty)^2 = (0,\infty)\times(0,\infty)$ and f is an arbitrary real function on $(0,\infty)^2$.

Definition 2.1 ([5]). We denote by $L_{(p_1,p_2,\rho_1,\rho_2)}[(0,\infty)^2]$ the space of all measurable functions f on $(0,\infty)^2$ such that the mixed norm

$$\|f\|_{L_{(p_1,p_2,\rho_1,\rho_2)}[(0,\infty)^2]} = \Big(\int_0^\infty (\int_0^\infty |f(x_1,x_2)|^{p_1}\rho_1(x_1)dx_1)^{p_2/p_1}\rho_2(x_2)dx_2\Big)^{1p_2}$$

is finite.

The variable Lebesgue space with mixed norm was introduced and studied in [1].

We denote by $C^1(0,\infty)$ the space of continuously differentiable functions on $(0,\infty)$. The set of all locally absolutely continuous functions on $(0,\infty)$ is denoted by $AC_{\rm loc}(0,\infty)$.

Let $v_i(t), \omega_i(t)$ be weight functions defined on $(0,\infty)$, $v_i \in C^1(0,\infty)$ and $\lambda_i > 0$, where $i = 1,2$. We consider two nonlinear differential equations

$$\lambda_1 \frac{d}{dt}\Big([v_1(t)]^{q_1/p_1}[y_1'(t)]^{q_1/p_1'}\Big) + \omega_1(t)[y_1(t)]^{q_1/p_1'} = 0, \tag{2.1}$$

$$\lambda_2 \frac{d}{dt}\Big([v_2(t)]^{q_2/p_2}[y_2'(t)]^{q_2/p_2'}\Big) + \omega_2(t)[y_2(t)]^{q_2/p_2'} = 0, \tag{2.2}$$

where

$$y_i(t) > 0, \quad y_i'(t) > 0 \quad (t>0), \quad y_i' \in AC_{\rm loc}(0,\infty), \quad i = 1,2. \tag{2.3}$$

We say that a pair of functions $(y_1(t), y_2(t))$ is a solution of problem (2.1)-(2.3), if these functions a.e. on $(0,\infty)$ satisfy the equations (2.1), (2.2) and condition (2.3). We denote $y_i(0) = \lim_{t\to+0} y_i(t)$, $i = 1,2$.

First we prove the following theorem.

Theorem 2.2. *Let $1 < p_i \le q_i < \infty$ and $v_i(t), \omega_i(t)$ be weight functions on $(0,\infty)$, $v_i \in C^1(0,\infty)$, $\lambda_i > 0$, where $i = 1,2$. If the problem (2.1)-(2.3) has a solution $(y_1(t), y_2(t))$ then the inequality*

$$\|u\|_{L_{(q_1,q_2,\omega_1,\omega_2)}[(0,\infty)^2]} \le \lambda_1^{1/q_1}\lambda_2^{1/q_2}\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L_{(p_1,p_2,v_1,v_2)}[(0,\infty)^2]} \tag{2.4}$$

holds, where $u : (0,\infty)^2 \mapsto R$ is an arbitrary absolutely continuous function of two variables satisfying condition

$$\begin{aligned} u(x_1,0) &= \lim_{t_2\to+0} u(x_1,t_2) = 0 \\ u(0,x_2) &= \lim_{t_1\to+0} u(t_1,x_2) = 0. \end{aligned} \tag{2.5}$$

Proof. It is well known that for any absolutely continuous function of two variables the representation

$$\begin{aligned} u(x_1,x_2) &= u(0,0) + \int_0^{x_1} \frac{\partial u(\alpha_1,0)}{\partial \alpha_1} d\alpha_1 + \int_0^{x_2} \frac{\partial u(0,\alpha_2)}{\partial \alpha_2} d\alpha_2 \\ &\quad + \int_0^{x_1}\int_0^{x_2} \frac{\partial^2 u(\alpha_1,\alpha_2)}{\partial\alpha_1\partial\alpha_2} d\alpha_1 d\alpha_2 \end{aligned} \tag{2.6}$$

is valid (see [11]). Obviously, condition (2.5) implies $u(0,0) = \lim_{\substack{t_1\to+0\\ t_2\to+0}} u(t_1,t_2) = 0$. Therefore, by (2.6) and (2.5), we have

$$u(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} \frac{\partial^2 u(\alpha_1,\alpha_2)}{\partial\alpha_1\partial\alpha_2} d\alpha_1 d\alpha_2.$$

Note that the last integral representation defines the two-dimensional Hardy operator.

Let a vector function $y(x_1,x_2) = (y_1(x_1), y_2(x_2))$ be a solution of (2.1)-(2.3). Then by Hölder's inequality, we have

$$\begin{aligned}
&\Big|u(x_1,x_2)\Big|^{q_1}\omega_1(x_1) \\
&= \Big|\int_0^{x_1}\int_0^{x_2} \frac{\partial^2 u}{\partial t_1\partial t_2} dt_1 dt_2\Big|^{q_1}\omega_1(x_1) \\
&\le \Big(\int_0^{x_1}\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| dt_1 dt_2\Big)^{q_1}\omega_1(x_1) \\
&= \Big(\int_0^{x_1}\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{1/p_1'}[y_1'(t_1)]^{-1/p_1'} dt_1\, dt_2\Big)^{q_1}\omega_1(x_1) \\
&= \Big[\int_0^{x_1}\Big(\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{-1/p_1'} dt_2\Big)[y_1'(t_1)]^{1/p_1'} dt_1\Big]^{q_1}\omega_1(x_1) \\
&\le \Big(\int_0^{x_1} y_1'(t_1)dt_1\Big)^{q_1/p_1'}\Big(\int_0^{x_1}\Big(\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{-1/p_1'} dt_2\Big)^{p_1} dt_1\Big)^{q_1/p_1}\omega_1(x_1) \\
&\le \omega_1(x_1)\,(y_1(x_1))^{q_1/p_1'}\Big(\int_0^{x_1}(\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{-1/p_1'} dt_2)^{p_1} dt_1\Big)^{q_1/p_1} \\
&= -\lambda_1 \frac{d}{dx_1}\Big([v_1(x_1)]^{q_1/p_1}[y_1'(x_1)]^{\frac{q_1}{p_1'}}\Big) \\
&\quad \times \Big(\int_0^{x_1}\Big(\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{-1/p_1'} dt_2\Big)^{p_1} dt_1\Big)^{q_1/p_1} \\
&= \Big(\int_0^{x_1}\Big[-\lambda_1 \frac{d}{dx_1}\Big([v_1(x_1)]^{\frac{q_1}{p_1}}[y_1'(x_1)]^{\frac{q_1}{p_1'}}\Big)\Big]^{p_1/q_1} \\
&\quad \times \Big(\int_0^{x_2} \Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big| [y_1'(t_1)]^{-1/p_1'} dt_2\Big)^{p_1} dt_1\Big)^{q_1/p_1}.
\end{aligned}$$

Integrating both with respect to x_1 and applying the generalized Minkowskii inequality, we obtain

$$
\begin{aligned}
&\Big(\int_0^\infty |u(x_1,x_2)|^{q_1}\,\omega_1(x_1)dx_1\Big)^{p_1/q_1}\\
&\le \Big\{\int_0^\infty \Big(\int_0^{x_1}\Big[-\lambda_1\frac{d}{dx_1}([v_1(x_1)]^{q_1/p_1}[y_1'(x_1)]^{q_1/p_1'})\Big]^{p_1/q_1}\\
&\quad\times\Big(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|[y_1'(t_1)]^{-1/p_1'}dt_2\Big)^{p_1}dt_1\Big)^{q_1/p_1}dx_1\Big\}^{p_1/q_1}\\
&\le \int_0^\infty\Big(\int_{t_1}^\infty -\lambda_1\frac{d}{dx_1}\big([v_1(x_1)]^{q_1/p_1}[y_1'(x_1)]^{q_1/p_1'}\big)\\
&\quad\times\Big(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|[y_1'(t_1)]^{-1/p_1'}dt_2\Big)^{q_1}dx_1\Big)^{p_1/q_1}dt_1\\
&= \int_0^\infty\Big(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|[y_1'(t_1)]^{-1/p_1'}dt_2)^{p_1}\\
&\quad\times\Big(\int_{t_1}^\infty -\lambda_1\frac{d}{dx_1}([v_1(x_1)]^{q_1/p_1}[y_1'(x_1)]^{q_1/p_1'})dx_1\Big)^{p_1/q_1}dt_1\\
&\le \int_0^\infty\Big(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|[y_1'(t_1)]^{-1/p_1'}dt_2\Big)^{p_1}[\lambda_1([v_1(t_1)]^{q_1/p_1}[y_1'(t_1)]^{\frac{q_1}{p_1'}})]^{p_1/q_1}dt_1\\
&= \lambda_1^{p_1/q_1}\int_0^\infty\Big(\int_0^{x_2}\Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big|dt_2\Big)^{p_1}[y_1'(t_1)]^{-p_1/p_1'}v_1(t_1)[y_1'(t_1)]^{p_1/p_1'}dt_1\\
&= \lambda_1^{p_1/q_1}\int_0^\infty\Big(\int_0^{x_2}\Big|\frac{\partial^2 u}{\partial t_1\partial t_2}\Big|dt_2\Big)^{p_1}v_1(t_1)dt_1.
\end{aligned}
$$

We have

$$
\Big(\int_0^\infty |u(x_1,x_2)|^{q_1}\omega_1(x_1)dx_1\Big)^{1/q_1} \le \lambda_1^{1/q_1}\Big(\int_0^\infty(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|dt_2)^{p_1}v_1(t_1)dt_1\Big)^{1/p_1}.
$$

Again applying the generalized Minkowskii inequality, we obtain

$$
\begin{aligned}
&\Big(\int_0^\infty\Big(\int_0^{x_2}\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|dt_2\Big)^{p_1}v_1(t_1)dt_1\Big)^{1/p_1}\\
&\le \int_0^{x_2}\Big(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)dt_1\Big)^{1/p_1}dt_2.
\end{aligned}
$$

Thus,

$$
\Big(\int_0^\infty |u(x_1,x_2)|^{q_1}\omega_1(x_1)dx_1\Big)^{1/q_1} \le \lambda_1^{1/q_1}\int_0^{x_2}\Big(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)dt_1\Big)^{1/p_1}dt_2.
$$

From the above estimate we obtain

$$
\begin{aligned}
&\Big(\int_0^\infty |u(x_1,x_2)|^{q_1}\omega_1(x_1)dx_1\Big)^{q_2/q_1}\omega_2(x_2)\\
&\le \lambda_1^{q_2/q_1}\omega_2(x_2)\Big(\int_0^{x_2}(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)dt_1)^{1/p_1}dt_2\Big)^{q_2}.
\end{aligned}
$$

Analogously, we have the estimate

$$
\omega_2(x_2)\Big(\int_0^{x_2}\Big(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)dt_1\Big)^{1/p_1}dt_2\Big)^{q_2}
$$

$$
\begin{aligned}
&\le \Big(\int_0^{x_2}\Big[-\lambda_2\frac{d}{dx_2}\Big([v_2(x_2)]^{q_2/p_2}[y_2'(x_2)]^{q_2/p_2'}\Big)\Big]^{p_2/q_2}\\
&\quad\times\Big(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)[y_2'(t_2)]^{-1/p_2'}dt_1\Big)^{p_2/p_1}dt_2\Big)^{q_2/p_2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big(\int_0^\infty|u(x_1,x_2)|^{q_1}\,\omega_1(x_1)dx_1\Big)^{q_2/q_1}\omega_2(x_2)\\
&\le \lambda_1^{q_2/q_1}\Big(\int_0^{x_2}\Big[-\lambda_2\frac{d}{dx_2}\Big([v_2(x_2)]^{q_2/p_2}[y_2'(x_2)]^{\frac{q_2}{p_2'}}\Big)\Big]^{p_2/q_2}\\
&\quad\times\Big(\int_0^\infty\big|\frac{\partial^2 u}{\partial t_1\partial t_2}\big|^{p_1}v_1(t_1)[y_2'(t_2)]^{-\frac{1}{p_2'}}dt_1\Big)^{p_2/p_1}dt_2\Big)^{q_2/p_2}.
\end{aligned}
$$

Integrating both sides of the estimate with respect to x_2 and applying the generalized Minkowskii inequality, we finally obtain (2.4). This completes the proof. □

We put

$$
M_i=\frac{p_i'}{q_i}\inf_g\sup_{t>0}\frac{1}{g_i(t)-\int_0^t(v_i(s))^{1-p_i'}ds}\int_0^t\omega_i(s)(g_i(s))^{\frac{q_i}{p_i'}+1}ds, \tag{2.7}
$$

where the infimum is taken over the class of all measurable functions g such that

$$
g_i(t)>\int_0^t(v_i(s))^{1-p_i'}ds,\quad \text{for all } t>0 \text{ and } i=1,2.
$$

The following lemma establishes a connection between problem (2.1)-(2.3) and M_i $(i=1,2)$.

Lemma 2.3. *Let $\lambda_i>0$ be given numbers satisfying the conditions of Theorem 2.2 and the expressions M_i be defined by (2.7), where $i=1,2$. Suppose v_i and ω_i are weight functions on $(0,\infty)$ and the derivative $v_i'(t)$ exists for all $t\in(0,\infty)$. Then the following statements are equivalent:*

(a) *if the problem (2.1)-(2.3) has a solution with a locally absolutely continuous first derivative, then $\lambda_i\ge M_i$;*
(b) *if $M_i<+\infty$, then the problem (2.1)-(2.3) has a solution for every $\lambda_i>M_i$, where $i=1,2$.*

Proof. Assume that (a) holds. Let $y(t)=(y_1(t),y_2(t))$ be a solution of (2.1)-(2.3). Let us take $w_i=\frac{y_i}{y_i'}v_i^{1-p_i'}$. Then the functions $w_1(t),w_2(t)$ are positive solutions of the equations

$$
w_1'(t)=\frac{p_1'}{q_1\lambda_1}\omega_1(t)(w_1(t))^{\frac{q_1}{p_1'}+1}+[v_1(t)]^{1-p_1'}, \tag{2.8}
$$

$$
w_2'(t)=\frac{p_2'}{q_2\lambda_2}\,\omega_2(t)(w_2(t))^{\frac{q_2}{p_2'}+1}+[v_2(t)]^{1-p_2'}. \tag{2.9}
$$

By (2.8) and (2.9), we obtain

$$
w_i(t)\ge\int_0^t w_i'(s)ds=\frac{p_i'}{q_i\lambda_i}\int_0^t\omega_i(s)(w_i(s))^{\frac{q_i}{p_i'}+1}ds+\int_0^t[v_i(s)]^{1-p_i'}ds, \tag{2.10}
$$

for $i=1,2$. This implies $w_i(t)\ge\int_0^t[v_i(s)]^{1-p_i'}ds$ and

$$\lambda_i\ge\frac{p_i'}{q_i}\frac{1}{w_i(t)-\int_0^t(v_i(s))^{1-p_i'}ds}\int_0^t\omega_i(s)(w_i(s))^{\frac{q_i}{p_i'}+1}ds. \tag{2.11}$$

From this inequality and (2.7) we obtain that $\lambda_i\ge M_i$ and the proof of (a) implies (b) is complete.

Now let (b) hold. Let us fix $\lambda_i>M_i$, where $i=1,2$. By the definition of M_i there exists a measurable functions $g_i(t)$ such that

$$g_i(t)\ge\int_0^t(v_i(s))^{1-p_i'}ds+\frac{p_i'}{q_i\,\lambda_i}\int_0^t\omega_i(s)(g_i(s))^{\frac{q_i}{p_i'}+1}ds. \tag{2.12}$$

We define a sequence of functions $w_{n,i}(t)$ $(i=1,2)$ by setting

$$\begin{gathered}w_{0,i}(t)=g_i(t),\\ w_{n+1,i}(t)=\int_0^t(v_i(s))^{1-p_i'}ds+\frac{p_i'}{q_i\lambda_i}\int_0^t\omega_i(s)(w_{n,i}(s))^{\frac{q_i}{p_i'}+1}ds,\end{gathered} \tag{2.13}$$

where $n=0,1,2,\dots$. Also (2.12) implies $w_{0,i}(t)\ge w_{1,i}(t)$. We put $w_{n-1,i}(t)\ge w_{n,i}(t)$. Now we prove that the sequences $\{w_{n,1}(t)\}$ and $\{w_{n,2}(t)\}$ are non-increasing with respect to n on $t\in(0,\infty)$. We have

$$w_{n,i}(t)-w_{n+1,i}(t)=\frac{p_i'}{q_i\lambda_i}\int_0^t\omega_i(s)[(w_{n-1,i}(s))^{\frac{q_i}{p_i'}+1}-(w_{n,i}(s))^{\frac{q_i}{p_i'}+1}]ds\ge 0.$$

Since $w_{n,i}(t)\ge 0$, the sequences in (2.13) converge. For $i=1,2$, we denote $w_i(t):=\lim_{n\to\infty}w_{n,i}(t)$. By the Levi monotone convergence theorem it follows that w_i is a nonnegative solution of the equation

$$w_i(t)=\int_0^t(v_i(s))^{1-p_i'}ds+\frac{p_i'}{q_i\,\lambda_i}\int_0^t\omega_i(s)(w_i(s))^{\frac{q_i}{p_i'}+1}ds,$$

where $i=1,2$. Hence, w_i is absolutely continuous and satisfies the differential equation

$$w_i'(t)=(v_i(t))^{1-p_i'}+\frac{p_i'}{q_i\lambda_i}\omega_i(t)(w_i(t))^{\frac{q_i}{p_i'}+1}.$$

Therefore, for any fixed numbers $a_i>0$ $(i=1,2)$ the functions

$$y_i(t)=\exp\Big(\int_{a_i}^t[w_i(s)]^{-1}(v_i(s))^{1-p_i'}\,ds\Big)$$

satisfy problem (2.1)-(2.3). This completes the proof. □

Theorem 2.4. *Let $1<p_i\le q_i<\infty$, $M_i<\infty$, $v_i(t)$ and $\omega_i(t)$ be weight functions on $(0,\infty)$, where $i=1,2$. Suppose $C>0$ is the best possible constant, such that*

$$\|u\|_{L_{(q_1,q_2,\omega_1,\omega_2)}[(0,\infty)^2]}\le C\|\frac{\partial^2u}{\partial x_1\partial x_2}\|_{L_{(p_1,p_2,v_1,v_2)}[(0,\infty)^2]}\,, \tag{2.14}$$

where $u:(0,\infty)^2\mapsto R$ is an arbitrary absolutely continuous function of two variables satisfying condition (2.5). Then

$$C\le M_1^{1/q_1}M_2^{1/q_2}.$$

Proof. We denote $\frac{\partial^2 u}{\partial x_1 \partial x_2} = h(x_1, x_2)$. Under condition (2.5) we have

$$u(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} h(t_1, t_2) dt_1 dt_2.$$

Obviously,

$$C = \sup \Big(\int_0^\infty \Big(\int_0^\infty \Big| \int_0^{x_1} \int_0^{x_2} h(t_1, t_2) dt_1 dt_2 \Big|^{q_1} \omega_1(x_1) dx_1 \Big)^{\frac{q_2}{q_1}} \omega_2(x_2) dx_2 \Big)^{1/q_2}, \tag{2.15}$$

where the supremum is taken over the class of all measurable functions h such that

$$\int_0^\infty \big(\int_0^\infty |h(x_1, x_2)|^{p_1} v_1(x_1) dx_1 \Big)^{p_2/p_1} v_2(x_2) dx_2 = 1. \tag{2.16}$$

Assume the contrary. Let $C > M_1^{1/q_1} M_2^{1/q_2}$. Then there exists a number $\mu > 0$ such that $C > \mu > M_1^{1/q_1} M_2^{1/q_2}$. Since $M_i < \infty$, by Lemma 2.3, problem (2.1)-(2.3) has a solution. Therefore, by Theorem 2.2, the inequality

$$\|u\|_{L_{(q_1,q_2,\omega_1,\omega_2)}[(0,\infty)^2]} \leq \mu \Big\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \Big\|_{L_{(p_1,p_2,v_1,v_2)}[(0,\infty)^2]}$$

holds for all absolutely continuous function $u(x_1, x_2)$ satisfying (2.5). Hence C is not the best possible constant in (2.14). This contradiction completes the proof. $\square$

Corollary 2.5. *Let*

$$B_i = \sup_{x_i > 0} \int_{x_i}^\infty \omega_i(s) ds \Big[\int_0^{x_i} (v_i(s))^{1-p_i'} ds \Big]^{q_i/p_i'}, \quad i = 1, 2.$$

Then

$$B_1^{\frac{1}{q_1}} B_2^{1/q_2} \leq C \leq M_1^{1/q_1} M_2^{1/q_2} \leq \prod_{i=1}^2 (q_i^{1/q_i} (q_i')^{1/p_i'}) B_1^{1/q_1} B_2^{1/q_2}, \tag{2.17}$$

$$M_1^{1/q_1} M_2^{1/q_2} \leq \prod_{i=1}^2 \Big(q_i^{1/q_i} (q_i')^{1/p_i'} \Big) C. \tag{2.18}$$

Proof. Let $(\xi_1, \xi_2) \in (0, \infty)^2$ and $P_{\xi_1 \xi_2} = \{(x_1, x_2) \in (0, \infty)^2 : 0 < x_1 < \xi_1, 0 < x_2 < \xi_2\}$. We put

$$s(x_1, x_2) = \begin{cases} \prod_{i=1}^2 \Big(\int_0^{\xi_i} [v_i(t_i)]^{1-p_i'} dt_i \Big)^{-1/p_i} [v_i(x_i)]^{1-p_i'}, & \text{for } (x_1, x_2) \in P_{\xi_1 \xi_2} \\ 0, & \text{for } (x_1, x_2) \notin P_{\xi_1 \xi_2}, \end{cases}$$

Obviously,

$$\begin{aligned} &\int_0^\infty \Big(\int_0^\infty (s(x_1, x_2))^{p_1} v_1(x_1) dx_1 \Big)^{p_2/p_1} v_2(x_2) dx_2 \\ &= \int_0^{\xi_2} \Big(\int_0^{\xi_1} \Big[\prod_{i=1}^2 \Big(\int_0^{\xi_i} [v_i(t_i)]^{1-p_i'} dt_i \Big)^{-1/p_i} [v_i(x_i)]^{1-p_i'} \Big]^{p_1} v_1(x_1) dx_1 \Big)^{p_2/p_1} \\ &\quad \times v_2(x_2) dx_2 = 1. \end{aligned}$$

Then, by (2.15), we have

$$C \geq \Big(\int_0^\infty \Big(\int_0^\infty \Big(\int_0^{x_1} \int_0^{x_2} s(t_1, t_2) dt_1 dt_2 \Big)^{q_1} \omega_1(x_1) dx_1 \Big)^{q_2/q_1} \omega_2(x_2) dx_2 \Big)^{1/q_2}$$

$$
\begin{aligned}
&\geq \Big(\int_{\xi_2}^\infty \Big(\int_{\xi_1}^\infty (\int_0^{x_1}\int_0^{x_2} s(t_1,t_2)dt_1dt_2)^{q_1}\omega_1(x_1)dx_1\Big)^{q_2/q_1}\omega_2(x_2)dx_2\Big)^{1/q_2}\\
&= \Big(\int_{\xi_2}^\infty \Big(\int_{\xi_1}^\infty \Big(\int_0^{\xi_1}\int_0^{\xi_2}\prod_{i=1}^2\Big(\int_0^{\xi_i}[v_i(y_i)]^{1-p_i'}dy_i\Big)^{-1/p_i}[v_i(t_i)]^{1-p_i'}dt_1dt_2\Big)^{q_1}\\
&\quad\times\omega_1(x_1)dx_1\Big)^{q_2/q_1}\omega_2(x_2)dx_2\Big)^{1/q_2}\\
&= \Big(\int_0^{\xi_1}[v_1(y_1)]^{1-p_1'}dy_1\Big)^{1/p_1'}\Big(\int_{\xi_1}^\infty \omega_1(x_1)dx_1\Big)^{1/q_1}\Big(\int_0^{\xi_2}[v_2(y_2)]^{1-p_2'}dy_2\Big)^{1/p_2'}\\
&\quad\times\Big(\int_{\xi_2}^\infty\omega_2(x_2)dx_2\Big)^{1/q_2}.
\end{aligned}
$$

Passing to the supremum in the last inequality (with respect to ξ_1 and ξ_2), we obtain $C \geq B_1^{1/q_1}B_2^{1/q_2}$. The inequality $C \leq M_1^{1/q_1}M_2^{1/q_2}$ follows from Theorem 2.4.

Now we prove that $M_i \leq q_i(q_i')^{q_i/p_i'}B_i$, where $i=1,2$. First, by the definition of B_i we obtain that

$$
B_i^{p_i'/q_i}\Big(\int_{x_i}^\infty \omega_i(y_i)dy_i\Big)^{-p_i'/q_i} > \int_0^{x_i}[v_i(t_i)]^{1-p_i'}dt_i.
$$

Putting

$$
g_i(x_i) = \frac{q_i'B_i^{p_i'/q_i}}{(\int_{x_i}^\infty \omega_i(y_i)dy_i)^{p_i'/q_i}}
$$

for $i=1,2$, in (2.7), we obtain

$$
\begin{aligned}
M_i &= \frac{p_i'}{q_i}\inf\sup_{x_i>0}\frac{1}{q_i'B_i^{p_i'/q_i}\big(\int_{x_i}^\infty \omega_i(t_i)dt_i\big)^{-p_i'/q_i} - \int_0^{x_i}[v_i(t_i)]^{1-p_i'}dt_i}\\
&\quad\times\int_0^{x_i}\omega_i(t_i)\Big(q_i'B_i^{p_i'/q_i}\Big(\int_{t_i}^\infty\omega_i(y_i)dy_i\Big)^{-p_i'/q_i}\Big)^{\frac{q_i}{p_i'}+1}dt_i\\
&\leq \sup_{x_i>0}\frac{(q_i')^{\frac{q_i}{p_i'}+1}B^{1+\frac{p_i'}{q_i}}}{q_i'B_i^{p_i'/q_i}\big(\int_{x_i}^\infty \omega_i(t_i)dt_i\big)^{-p_i'/q_i} - \int_0^{x_i}[v_i(t_i)]^{1-p_i'}dt_i}\\
&\quad\times\int_0^{x_i}\frac{d}{dt_i}\Big[\Big(\int_{t_i}^\infty\omega_i(y_i)dy_i\Big)^{-p_i'/q_i}\Big]\\
&\leq (q_i')^{\frac{q_i}{p_i'}+1}B^{1+\frac{p_i'}{q_i}}\sup_{x_i>0}\frac{\big(\int_{x_i}^\infty \omega_i(t_i)dt_i\big)^{-p_i'/q_i}}{q_i'B_i^{p_i'/q_i}\big(\int_{x_i}^\infty \omega_i(t_i)dt_i\big)^{-p_i'/q_i} - \int_0^{x_i}[v_i(t_i)]^{1-p_i'}dt_i}.
\end{aligned}
$$

The definition of B_i implies that

$$
\int_0^{x_i}[v_i(y_i)]^{1-p_i'}dy_i \leq B^{p_i'/q_i}\Big(\int_{x_i}^\infty\omega_i(y_i)dy_i\Big)^{-p_i'/q_i}.
$$

Hence

$$
\begin{aligned}
&q_i'B_i^{p_i'/q_i}\Big(\int_{x_i}^\infty \omega_i(t_i)dt_i\Big)^{-p_i'/q_i} - \int_0^{x_i}[v_i(t_i)]^{1-p_i'}dt_i\\
&\geq (q_i'-1)B_i^{p_i'/q_i}\Big(\int_{x_i}^\infty\omega_i(t_i)dt_i\Big)^{-p_i'/q_i}.
\end{aligned}
$$

Thus

$$M_i \le \frac{(q_i')^{\frac{q_i}{p_i'}+1} B_i^{1+\frac{p_i'}{q_i}}}{(q_i'-1)B_i^{p_i'/q_i}} = q_i(q_i')^{q_i/p_i'}B_i,$$

where $i = 1, 2$. The inequality (2.17) is proved. Clearly, (2.17) implies (2.18). The proof is complete. □

It follows from Corollary 2.5 that (2.14) holds if and only if $B_i < +\infty$, where $i = 1, 2$ (see [2]).

3. Main result

Theorem 3.1. *Let $1 < p_i \le q_i < \infty$ and $v_i(t)$ and $\omega_i(t)$ be weight functions on $(0, \infty)$, $v_i \in C^1(0, \infty)$, where $i = 1, 2$. Then for the solvability of problem* (2.1)-(2.3)*, it is necessary and sufficient that there exists a constant $C_0 > 0$ such that the inequality*

$$\|u\|_{L_{(q_1,q_2,\omega_1,\omega_2)}[(0,\infty)^2]} \le C_0 \Big\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \Big\|_{L_{(p_1,p_2,v_1,v_2)}[(0,\infty)^2]}, \tag{3.1}$$

holds, where $u : (0, \infty)^2 \mapsto R$ is an arbitrary absolutely continuous function of two variables satisfying condition (2.5).

Proof. The sufficiency part of the statement follows from Theorem 2.4. We shall prove only the necessity part. Let $u(x_1, x_2)$ be an absolutely continuous function satisfying condition (2.5) and let the inequality (3.1) holds. Then $C \le C_0 < \infty$, where C is the constant in (2.14). (2.17) implies that $M_i < \infty$, where $i = 1, 2$. Then, by Lemma 2.3, problem (2.1)-(2.3) has a solution for any $\lambda_i > M_i$, where $i = 1, 2$. This completes the proof. □

Corollary 3.2. *Let $p_1 = p_2 = p$, $q_1 = q_2 = q$ and $v(x_1, x_2) = v_1(x_1)v_2(x_2)$, $\omega(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$ be weight functions on $(0, \infty)^2$ satisfy all conditions of Theorem 3.1. Then* (3.1) *holds for any absolutely continuous function of two variables satisfying* (2.5).

Example 3.3. Let $1 < p_i \le q_i < \infty$, $v_i(x_i) = x_i^{\alpha_i}$ and $\omega_i(x_i) = x_i^{\frac{q_i}{p_i}(p_i-1-\alpha_i)-1}$, where $i = 1, 2$. Then (3.1) holds, if and only if $\alpha_i < p_i - 1$.

We remark that some sufficient condition on weight functions connecting certain nonlinear differential equation with one-dimensional Hardy operator in weighted variable Lebesgue spaces was found in [3].

Acknowledgements. This research was partially supported by the grant from the Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2013-9(15)-46/10/1, and by a grant from the Presidium of Azerbaijan National Academy of Science 2015. The author is grateful to the referees for his their careful reading and valuable comments.

References

[1] M. M. Abbasova, R. A. Bandaliyev; *On an inequality and p(x)-mean continuity in the variable Lebesgue space with mixed norm*, Trans. Nat. Acad. Sci. of Azerb. **26** (2006), 47-56.

[2] J. Appell, A. Kufner; *On the two-dimensional Hardy operator in Lebesgue spaces with mixed norms*, Analysis, **15** (1995), 91-98.

[3] R. A. Bandaliyev; *Applications of multidimensional Hardy operator and its connection with a certain nonlinear differential equation in weighted variable Lebesgue spaces*, Annals of Funct. Analysis, **4** (2013), 118-130..

[4] P. R. Beesack; *Hardy's inequality and its extensions*, Pacific J. Math., **11**(1961), 39-61.

[5] A. Benedek, R. Panzone; *The spaces L_p, with mixed norm*, Duke Math. J. **28** (1961), 302-324.

[6] J. S. Bradley; *The Hardy inequalities with mixed norms*, Canad. Math. Bull., **21** (1978), 405-408.

[7] D. E. Edmunds, W. D. Evans; *Hardy operators, function spaces and embeddings*, Springer-Verlag, Berlin, 2004.

[8] P. Gurka; *Generalized Hardy's inequality*, Čas. pro Pêstování Mat., **109**(1984), 194-203.

[9] G. H. Hardy, J. E. Littlewood, G. Pólya; *Inequalities*, Cambridge University Press, 1934.

[10] A. Kufner, L.-E. Persson; *Weighted inequalities of Hardy type*, World Scientific Publishing Co, New Jersey- London- Singapore- Hong Kong, 2003.

[11] J. Malý; *Absolutely continuous functions of several variables*, J. Math. Anal. Appl., **231** (1999), 492-508.

[12] V. G. Maz'ya; *Sobolev spaces*, Springer-Verlag, Berlin, 1985.

[13] B. Muckenhoupt; *Hardy's inequalities with weights*, Studia Math. **44** (1972), 31-38.

[14] G. Talenti; *Osservazione sopra una classe di disuguaglianze*, Rend. Sem. Mat. Fiz. Milano, **39** (1969), 171-185.

[15] G. Tomaselli; *A class of inequalities*, Boll. Unione Mat. Ital., **2** (1969), 622-631.

Rovshan A. Bandaliyev
Department of Mathematical Analysis, Institute of Mathematics and Mechanics of ANAS, Az 1141, B. Vahabzade str., 9, Baku, Azerbaijan
E-mail address: bandaliyev.rovshan@math.ab.az