Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 317, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SIGN-CHANGING SOLUTIONS FOR ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION IN BOUNDED DOMAINS

SITONG CHEN, YINBIN LI, XIANHUA TANG

Abstract. In this article we study the Schrödinger equation

$$-\Delta u = f(x,u), \quad x \in \Omega, \quad u \in H_0^1(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f(x,u)$ is asymptotically linear at infinity with respect to u. Inspired by the works of Salvatore [14] on sign-changing solutions, in which $f(x,u)$ is asymptotically linear at zero with respect to u, we prove, via the constraint variational method and the quantitative deformation lemma, that the equation possesses one sign-changing solution with exactly two nodal domains.

1. Introduction and statement of main results

In this article, we consider the Schrödinger equation

$$\begin{gathered} -\Delta u = f(x,u), \quad x \in \Omega, \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. The main aim of this paper is to find sign-changing solutions of (1.1) when f is asymptotically linear. Precisely, we assume that f satisfies the following assumptions:

- (A1) $f \in C(\Omega \times \mathbb{R})$, $F(x,t) := \int_0^t f(x,s)\mathrm{d}s \geq 0$ and $f(x,t) = o(|t|)$ as $|t| \to 0$, uniformly in $x \in \Omega$;
- (A2) $f(x,t) = V_\infty(x)t + f_1(x,t)$, $V_\infty \in C(\Omega)$, and $f_1(x,t) = o(|t|)$ as $|t| \to +\infty$, uniformly in $x \in \Omega$;
- (A3) $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty,0) \cup (0,\infty)$ for every $x \in \Omega$;
- (A4) $\widetilde{F}(x,t) := \frac{1}{2}f(x,t)t - F(x,t) \to +\infty$ as $t \to +\infty$ uniformly in $x \in \Omega$.

The nonlinear Schrödinger equation is of interest in many branches of physics. As we know, the solutions of problems like (1.1) are related to the existence of standing wave solutions for nonlinear Schrödinger equation like

$$i\hbar\frac{\partial \Psi}{\partial t} = -\hbar^2 \triangle \Psi + V(x)\Psi - f(x,\Psi) \quad \text{for all } x \in \Omega, \tag{1.2}$$

where Ω is a domain in $\mathbb{R}^N$, $\hbar > 0$ and Ψ is the amplitude of the wave. Equation (1.2) is one of the main objects of quantum physics, for it appears in problems

---

2010 *Mathematics Subject Classification.* 35J10, 35J20.

Key words and phrases. Schrödinger equation; sign-changing solutions; asymptotically linear.

Submitted June 22, 2016. Published December 14, 2016.

involving nonlinear optics, plasma physics and condensed matter physics, see Anderson and Bonnedal [1], Chen [6], Chiao et al [8], Gatz and Herrmann [9], Karlsson [10], Sodha et al [17], Stuart [19] and the references therein.

In recent years, problems like (1.1) have been widely studied under variant assumptions on f, and the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states were obtained in many papers, see for example [3, 7, 12, 13, 20, 21, 22, 25, 26] and the references therein. When f is superlinear at infinity in u, the existence of sign-changing solutions of (1.1) was established by Bartsch, Liu and Weth in [2]. For more discussions on the existence of sign-changing solutions of (1.1), in this case, we refer the readers to [4, 5, 11, 27] and the references therein. When f is asymptotically linear at zero in u, that is, f satisfies the condition:

$$\mu_1 < \liminf_{t\to 0} \frac{f(x,t)}{t} \le \limsup_{t\to 0} \frac{f(x,t)}{t} < \mu_k \quad \text{uniformly for } x\in\Omega, \tag{1.3}$$

where $\{\mu_j\}$ is the sequence of eigenvalues of the Schrödinger operator $-\Delta+V(x)$ and V is a linear potential, Salvatore [14] proved the existence of sign-changing solutions. Note that conditions (A1) and (1.3) are quite different and were considered in different situations. To the best of our knowledge, there are no works concerning the least energy sign-changing solutions for Problem (1.1) with asymptotically linear case at infinity, and it is an interesting problem.

Let $H^1(\Omega)$ be the usual Sobolev space with the standard scalar product and norm

$$(u,v)=\int_\Omega (\nabla u\nabla v+uv)\mathrm{d}x, \quad \|u\|^2=\int_\Omega \left(|\nabla u|^2+u^2\right)\mathrm{d}x.$$

Define the energy functional $\Phi: H_0^1(\Omega)\to\mathbb{R}$ by

$$\Phi(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x. \tag{1.4}$$

Conditions (A1) and (A2) imply that Φ is a well-defined of class C^1 functional, and that

$$\langle \Phi'(u),\varphi\rangle=\int_\Omega \nabla u\nabla\varphi\mathrm{d}x-\int_\Omega f(x,u)\varphi\mathrm{d}x, \quad \forall u,\varphi\in H_0^1(\Omega). \tag{1.5}$$

Clearly, critical points of Φ are the weak solutions of (1.1). Furthermore, if $u\in H_0^1(\Omega)$ is a solution of (1.1) and $u^\pm\neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x):=\max\{u(x),0\}, \quad u^-(x):=\min\{u(x),0\}.$$

Using (1.4) and (1.5), it is obvious that

$$\begin{gathered}\Phi(u)=\Phi(u^+)+\Phi(u^-),\\ \langle\Phi'(u),u^+\rangle=\langle\Phi'(u^+),u^+\rangle, \quad \langle\Phi'(u),u^-\rangle=\langle\Phi'(u^-),u^-\rangle.\end{gathered}$$

To obtain a sign-changing solution of (1.1), we first seek a minimizer of the energy functional Φ under the constraint

$$\mathcal{M}=\{u\in H_0^1(\Omega): u^\pm\neq 0,\ \langle\Phi'(u),u^+\rangle=\langle\Phi'(u),u^-\rangle=0\},$$

then show that the minimizer is a sign-changing solution of (1.1).

To state our results, we make the following assumption:

(A5) $\inf_{x\in\Omega} V_\infty(x) > \mu := \inf_{u\in\Pi} \max\{\|\nabla u^+\|_2^2,\ \|\nabla u^-\|_2^2\}$, where

$$\Pi := \{u \in H_0^1(\Omega) : u^\pm \neq 0,\ \int_\Omega |u^\pm|^2 \mathrm{d}x = 1\}.$$

Let λ_1 be the first eigenvalue of $-\Delta$, then for any $u \in H_0^1(\Omega)$ and $u \neq 0$,

$$\begin{aligned}\lambda_1 &\le \frac{\|\nabla u\|_2^2}{\|u\|_2^2}\\ &= \frac{\|\nabla u^+\|_2^2 + \|\nabla u^-\|_2^2}{\|u^+\|_2^2 + \|u^-\|_2^2}\\ &\le \frac{2}{\|u^+\|_2^2 + \|u^-\|_2^2} \max\{\|\nabla u^+\|_2^2, \|\nabla u^-\|_2^2\}.\end{aligned}$$

By the definition of μ and (A5), one has $\lambda_1 \le \mu < +\infty$.

Remark 1.1. Using (A1) and (A2), it is obvious that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1} \quad \text{and} \quad |F(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p \tag{1.6}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $2 < p < 2^* = \frac{2N}{N-2}$. Furthermore, (A1) and (A3) imply

$$\frac{1}{2} f(x,t)t > F(x,t) > 0, \quad \forall t \neq 0,\ x \in \Omega. \tag{1.7}$$

It follows from (A1)–(A3) and (A5) that

$$\frac{f_1(x,t)}{|t|} \to -V_\infty(x) < 0 \quad \text{as } |t| \to 0,$$

$$t \mapsto \frac{f_1(x,t)}{|t|} \text{ is negative, strictly increasing on } (-\infty,0) \cup (0,\infty),$$

which, together with $f_1(x,t) = o(|t|)$ as $|t| \to \infty$ uniform in x, yields

$$tf_1(x,t) < 0, \quad \forall t \neq 0. \tag{1.8}$$

Theorem 1.2. *Assume* (A1)–(A5) *are satisfied. Then* (1.1) *has a sign-changing solution* $u \in \mathcal{M}$ *such that* $\Phi(u) = \inf_{\mathcal{M}} \Phi > 0$*, which has precisely two nodal domains.*

Now, we give an example to illustrate the feasibility of assumptions (A1)–(A5). Let

$$F(x,t) = \frac{V_\infty(x)}{2} t^2 \big(1 - \frac{1}{1+|t|^\alpha}\big), \forall\ x \in \Omega,\ t \in \mathbb{R},$$

where $\alpha \in (0,2)$, $V_\infty \in C(\Omega)$, $\inf_\Omega V_\infty > \mu$. By elementary computations, it is easy to check that f satisfies (A1)–(A5).

The main tools this article are the minimization argument and the quantitative deformation lemma. We must point out that the difficulty in proving Theorem 1.2 is to show that $\mathcal{M} \neq \emptyset$ and the minimizer is a critical point of Φ.

This article organized as follows. In Section 2, we prove several preliminary lemmas. The proof of Theorem 1.2 will be given in the last section.

2. Preliminaries

In this section, we prove that the minimizer of the energy functional Φ under the constraint $\mathcal{M}$ is a critical point. To this end, we show $\mathcal{M} \neq \emptyset$ with the aid of an important behavior of strictly increasing functions.

Lemma 2.1 ([20, Lemma 2.3]). *Suppose that $h(x,t)$ is strictly increasing in $t \in \mathbb{R}$ and $h(x,0)=0$ for any $x \in \mathbb{R}^N$. Then*

$$\frac{1-\theta^2}{2}h(x,\tau)\tau|\tau| > \int_{\theta\tau}^{\tau} h(x,s)|s|\mathrm{d}s, \quad \forall \theta \in [0,1)\cup(1,\infty),\ \tau \in \mathbb{R}\backslash\{0\}.$$

Lemma 2.2. *Suppose that (A1)–(A3) are satisfied. Then for any $u = u^+ + u^- \in H_0^1(\Omega)$ with $u^\pm \neq 0$, $s,\ t \geq 0$ and $(s-1)^2 + (t-1)^2 \neq 0$,*

$$\Phi(u) > \Phi(su^+ + tu^-) + \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle. \tag{2.1}$$

Proof. For any $x \in \Omega$, from (A3) and Lemma 2.1 it follows that

$$\frac{1-\theta^2}{2}f(x,\tau)\tau > \int_{\theta\tau}^{\tau} f(x,\xi)\mathrm{d}\xi, \quad \forall \theta \in [0,1)\cup(1,\infty),\ \tau \in \mathbb{R}\backslash\{0\}. \tag{2.2}$$

By (1.4), (1.5) and (2.2), for any $u = u^+ + u^- \in H_0^1(\Omega)$ with $u^\pm \neq 0$, $s,\ t \geq 0$ and $(s-1)^2 + (t-1)^2 \neq 0$, we have

$$\begin{aligned}
&\Phi(u) - \Phi(su^+ + tu^-)\\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x + \int_\Omega F(x, su^+ + tu^-)\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla(su^+ + tu^-)|^2\mathrm{d}x\\
&= \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle + \int_\Omega \Big[\frac{1-t^2}{2}f(x,u^-)u^-\\
&\quad - \int_{tu^-}^{u^-} f(x,\xi)\mathrm{d}\xi\Big]\mathrm{d}x + \int_\Omega \Big[\frac{1-s^2}{2}f(x,u^+)u^+ - \int_{su^+}^{u^+} f(x,\xi)\mathrm{d}\xi\Big]\mathrm{d}x\\
&> \frac{1-s^2}{2}\langle \Phi'(u), u^+\rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^-\rangle.
\end{aligned}$$

This shows that (2.1) holds. $\square$

From Lemma 2.2, we have the following two corollaries.

Corollary 2.3. *Suppose that* (A1)–(A3) *are satisfied. Then for any $u = u^+ + u^- \in \mathcal{M}$,*

$$\Phi(u) \geq \Phi(su^+ + tu^-), \quad \forall s,t \geq 0.$$

Corollary 2.4. *Suppose that* (A1)–(A3) *are satisfied. Then for any $u = u^+ + u^- \in \mathcal{M}$,*

$$\Phi(u^+ + u^-) = \max_{s,t\geq 0} \Phi(su^+ + tu^-).$$

Define the set

$$E_0 = \big\{u \in H_0^1(\Omega) : \|\nabla u^\pm\|_2^2 - \int_\Omega V_\infty(x)|u^\pm|^2\mathrm{d}x < 0\big\}. \tag{2.3}$$

Lemma 2.5. *Suppose that* (A1)–(A3), (A5) *are satisfied. Then $E_0 \neq \emptyset$ and $\mathcal{M} \subset E_0$.*

Proof. In view of (A5), the definition of μ implies that there exists $v \in \Pi$ such that

$$\max\{\|\nabla v^+\|_2^2, \|\nabla v^-\|_2^2\} \le \mu + \frac{\inf_\Omega V_\infty - \mu}{2} = \frac{\inf_\Omega V_\infty + \mu}{2}.$$

It follows that

$$\begin{aligned}\|\nabla v^\pm\|_2^2 - \int_\Omega V_\infty(x)|v^\pm|^2 \mathrm{d}x &\le \max\{\|\nabla v^+\|_2^2, \|\nabla v^-\|_2^2\} - \inf_\Omega V_\infty \\ &\le \frac{\mu - \inf_\Omega V_\infty}{2} < 0.\end{aligned}$$

Hence, we have $v \in E_0$. This shows that $E_0 \neq \emptyset$ because of (A5). Moreover, by (1.5) and (1.8), we can easily derive that for any $u \in \mathcal{M}$,

$$\|\nabla u^\pm\|_2^2 - \int_\Omega V_\infty(x)|u^\pm|^2 \mathrm{d}x = \int_\Omega f_1(x, u^\pm)u^\pm \mathrm{d}x < 0.$$

This shows that $\mathcal{M} \subset E_0$. □

Lemma 2.6. *Suppose that* (A1)–(A3), (A5) *are satisfied. If $u \in E_0$, then there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$.*

Proof. Let

$$g_1(s) = s^2 \int_\Omega |\nabla u^+|^2 \mathrm{d}x - \int_\Omega f(x, su^+)su^+ \mathrm{d}x, \tag{2.4}$$

$$g_2(t) = t^2 \int_\Omega |\nabla u^-|^2 \mathrm{d}x - \int_\Omega f(x, tu^-)tu^- \mathrm{d}x. \tag{2.5}$$

Clearly, $g_1(0) = g_2(0) = 0$. Using (A1), (A2), (1.6) and (2.3), we conclude that $g_1(s) > 0$ for $s > 0$ small, and

$$\begin{aligned}g_1(s) &= s^2 \int_\Omega |\nabla u^+|^2 \mathrm{d}x - \int_\Omega f(x, su^+)su^+ \mathrm{d}x \\ &= s^2 \int_\Omega [|\nabla u^+|^2 - V_\infty(x)|u^+|^2] \mathrm{d}x - \int_\Omega \frac{f_1(x, su^+)}{su^+}(su^+)^2 \mathrm{d}x < 0\end{aligned}$$

for s large. From the continuity of $g_1(\cdot)$, there is a $s_u > 0$ such that $g_1(s_u) = 0$. Using (A3), it is easy to verify that s_u is unique. Then it follows from (1.5) and (2.4) that $\langle \Phi'(s_u u^+), u^+ \rangle = 0$. Similarly, there is a unique $t_u > 0$ such that $g_2(t_u) = 0$, and so $\langle \Phi'(t_u u^-), u^- \rangle = 0$. □

Lemma 2.7. *Suppose that* (A1)–(A3), (A5) *satisfied. Then*

$$\inf_{u \in \mathcal{M}} \Phi(u) = m = \inf_{u \in E_0} \max_{s,t \ge 0} \Phi(su^+ + tu^-).$$

Combining Corollary 2.4, Lemmas 2.5 and 2.6, we obtain the proof of the above lemma.

Lemma 2.8. *Suppose that* (A1)–(A5) *are satisfied. Then $m > 0$ is achieved.*

Proof. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. Next, we prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. By Sobolev imbedding theorem, passing to a subsequence, we may assume that there exists $v \in H_0^1(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$, $v_n \to v$ strongly in $L^s(\Omega)$, $2 \le s < 2^*$. If $v = 0$, then $v_n \to 0$ in $L^s(\Omega)$, $2 \le s < 2^*$. Fix $R > [2(1+m)]^{1/2}$, by (1.6), one has

$$\limsup_{n\to\infty} \int_\Omega F(x, Rv_n)\mathrm{d}x \le R^2\varepsilon \lim_{n\to\infty} \|v_n\|_2^2 + R^p C_\varepsilon \lim_{n\to\infty} \|v_n\|_p^p = 0. \tag{2.6}$$

Let $t_n = R/\|u_n\|$. Then by (2.6) and Corollary 2.3, one has

$$\begin{aligned} m &= \Phi(u_n) + o(1) \ge \Phi(t_n u_n) + o(1) \\ &= \frac{t_n^2}{2}\|u_n\|^2 - \int_\Omega F(x, t_n u_n)\mathrm{d}x + o(1) \\ &= \frac{R^2}{2} - \int_\Omega F(x, Rv_n)\mathrm{d}x + o(1) \\ &= \frac{R^2}{2} + o(1) > m + 1 + o(1), \end{aligned}$$

which is a contradiction. Thus $v \ne 0$.

For $x \in \Omega_0 := \{y \in \Omega : \ v(y) \ne 0\}$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Thus, from (1.4), (1.5), (A3), (A4) and Fatou's lemma it follows that

$$m + 1 \ge \lim_{n\to\infty} \big[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\big] \ge \liminf_{n\to\infty} \int_{\Omega_0} \widetilde{F}(x, u_n)\mathrm{d}x = +\infty.$$

This contradiction shows that $\{\|u_n\|\}$ is bounded. Hence, passing to a subsequence, there exists $\widetilde{u} \in H_0^1(\Omega)$ such that $u_n^\pm \rightharpoonup \widetilde{u}^\pm$ weakly in $H_0^1(\Omega)$, $u_n^\pm \to \widetilde{u}^\pm$ strongly in $L^s(\Omega)$, $2 \le s < 2^*$. Since $u_n \in \mathcal{M}$, we have $\langle \Phi'(u_n), u_n^\pm\rangle = 0$. In view of (1.6) and Sobolev embedding theorem, there exists $C_1 > 0$ such that

$$\|u_n^\pm\|^2 = \int_\Omega f(x, u_n^\pm)u_n^\pm \mathrm{d}x \le \frac{1}{2}\|u_n^\pm\|^2 + C_1\|u_n^\pm\|^2\|u_n^\pm\|_p^{p-2},$$

which implies

$$\int_\Omega |u_n^\pm|^p \mathrm{d}x \ge (\frac{1}{2C_1})^{\frac{p}{p-2}}.$$

By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \le s < 2^*$, we obtain

$$\int_\Omega |\widetilde{u}^\pm|^p \mathrm{d}x \ge (\frac{1}{2C_1})^{\frac{p}{p-2}}.$$

Thus, $\widetilde{u}^\pm \ne 0$. Moreover, (A1), (A2) and [24, A.2] imply

$$\lim_{n\to\infty} \int_\Omega f(x, u_n^\pm)u_n^\pm \mathrm{d}x = \int_\Omega f(x, \widetilde{u}^\pm)\widetilde{u}^\pm \mathrm{d}x, \tag{2.7}$$

$$\lim_{n\to\infty} \int_\Omega F(x, u_n^\pm)\mathrm{d}x = \int_\Omega F(x, \widetilde{u}^\pm)\mathrm{d}x, \tag{2.8}$$

$$\lim_{n\to\infty} \int_\Omega f_1(x, u_n^\pm)u_n^\pm \mathrm{d}x = \int_\Omega f_1(x, \widetilde{u}^\pm)\widetilde{u}^\pm \mathrm{d}x. \tag{2.9}$$

From (1.8), (2.9) and the weak semicontinuity of norm, we have

$$\begin{aligned} \|\nabla\widetilde{u}^\pm\|_2^2 - \int_\Omega V_\infty(x)|\widetilde{u}^\pm|^2\mathrm{d}x &\le \liminf_{n\to\infty} \big\{\|\nabla u_n^\pm\|_2^2 - \int_\Omega V_\infty(x)|u_n^\pm|^2\mathrm{d}x\big\} \\ &= \liminf_{n\to\infty} \int_\Omega f_1(x, u_n^\pm)u_n^\pm \mathrm{d}x \\ &= \int_\Omega f_1(x, \widetilde{u}^\pm)\widetilde{u}^\pm \mathrm{d}x < 0, \end{aligned}$$

which shows that $\widetilde{u} \in E_0$. By Lemma 2.6, there exist $s_0 > 0$ and $t_0 > 0$ such that $s_0\widetilde{u}^+ + t_0\widetilde{u}^- \in \mathcal{M}$ and $\Phi(s_0\widetilde{u}^+ + t_0\widetilde{u}^-) \geq m$. By (1.5), (2.7) and the weak semicontinuity of norm, we have

$$\begin{aligned}\langle \Phi'(\widetilde{u}), \widetilde{u}^{\pm}\rangle &= \|\widetilde{u}^{\pm}\|^2 - \int_\Omega f(x,\widetilde{u}^{\pm})\widetilde{u}^{\pm}\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \big\{\|u_n^{\pm}\|^2 - \int_\Omega f(x,u_n^{\pm})u_n^{\pm}\mathrm{d}x\big\} = 0.\end{aligned} \tag{2.10}$$

From (1.4), (1.5), (2.1), (2.7), (2.10), Fatou's Lemma and Lemma 2.7 it follows that

$$\begin{aligned} m &= \lim_{n\to\infty}\big[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\big] = \lim_{n\to\infty}\int_\Omega \big[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\big]\mathrm{d}x \\ &= \lim_{n\to\infty}\int_\Omega \big[\frac{1}{2}f(x,\widetilde{u})\widetilde{u} - F(x,\widetilde{u})\big]\mathrm{d}x = \Phi(\widetilde{u}) - \frac{1}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}\rangle \\ &\geq \Phi(s_0\widetilde{u}^+ + t_0\widetilde{u}^-) + \frac{1-s_0^2}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}^+\rangle + \frac{1-t_0^2}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}^-\rangle - \frac{1}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}\rangle \\ &\geq m - \frac{s_0^2}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}^+\rangle - \frac{t_0^2}{2}\langle \Phi'(\widetilde{u}), \widetilde{u}^-\rangle. \end{aligned}$$

This implies that $\widetilde{u} \in \mathcal{M}$ and $\Phi(\widetilde{u}) = m$. □

Lemma 2.9. *Suppose that* (A1)–(A5) *are satisfied. If $\hat{u} \in \mathcal{M}$ and $\Phi(\hat{u}) = m$, then $\hat{u}$ is a critical point of Φ.*

Proof. Assume that $\hat{u} = \hat{u}^+ + \hat{u}^- \in \mathcal{M}$, $\Phi(\hat{u}) = m$ and $\Phi'(\hat{u}) \neq 0$. Then there exist $\delta > 0$ and $\lambda > 0$ such that

$$\|\Phi'(u)\| \geq \lambda, \quad \text{for all } \|u - \hat{u}\| \leq 3\delta \text{ and } u \in H_0^1(\Omega).$$

Let $D = (1/2, 3/2) \times (1/2, 3/2)$. It follows from Lemma 2.2 that

$$\chi := \max_{(s,t)\in\partial D} \Phi(s\hat{u}^+ + t\hat{u}^-) < m. \tag{2.11}$$

For $\varepsilon := \min\{(m-\chi)/3, \lambda\delta/8\}$, $S := B(\hat{u}, \delta)$, [24, Lemma 2.3] yields a deformation $\eta \in C([0,1] \times H_0^1(\Omega))$ such that

(i) $\eta(1,u) = u$ if $\Phi(u) < m - 2\varepsilon$ or $\Phi(u) > m + 2\varepsilon$;
(ii) $\eta(1, \Phi^{m+\varepsilon} \cap B(\hat{u},\delta)) \subset \Phi^{m-\varepsilon}$;
(iii) $\Phi(\eta(1,u)) \leq \Phi(u)$ for all $u \in H_0^1(\Omega)$.

We claim that

$$\max_{(s,t)\in\overline{D}} \Phi(\eta(1, s\hat{u}^+ + t\hat{u}^-)) < m. \tag{2.12}$$

Indeed, by Lemma 2.2 and (iii), we have

$$\Phi(\eta(1, s\hat{u}^+ + t\hat{u}^-)) \leq \Phi(s\hat{u}^+ + t\hat{u}^-) < \Phi(\hat{u}) = m, \tag{2.13}$$

for all $s, t \geq 0$, $|s-1|^2 + |t-1|^2 \geq \delta^2/\|\hat{u}\|^2$.

On the other hand, by Corollary 2.4, we have $\Phi(s\hat{u}^+ + t\hat{u}^-) \leq \Phi(\hat{u}) = m$ for $s, t \geq 0$, then it follows from (ii) that

$$\Phi(\eta(1, s\hat{u}^+ + t\hat{u}^-)) \leq m - \varepsilon, \quad \forall s,t \geq 0,\ |s-1|^2 + |t-1|^2 < \delta^2/\|\hat{u}\|^2. \tag{2.14}$$

Both (2.13) and (2.14) imply that (2.12) holds. Define $h(s,t) = s\hat{u}^+ + t\hat{u}^-$. We now prove that $\eta(1, h(D)) \cap \mathcal{M} \neq \emptyset$, contradicting to the definition of m. We adopt the idea from [16]. Let $\beta(s,t) := \eta(1, h(s,t))$ and

$$\Psi_0(s,t) := \big(\Phi'(h(s,t))\hat{u}^+, \Phi'(h(s,t))\hat{u}^-\big),$$

$$\Psi_1(s,t) := \big(\frac{1}{s}\Phi'(\beta(s,t))(\beta(s,t))^+, \frac{1}{t}\Phi'(\beta(s,t))(\beta(s,t))^-\big).$$

By Lemma 2.6 and degree theory, we can derive that $\deg(\Psi_0, D, 0) = 1$. From (2.11) and (i) it follows that $\beta = h$ on ∂D. Consequently, $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$, and so, $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, that is $\eta(1, h(s_0, t_0)) = \beta(s_0, t_0) \in \mathcal{M}$, which contradicts (2.12). From this, we conclude that $\hat{u}$ is a critical point of Φ. □

3. Sign-changing solutions

Proof of Theorem 1.2. In view of Lemmas 2.8 and 2.9, there exists a $u \in \mathcal{M}$ such that $\Phi(u) = m$ and $\Phi'(u) = 0$. Now, we show that u has exactly two nodal domains. Set $u = u_1 + u_2 + u_3$ and $\langle \Phi'(u), u_i\rangle = 0$ $(i = 1, 2, 3)$, where

$$\begin{gathered} u_1 \geq 0, \quad u_2 \leq 0, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad u_3|_{\Omega_1\cup\Omega_2} = 0, \\ \Omega_1 := \{x \in \Omega : u_1(x) > 0\}, \quad \Omega_2 := \{x \in \Omega : u_2(x) < 0\}, \end{gathered} \tag{3.1}$$

and Ω_1, Ω_2 are connected open subsets of Ω.

Let $v = u_1 + u_2$, then $v^+ = u_1$, $v^- = u_2$, $v^\pm \neq 0$ and $\langle \Phi'(v), v^\pm\rangle = 0$. By (1.4), (1.5), (1.7), (2.1), (3.1) and Lemma 2.7, we have

$$\begin{aligned} m = \Phi(u) &= \Phi(u) - \frac{1}{2}\langle \Phi'(u), u\rangle \\ &= \Phi(v) + \Phi(u_3) - \frac{1}{2}[\langle \Phi'(v), v\rangle + \langle \Phi'(u_3), u_3\rangle] \\ &\geq \sup_{s,t\geq 0}\big\{\Phi(sv^+ + tv^-) + \frac{1-s^2}{2}\langle \Phi'(v), v^+\rangle + \frac{1-t^2}{2}\langle \Phi'(v), v^-\rangle\big\} \\ &\quad + \Phi(u_3) - \frac{1}{2}[\langle \Phi'(v), v\rangle + \langle \Phi'(u_3), u_3] \\ &= \sup_{s,t\geq 0}\Phi(sv^+ + tv^-) + \int_\Omega \big[\frac{1}{2}f(x, u_3)u_3 - F(x, u_3)\big]\mathrm{d}x \geq m, \end{aligned}$$

which shows that $u_3 = 0$. Therefore, u has exactly two nodal domains. □

Acknowledgements. This work is partially supported by the National Natural Science Foundation of China (No: 11571370). The authors thank the anonymous referees for their valuable suggestions and comments.

References

[1] D. Anderson, M. Bonnedal; *Variational approach to nonlinear self-focusing of Gaussian laser beams*, Physics of Fluids (1958-1988) 22(1) (1979), 105-109.
[2] T. Bartsch, Z. L. Liu, T. Weth; *Sign changing solutions of superlinear Schrödinger equations*, Commun. Partial Differential Equations, 29 (2004), 25-42.
[3] T. Bartsch, Z. Q. Wang; *Existence and multiplicity results for some superlinear elliptic problemson $\mathbb{R}^N$*, Comm. PDE., 20 (1995), 1725-1741.
[4] T. Bartsch, T. Weth; *Three nodal solutions of singularly perturbed elliptic equations on domains without topology.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), 259-281.
[5] A. Castro, J. Cossio, J. M. Neuberger; *A sign-changing solution for a superlinear Dirichlet problem*, 1997.
[6] Y. Chen; *Self-trapped light in saturable nonlinear media*, Optics letters, 16(1) (1991), 4-6.
[7] S. T. Chen, X. H. Tang; *Semiclassical solutions for linearly coupled Schrodinger equations*, Electron. J. Differ Eq., 251 (2014), 1-11.
[8] R. Y. Chiao, E. Garmire, C. H. Townes; *Self-trapping of optical beams*, Phys. Rev. Lett., 13 (1964), 479-482.

[9] S. Gatz, J. Herrmann; *Soliton propagation in materials with saturable nonlinearity*, JOSA B, 8(11) (1991), 2296-2302.

[10] M. Karlsson; *Optical beams in saturable self-focusing media*, Physical Review A, 46(5) (1992), 2726.

[11] E. S. Noussair, J. Wei; *On the effect of the domain geometry on the existence and profile of nodal solution of some singularly perturbed semilinear Dirichlet problem*, Indiana Univ. Math. J., 46 (1997), 1321-1332.

[12] D. D. Qin, X. H. Tang; *Asymptotically linear Schrödinger equation with zero on the boundary of the spectrum*, Electronic J. Differ. Equ., 213 (2015), 1-15.

[13] D. D. Qin, X. H. Tang, J. Zhang; *Multiple solutions for semilinear elliptic equations with sign-changing potential and nonlinearity*, Electron. J. Differential Equations, 207 (2013), 1-9.

[14] A. Salvatore; *Sign-changing solutions for an asympotically linear Schrödinger equation*, Discrete Contin Dyn Syst Suppl., (2009), 669-677.

[15] A. Szulkin, T. Weth; *Ground state solutions for some indefinite variational problems*, J. Func. Anal., 257 (2009), 3802-3822.

[16] W. Shuai; *Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains*, J. Differential Equations, 259.4 (2015), 1256-1274.

[17] M. S. Sodha, A. K. Ghatak, V. K. Tripathi; *Self Focusing of Laser Beams in Plasmas and Semiconductors*, Progress in optics, 13 (1976), 169-265.

[18] M. Struwe; *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems*, Berlin, Springer, 1990.

[19] C. A. Stuart; *Guidance properties of nonlinear planar waveguides*, Archive for rational mechanics and analysis, 125(2) (1993), 145-200.

[20] X. H. Tang; *Non-Nehari manifold method for asymptotically linear Schrödinger equation*, J. Aust. Math. Soc., 98 (2015), 104-116.

[21] X. H. Tang; *Infinitely many solutins for semilinear Schrödinger equation with sign-changing potential and nonlinearity*, J. Math. Anal. Appl., 401 (2013), 407-415.

[22] X. H. Tang; *Non-Nehari manifold method for asymptotically periodic Schrödinger equations*, Science China Math., 58 (2015), 715-728.

[23] T. Weth; *Energy bounds for entire nodal solutions of autonomous superlinear equations*, Calc. Var. Partial Differential Equations, 27 (2006), 421-437.

[24] M. Willem; *Minimax Theorems*, Birkhäuser, Boston, 1996.

[25] J. Zhang X. H. Tang, W. Zhang; *Existence and multiplicity of stationary solutions for a class of Maxwell-Dirac system*, Nonlinear Anal., 127 (2015) 298-311.

[26] J. Zhang X. H. Tang, W. Zhang; *Ground state solutions for a class of nonlinear Maxwell-Dirac system*, Topol. Meth. Nonl. Anal., 46 (2015), 785-798.

[27] W. M. Zou; *Sign-Changing Critical Point Theory*, Springer, New York, 2008.

Sitong Chen
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: `mathsitongchen@163.com`

Yinbin Li
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: `liyinbin1991@163.com`

Xianhua Tang
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: `tangxh@mail.csu.edu.cn`