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ENTIRE FUNCTIONS THAT SHARE A SMALL FUNCTION WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this article, we study the uniqueness of entire functions that share small functions of finite order with their difference operators. In particular, we give a generalization of results in [3, 4, 13].

1. Introduction and statement of results

In this article, by meromorphic functions we mean meromorphic functions in the complex plane. In what follows, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions [9, 11, 17]. In addition, we will use $\rho(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros of f, we say that a meromorphic function $\varphi(z)$ is a small function of $f(z)$ if $T(r,\varphi) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z+c)$ and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N},\ n \geq 2.$$

In particular, $\Delta_c^n f(z) = \Delta^n f(z)$ for the case $c = 1$.

Let f and g be two meromorphic functions and let a be a finite nonzero value. We say that f and g share the value a CM provided that $f - a$ and $g - a$ have the same zeros counting multiplicities. Similarly, we say that f and g share a IM provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. Rubel and Yang [15] proved that if an entire function f shares two distinct complex numbers CM with its derivative f', then $f \equiv f'$. In 1986, Jank et al [10] proved that for a nonconstant meromorphic function f, if f, f' and f'' share a finite nonzero value CM, then $f' \equiv f$. This result suggests the following question:

> Question 1 in [17]. Let f be a nonconstant meromorphic function, let a be a finite nonzero constant, and let n and m $(n < m)$ be

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positive integers. If f, $f^{(n)}$ and $f^{(m)}$ share a CM, then can we get the result $f^{(n)} \equiv f$?

The following example from [18] shows that the answer to the above question is, in general, negative. Let n and m be positive integers satisfying $m > n+1$, and let b be a constant satisfying $b^n = b^m \neq 1$. Set $a = b^n$ and $f(z) = e^{bz} + a - 1$. Then f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, and $f^{(n)} \not\equiv f$. However, when f is an entire function of finite order and $m = n+1$, the answer to Question 1 is positive. In fact, P. Li and C. C. Yang proved the following:

Theorem 1.1 ([14]). *Let f be a nonconstant entire function, let a be a finite nonzero constant, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f \equiv f'$.*

Recently several papers have focussed on the Nevanlinna theory with respect to difference operators see, e.g. [1, 5, 7, 8]. Many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts or difference operators. Chen et al [3, 4] proved a difference analogue of result of Jank et al and obtained the following results.

Theorem 1.2 ([3]). *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Theorem 1.3 ([4]). *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ ($n \geq 2$) share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^n f$.*

Theorem 1.4 ([4]). *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ share 0 CM, then $\Delta_c^n f(z) = C\Delta_c f(z)$, where C is a nonzero constant.*

Recently Latreuch et al [13] proved the following results.

Theorem 1.5 ([13]). *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ ($n \geq 1$) share $a(z)$ CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Theorem 1.6 ([13]). *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) = C\Delta_c^n f(z)$, where C is a nonzero constant.*

For the case $n = 1$, El Farissi and others gave the following result.

Theorem 1.7 ([6]). *Let $f(z)$ be a non-periodic entire function of finite order, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

We remark that Theorem 1.7 is essentially known in [6]. For the convenience of readers, we give his proof in the Lemma 2.4. Now It is natural to ask the following question:

Under the hypotheses of Theorem 1.5, can we obtain $\Delta_c f(z) \equiv f(z)$?

The aim of this article is to answer this question and to give a difference analogue of result of Li and Yang [14]. In fact we obtain the following results:

Theorem 1.8. *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z) \in S(f)$ $(\not\equiv 0)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

The condition $\Delta_c^n f(z) \not\equiv 0$ is necessary. Let us take for example the entire function $f(z) = 1 + e^{2\pi i z}$ and $c = a = 1$, then $f - a$ and $\Delta^n f - a = \Delta^{n+1} f - a = -1$ have the same zeros but $\Delta f \neq f$. On the other hand, under the conditions of Theorem 1.8, $\Delta_c^n f(z) \not\equiv 0$ can not be a periodic entire function with periodic c because $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ [13, Theorem 1.5].

Example 1.9. Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share a CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c f(z) \equiv f(z)$. This example satisfies Theorem 1.8.

Theorem 1.10. *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z)$, $b(z) \in S(f)$ $(\not\equiv 0)$ such that $b(z)$ is a periodic entire function with period c and $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$. If $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $\Delta_c f(z) \equiv f(z) + b(z) + \Delta_c a(z) - a(z)$.*

The condition $b(z) \not\equiv 0$ is necessary in the proof of Theorem 1.10, for the case $b(z) \equiv 0$, please see Theorem 1.17. The condition $\Delta_c^m a(z) \equiv 0$ in Theorem 1.10 is more general than the condition "periodic entire function of period c". For the case $m = 1$, we deduce the following result.

Corollary 1.11. *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z)$, $b(z) \in S(f)$ $(\not\equiv 0)$ be periodic entire functions with period c. If $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $\Delta_c f(z) \equiv f(z) + b(z) - a(z)$.*

Example 1.12. Let $f(z) = e^{z \ln 2} - 2$, $a = -1$ and $b = 1$. It is clear that $f(z) - a$, $\Delta^n f(z) - b$ and $\Delta^{n+1} f(z) - b$ share 0 CM. Here, we also get $\Delta f(z) = f(z) + b - a$.

Example 1.13. Let $f(z) = e^{z \ln 2} + z^3 - 1$, $a(z) = z^3$ and $b = 1$. It is clear that $f(z) - z^3$, $\Delta^4 f(z) - 1$ and $\Delta^5 f(z) - 1$ share 0 CM. On the other hand, we can verify that $\Delta f(z) = f(z) + 1 + \Delta z^3 - z^3$ which satisfies Theorem 1.10.

Theorem 1.14. *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c f(z) \equiv C f(z)$, where C is a nonzero constant.*

Example 1.15. Let $f(z) = e^{az}$ and $c = 1$ where $a \neq 2k\pi i$ $(k \in \mathbb{Z})$, it is clear that $\Delta_c^n f(z) = (e^a - 1)^n e^{az}$ for any integer $n \geq 1$. So, $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c f(z) \equiv C f(z)$ where $C = e^a - 1$. This example satisfies Theorem 1.14.

Corollary 1.16. *Let $f(z)$ be a nonconstant entire function of finite order such that $f(z)$, $\Delta_c^n f(z)$ $(\not\equiv 0)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share 0 CM. If there exists a point z_0 and an integer $m \geq 1$ such that $\Delta_c^m f(z_0) = f(z_0) \neq 0$, then $\Delta_c^m f(z) \equiv f(z)$.*

By combining Theorem 1.10 and Theorem 1.14 we can prove the following result.

Theorem 1.17. *Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z) \in S(f)$ such that $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$. If $f(z) - a(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c f(z) \equiv C f(z) + \Delta_c a(z) - a(z)$, where C is a nonzero constant.*

2. Some lemmas

Lemma 2.1 ([5]). *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) = O(r^{\sigma-1+\varepsilon}).$$

By combining [2, Theorem 1.4] and [12, Theorem 2.2], we can prove the following lemma.

Lemma 2.2. *Let $a_0(z), a_1(z), \dots, a_n(z) (\not\equiv 0)$, $F(z)$ $(\not\equiv 0)$ be finite order meromorphic functions, c_k $(k = 0, \dots, n)$ be constants, unequal to each other. If f is a finite order meromorphic solution of the equation*

$$a_n(z)f(z+c_n) + \dots + a_1(z)f(z+c_1) + a_0(z)f(z+c_0) = F(z) \tag{2.1}$$

with

$$\max\{\rho(a_i), (i = 0, \dots, n), \rho(F)\} < \rho(f),$$

then $\lambda(f) = \rho(f)$.

Proof. By (2.1) we have

$$\frac{1}{f(z+c_0)} = \frac{1}{F}\Big(a_n \frac{f(z+c_n)}{f(z+c_0)} + \dots + a_1 \frac{f(z+c_1)}{f(z+c_0)} + a_0\Big). \tag{2.2}$$

Set $\max\{\rho(a_j)$ $(j = 0, \dots, n), \rho(F)\} = \beta < \rho(f) = \rho$. Then, for any given ε $(0 < \varepsilon < \frac{\rho-\beta}{2})$, we have

$$\sum_{j=0}^{n} T(r, a_j) + T(r, F) \leq (n+2)\exp\{r^{\beta+\varepsilon}\} = o(1)\exp\{r^{\rho-\varepsilon}\}. \tag{2.3}$$

By (2.2), (2.3) and Lemma 2.1, we obtain

$$\begin{aligned}
T(r,f) &= T\big(r, \frac{1}{f}\big) + O(1) \\
&= m(r, \frac{1}{f}) + N\big(r, \frac{1}{f}\big) + O(1) \\
&\leq N\big(r, \frac{1}{f}\big) + m\big(r, \frac{1}{F}\big) + \sum_{j=0}^{n} m(r, a_j) \\
&\quad + \sum_{j=1}^{n} m(r, \frac{f(z+c_j)}{f(z+c_0)}) + O(1) \\
&\leq N\big(r, \frac{1}{f}\big) + T(r, \frac{1}{F}) + \sum_{j=0}^{n} T(r, a_j) + \sum_{j=1}^{n} m(r, \frac{f(z+c_j)}{f(z+c_0)}) + O(1) \\
&\leq N\big(r, \frac{1}{f}\big) + O(r^{\rho-1+\varepsilon}) + o(1)\exp\{r^{\rho-\varepsilon}\}.
\end{aligned} \tag{2.4}$$

From this this inequality we obtain that $\rho(f) \leq \lambda(f)$ and since $\lambda(f) \leq \rho(f)$ for every meromorphic function, we deduce that $\lambda(f) = \rho(f)$. □

Recently, Wu and Zheng [16] obtained Lemma 2.2 by using a different proof.

Lemma 2.3 ([17]). *Suppose $f_j(z)$ $(j=1,2,\dots,n+1)$ and $g_j(z)$ $(j=1,2,\dots,n)$ $(n\geq 1)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$;

(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1\leq j\leq n+1$, $1\leq k\leq n$. Furthermore, the order of $f_j(z)$ is less than the order of $e^{g_h(z)-g_k(z)}$ for $n\geq 2$ and $1\leq j\leq n+1$, $1\leq h<k\leq n$.*

Then $f_j(z)\equiv 0$, $(j=1,2,\dots n+1)$.

Lemma 2.4 ([6]). *Let $f(z)$ be a non-periodic entire function of finite order, and let $a(z)\in S(f)$ $(\not\equiv 0)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z)\equiv f(z)$.*

Proof. Suppose that $\Delta_c f(z)\not\equiv f(z)$. Since f, $\Delta_c f$ and $\Delta_c^2 f$ share $a(z)$ CM, we have

$$\frac{\Delta_c f(z)-a(z)}{f(z)-a(z)}=e^{P(z)},\quad \frac{\Delta_c^2 f(z)-a(z)}{f(z)-a(z)}=e^{Q(z)}$$

where P $(e^P\not\equiv 1)$ and Q are polynomials. Using Theorem 1.2, we obtain that $\Delta_c^2 f\equiv\Delta_c f$, which means that

$$\alpha(z)=\Delta_c f(z)-f(z) \tag{2.5}$$

is entire periodic function of period c. By (2.5) we have

$$\Delta_c f(z)-a(z)=f(z)-a(z)+\alpha(z),$$

then

$$\frac{\Delta_c f(z)-a(z)}{f(z)-a(z)}=1+\frac{\alpha(z)}{f(z)-a(z)}=e^{P(z)},$$

which is equivalent to

$$f(z)-a(z)=\frac{\alpha(z)}{e^{P(z)}-1}. \tag{2.6}$$

Since $\alpha(z)$ and $a(z)$ are periodic functions of period c, we have

$$\Delta_c f(z)=\alpha(z)\Delta_c(\frac{1}{e^{P(z)}-1}), \tag{2.7}$$

$$\Delta_c^2 f(z)=\alpha(z)\Delta_c^2(\frac{1}{e^{P(z)}-1}). \tag{2.8}$$

We have the following two subcases:

(i) If $P\equiv K$ $(K\neq 2k\pi i,\ K\in\mathbb{Z})$, then by (2.7) we have $\Delta_c f(z)=0$. On the other hand, by using (2.5), (2.6) and $\Delta_c f(z)=0$, we deduce that

$$f(z)-a(z)=\frac{-f(z)}{e^K-1},\quad K\in\mathbb{C}-\{2k\pi i,k\in\mathbb{Z}\}.$$

So,

$$f(z)=\frac{e^K-1}{e^K}a(z).$$

Hence

$$T(r,f)=S(r,f),$$

which is a contradiction.

(ii) If P is nonconstant and since $\Delta_c^2 f(z)=\Delta_c f(z)$, then

$$e^{P_c(z)+P(z)}-3e^{P_{2c}(z)+P(z)}+2e^{P_{2c}(z)+P_c(z)}+e^{P_{2c}(z)}-3e^{P_c(z)}+2e^{P(z)}=0$$

which is equivalent to

$$e^{P_c(z)} + (2e^{\Delta_c P(z)} - 3)e^{P_{2c}(z)} = -e^{\Delta_c P_c(z)+\Delta_c P(z)} + 3e^{\Delta_c P(z)} - 2. \tag{2.9}$$

Since $\deg \Delta_c P = \deg P - 1$, we have

$$\rho(e^{P_c} + (2e^{\Delta_c P} - 3)e^{P_{2c}}) = \rho(-e^{\Delta_c P_c + \Delta_c P} + 3e^{\Delta_c P} - 2) \le \deg P - 1. \tag{2.10}$$

On the other hand,

$$\rho(e^{P_c} + (2e^{\Delta_c P} - 3)e^{P_{2c}}) = \rho(e^{P_c}) = \deg P \tag{2.11}$$

because if we have the contrary

$$\rho(e^{P_c} + (2e^{\Delta_c P} - 3)e^{P_{2c}}) < \rho(e^{P_c}),$$

we obtain the following contradiction

$$\deg P = \rho\Big(\frac{e^{P_c} + (2e^{\Delta_c P} - 3)e^{P_{2c}}}{e^{P_c}}\Big) = \rho(1 + (2e^{\Delta_c P} - 3)e^{\Delta P_c}) \le \deg P - 1.$$

By using (2.10) and (2.11), we obtain $\deg P \le \deg P - 1$ which is a contradiction. This leads to $\Delta_c f(z) = f(z)$. Thus, the proof is complete. $\square$

3. Proof of main results

Proof of the Theorem 1.8. Obviously, suppose that $\Delta_c f(z) \not\equiv f(z)$. By using Theorem 1.5, we have

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = e^{P(z)}, \tag{3.1}$$

$$\frac{\Delta_c^{n+1} f(z) - a(z)}{f(z) - a(z)} = e^{P(z)}, \tag{3.2}$$

where P ($e^P \not\equiv 1$) is a polynomial. We divide into two cases:

Case 1. P is a nonconstant polynomial. Setting now $g(z) = f(z) - a(z)$, from (3.1) and (3.2) we have

$$\Delta_c^n g(z) = e^{P(z)} g(z) + a(z), \tag{3.3}$$

$$\Delta_c^{n+1} g(z) = e^{P(z)} g(z) + a(z). \tag{3.4}$$

By (3.3) and (3.4), we have

$$g_c(z) = 2e^{P-P_c} g(z) + a(z)e^{-P_c}.$$

Using the principle of mathematical induction, we obtain

$$g_{ic}(z) = 2^i e^{P-P_{ic}} g(z) + a(z)(2^i - 1)e^{-P_{ic}}, \quad i \ge 1. \tag{3.5}$$

Now, we can rewrite (3.3) as

$$\begin{aligned}\Delta_c^n g(z) &= \sum_{i=1}^{n} C_n^i (-1)^{n-i} (2^i e^{P-P_{ic}} g(z) + a(z)(2^i - 1)e^{-P_{ic}}) + (-1)^n g(z)\\ &= e^P g(z) + a(z),\end{aligned}$$

which implies

$$\Big(\sum_{i=0}^{n} C_n^i (-1)^{n-i} 2^i e^{P-P_{ic}} - e^P\Big) g(z)$$

$$+ a(z)\Big(\sum_{i=0}^{n} C_n^i(-1)^{n-i}(2^i-1)e^{-P_{ic}}-1\Big)=0.$$

Hence

$$A_n(z)g(z)+B_n(z)=0, \tag{3.6}$$

where

$$\begin{gathered}
A_n(z)=\sum_{i=0}^{n} C_n^i(-1)^{n-i}2^i e^{P-P_{ic}}-e^P,\\
B_n(z)=a(z)\Big(\sum_{i=0}^{n} C_n^i(-1)^{n-i}(2^i-1)e^{-P_{ic}}-1\Big).
\end{gathered}$$

By the same method, we can rewrite (3.4) as

$$A_{n+1}(z)g(z)+B_{n+1}(z)=0, \tag{3.7}$$

where

$$\begin{gathered}
A_{n+1}(z)=\sum_{i=0}^{n+1} C_{n+1}^i(-1)^{n+1-i}2^i e^{P-P_{ic}}-e^P,\\
B_{n+1}(z)=a(z)\Big(\sum_{i=0}^{n+1} C_{n+1}^i(-1)^{n+1-i}(2^i-1)e^{-P_{ic}}-1\Big).
\end{gathered}$$

We can see easily from the equations (3.6) and (3.7) that

$$h(z)=A_n(z)B_{n+1}(z)-A_{n+1}(z)B_n(z)\equiv 0. \tag{3.8}$$

On the other hand, we remark that

$$\begin{aligned}
e^P B_n(z)&=a(z)e^P\Big(\sum_{i=0}^{n} C_n^i(-1)^{n-i}2^i e^{-P_{ic}}-\sum_{i=0}^{n} C_n^i(-1)^{n-i}e^{-P_{ic}}-1\Big)\\
&=a(z)e^P\Big(\sum_{i=0}^{n} C_n^i(-1)^{n-i}2^i e^{-P_{ic}}-1-\Delta_c^n(e^{-P})\Big)\\
&=a(z)(A_n(z)-e^P\Delta_c^n(e^{-P})).
\end{aligned}$$

Then

$$B_n(z)=a(z)(e^{-P}A_n(z)-\Delta_c^n(e^{-P})). \tag{3.9}$$

By the same method, we obtain

$$B_{n+1}(z)=a(z)(e^{-P}A_{n+1}(z)-\Delta_c^{n+1}(e^{-P})). \tag{3.10}$$

Now we return equation (3.8), by using (3.9) and (3.10), we obtain

$$\begin{aligned}
h(z)&=A_n(z)B_{n+1}(z)-A_{n+1}(z)B_n(z)\\
&=A_n(z)[a(z)(e^{-P}A_{n+1}(z)-\Delta_c^{n+1}(e^{-P}))]\\
&\quad -A_{n+1}(z)[a(z)(e^{-P}A_n(z)-\Delta_c^n(e^{-P}))]\\
&=a(z)[A_{n+1}(z)\Delta_c^n(e^{-P})-A_n(z)\Delta_c^{n+1}(e^{-P})]\equiv 0.
\end{aligned}$$

Hence

$$A_{n+1}(z)\Delta_c^n(e^{-P})-A_n(z)\Delta_c^{n+1}(e^{-P})\equiv 0.$$

Therefore,

$$\begin{aligned}&\Delta_c^n(e^{-P})\Big(\sum_{i=0}^{n+1} C_{n+1}^i(-1)^{n+1-i}2^i e^{-P_{ic}}-1\Big)\\&-\Delta_c^{n+1}(e^{-P})\Big(\sum_{i=0}^{n} C_{n}^i(-1)^{n-i}2^i e^{-P_{ic}}-1\Big)=0.\end{aligned}$$

Thus

$$\begin{aligned}&\Delta_c^n(e^{-P})\sum_{i=0}^{n+1} C_{n+1}^i(-1)^{n+1-i}2^i e^{-P_i c}-\Delta_c^{n+1}(e^{-P})\sum_{i=0}^{n} C_{n}^i(-1)^{n-i}2^i e^{-P_{ic}}\\&=\Delta_c^n(e^{-P})-\Delta_c^{n+1}(e^{-P})=\Delta_c^n(2e^{-P}-e^{-P_c}).\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=0}^{n}(\Delta_c^n(e^{-P})C_{n+1}^i(-1)^{n+1-i}-\Delta_c^{n+1}(e^{-P})C_{n}^i(-1)^{n-i})2^i e^{-P_{ic}}\\&+\Delta_c^n(e^{-P})2^{n+1}e^{-P_{(n+1)c}}\\&=\Delta_c^n(2e^{-P}-e^{-P_c}),\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{i=0}^{n}(\Delta_c^n(e^{-P})C_{n+1}^i+\Delta_c^{n+1}(e^{-P})C_{n}^i)(-1)^{n+1-i}2^i e^{P_{(n+1)c}-P_{ic}}\\&+\Delta_c^n(e^{-P})2^{n+1}=e^{P_{(n+1)c}}\Delta_c^n(2e^{-P}-e^{-P_c}).\end{aligned}\tag{3.11}$$

Let us denote

$$\alpha_i(z)=(-1)^{n+1-i}2^i e^{P_{(n+1)c}-P_{ic}},\ i=0,\dots,n$$

and

$$\alpha_{n+1}(z)=e^{P_{(n+1)c}}\Delta_c^n(2e^{-P}-e^{-P_c}).$$

It is clear that $\rho(\alpha_i)\le \deg P-1$ for all $i=0,2,\dots,n+1$. Then (3.11) becomes

$$\begin{aligned}&\sum_{i=0}^{n}(\Delta_c^n(e^{-P})C_{n+1}^i+\Delta_c^{n+1}(e^{-P})C_{n}^i)\alpha_i(z)+\Delta_c^n(e^{-P})2^{n+1}\\&=\Big(\sum_{i=0}^{n} C_{n+1}^i\alpha_i(z)+2^{n+1}\Big)\Delta_c^n(e^{-P})\\&\quad+\Big(\sum_{i=0}^{n} C_{n}^i\alpha_i(z)\Big)\Delta_c^{n+1}(e^{-P})=\alpha_{n+1}(z).\end{aligned}\tag{3.12}$$

For convenience, we denote

$$M(z)=\sum_{i=0}^{n} C_{n+1}^i\alpha_i(z)+2^{n+1},\quad N(z)=\sum_{i=0}^{n} C_{n}^i\alpha_i(z).$$

Then (3.12) is equivalent to

$$\begin{aligned} &M(z)\sum_{i=0}^{n} C_n^i(-1)^{n-i}e^{-P_{ic}} + N(z)\sum_{i=0}^{n+1} C_{n+1}^i(-1)^{n+1-i}e^{-P_{ic}} \\ &= \sum_{i=0}^{n}(C_n^i M(z) - C_{n+1}^i N(z))(-1)^{n-i}e^{-P_{ic}} + N(z)e^{-P_{(n+1)c}} \\ &= \alpha_{n+1}(z). \end{aligned} \tag{3.13}$$

As a conclusion, (3.13) can be written as

$$a_{n+1}(z)e^{-P(z+(n+1)c)} + a_n(z)e^{-P(z+nc)} + \cdots + a_0(z)e^{-P(z)} = \alpha_{n+1}(z), \tag{3.14}$$

where $a_0(z),\dots,a_{n+1}(z)$ and $\alpha_{n+1}(z)$ are entire functions. We distingue the following two subcases.

(i) If $\deg P > 1$, then

$$\max\{\rho(a_i)\ (i=0,\dots,n+1), \rho(\alpha_{n+1})\} < \deg P. \tag{3.15}$$

To prove that $\alpha_{n+1}(z) \not\equiv 0$, it suffices to show that $\Delta_c^n(2e^{-P} - e^{-P_c}) \not\equiv 0$. Suppose the contrary. Thus

$$\sum_{i=0}^{n} C_n^i(-1)^{n-i}(2e^{-P_{ic}} - e^{-P_{(i+1)c}}) \equiv 0. \tag{3.16}$$

The equation (3.16) can be written as

$$\sum_{i=0}^{n+1} b_i e^{-P_{ic}} \equiv 0,$$

where

$$b_i = \begin{cases} 2(-1)^n, & \text{if } i=0 \\ (2C_n^i + C_n^{i-1})(-1)^{n-i}, & \text{if } 1 \le i \le n \\ -1, & \text{if } i = n+1. \end{cases}$$

Since $\deg P = m > 1$, then for any two integers j and k such that $0 \le j < k \le n+1$, we have

$$\rho(e^{-P_{kc}+P_{jc}}) = \deg P - 1.$$

It is clear now that all the conditions of Lemma 2.3 are satisfied. So, by Lemma 2.3 we obtain $b_i \equiv 0$ for all $i = 0,\dots,n+1$, which is impossible. Then, $\alpha_{n+1}(z) \not\equiv 0$. By Lemma 2.2, (3.14) and (3.15) , we deduce that $\lambda(e^P) = \deg P > 1$, which is a contradiction.

(ii) $\deg P = 1$. Suppose now that $P(z) = \mu z + \eta$ ($\mu \neq 0$). Assume that $\alpha_{n+1}(z) \equiv 0$. It easy to see that

$$\Delta_c^n(2e^{-P} - e^{-P_c}) = (2 - e^{-\mu c})\Delta_c^n(e^{-P}).$$

In the following two subcases, we prove that both of $(2 - e^{-\mu c})$ and $\Delta_c^n(e^{-P})$ are not vanishing.

(A) Suppose that $2 = e^{-\mu c}$. Then for any integer i, we have $e^{-i\mu c} = 2^i$ and $e^{-P_{ic}} = 2^i e^{-P}$, applying that on (3.6), we obtain

$$A_n(z) = \sum_{i=0}^{n} C_n^i(-1)^{n-i}2^i e^{-i\mu c} - e^P = 3^n - e^P,$$

$$
\begin{aligned}
B_n(z) &= a(z)\Big(\sum_{i=0}^{n} C_n^i(-1)^{n-i}(2^i-1)e^{-P_{ic}}-1\Big)\\
&= a(z)(\sum_{i=0}^{n} C_n^i(-1)^{n-i}(4^i-2^i)e^{-P}-1) = a(z)((3^n-1)e^{-P}-1).
\end{aligned}
$$

Then

$$
(3^n-e^P)g(z)+a(z)((3^n-1)e^{-P}-1)=0,
$$

which is equivalent to

$$
g(z)=a(z)\frac{e^P-(3^n-1)}{e^P(3^n-e^P)}. \tag{3.17}
$$

By the same argument as before and (3.7), we obtain

$$
g(z)=a(z)\frac{e^P-(3^{n+1}-1)}{e^P(3^{n+1}-e^P)},
$$

which contradicts (3.17).

(B) Suppose now that $\Delta_c^n(e^{-P})\equiv 0$. Then

$$
\begin{aligned}
\Delta_c^n(e^{-P}) &= \sum_{i=0}^{n} C_n^i(-1)^{n-i}e^{-\mu(z+ic)-\eta}\\
&= e^{-P}\sum_{i=0}^{n} C_n^i(-1)^{n-i}e^{-\mu ic}\\
&= e^{-P}(e^{-\mu c}-1)^n.
\end{aligned}
$$

This together with $\Delta_c^n e^{-P}\equiv 0$ gives $(e^{-\mu c}-1)^n\equiv 0$, which yields $e^{\mu c}\equiv 1$. Therefore, for any $j\in\mathbb{Z}$,

$$
e^{P(z+jc)}=e^{\mu z+\mu jc+\eta}=(e^{\mu c})^j e^{P(z)}=e^{P(z)}. \tag{3.18}
$$

On the other hand, from (3.1) we have

$$
\Delta_c^n f(z)=e^{P(z)}(f(z)-a(z))+a(z). \tag{3.19}
$$

By (3.18) and (3.19), we have

$$
\Delta_c^{n+1}f(z)=e^{P(z)}\Delta_c f(z) \tag{3.20}
$$

Combining (3.2) and (3.20), we obtain

$$
\Delta_c f(z)=(f(z)-a(z))+a(z)e^{-P(z)}
$$

which means that $\Delta_c^{n+1}f(z)=\Delta_c^n f(z)$ for all $n\geq 1$. Therefore, $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM and by Lemma 2.4 we obtain $\Delta_c f(z)=f(z)$, which contradicts the hypothesis. Then $\Delta_c^n(e^{-P})\not\equiv 0$. From the subcases (A) and (B), we can deduce that $\alpha_{n+1}(z)\not\equiv 0$. It is clear that

$$
\max\{\rho(a_i),\rho(\alpha_{n+1}), i=0,\dots,n+1\}<\deg P=1.
$$

By using Lemma 2.2, we obtain $\lambda(e^P)=\deg P=1$, which is a contradiction, and P must be a constant.

Case 2. $P(z)\equiv K$, $K\in\mathbb{C}-\{2k\pi i, k\in\mathbb{Z}\}$. From (3.1) we have

$$
\Delta_c^n f(z)=e^K(f(z)-a(z))+a(z).
$$

Hence

$$
\Delta_c^{n+1}f(z)=e^K\Delta_c f(z). \tag{3.21}
$$

Combining (3.2) and (3.21), we obtain

$$\Delta_c f(z) = (f(z) - a(z)) + a(z)e^{-K}$$

which means that $\Delta_c^{n+1} f(z) = \Delta_c^n f(z)$ for all $n \geq 1$. Therefore, $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM and by Lemma 2.4 we obtain $\Delta_c f(z) = f(z)$, which contradicts the hypothesis. Then $e^P \equiv 1$ and the proof is complete. □

Proof of the Theorem 1.10. Setting $g(z) = f(z) + b(z) - a(z)$. Since $\Delta_c^m a(z) \equiv 0$ $(1 \leq m \leq n)$, we can remark that

$$\begin{gathered} g(z) - b(z) = f(z) - a(z), \\ \Delta_c^n g(z) - b(z) = \Delta_c^n f(z) - b(z), \\ \Delta_c^{n+1} g(z) - b(z) = \Delta_c^n f(z) - b(z), \quad n \geq 2. \end{gathered}$$

Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share $b(z)$ CM. By using Theorem 1.8, we deduce that $\Delta_c g(z) \equiv g(z)$, which leads to $\Delta_c f(z) \equiv f(z) + b(z) + \Delta_c a(z) - a(z)$ and the proof is complete. □

Proof of the Theorem 1.14. Note that $f(z)$ is a nonconstant entire function of finite order. Since $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, it follows from Theorem 1.6 that $\Delta_c^{n+1} f(z) = C\Delta_c^n f(z)$, where C is a nonzero constant. Then we have

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{P(z)}, \tag{3.22}$$

$$\frac{\Delta_c^{n+1} f(z)}{f(z)} = Ce^{P(z)}, \tag{3.23}$$

where P is a polynomial. By (3.22) and (3.23) we obtain

$$f_{ic}(z) = (C+1)^i e^{P - P_{ic}} f(z). \tag{3.24}$$

Then

$$\Delta_c^n f(z) = \Big(\sum_{i=0}^{n} C_n^i (-1)^{n-i} (C+1)^i e^{P - P_{ic}}\Big) f(z) = e^{P(z)} f(z). \tag{3.25}$$

This equality leads to $\deg P = 0$. Hence $P(z) - P_{ic}(z) \equiv 0$ and (3.25) will be

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} (C+1)^i = C^n = e^{P(z)}. \tag{3.26}$$

By (3.22), (3.23) and (3.26) we deduce that

$$\begin{gathered} \Delta_c^n f(z) = C^n f(z), \\ \Delta_c^{n+1} f(z) = C^{n+1} f(z). \end{gathered}$$

Then

$$\Delta_c^{n+1} f(z) = \Delta_c(\Delta_c^n f(z)) = \Delta_c(C^n f(z)) = C^n \Delta_c f(z) = C^{n+1} f(z),$$

which implies $\Delta_c f(z) = Cf(z)$. Thus, the proof is complete. □

Proof of Corollary 1.16. By Theorem 1.14 we have $\Delta_c f(z) = Cf(z)$, where C is a nonzero constant. Then

$$\Delta_c^m f(z) = C\Delta_c^{m-1} f(z) = C^m f(z), \ m \geq 1. \tag{3.27}$$

On the other hand, for $z_0 \in \mathbb{C}$ we have
$$\Delta_c^m f(z_0) = f(z_0). \tag{3.28}$$
By (3.27) and (3.28) we deduce that $C^m = 1$. Hence $\Delta_c^m f(z) = f(z)$. $\square$

Proof of the Theorem 1.17. Setting $g(z) = f(z) - a(z)$, we have
$$\begin{gathered} g(z) = f(z) - a(z), \\ \Delta_c^n g(z) = \Delta_c^n f(z) - b(z), \\ \Delta_c^{n+1} g(z) = \Delta_c^n f(z) - b(z), \quad n \geq 2. \end{gathered}$$
Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, it follows that $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share 0 CM. Using Theorem 1.14, we deduce that $\Delta_c g(z) \equiv C g(z)$, where C is a nonzero constant, which leads to $\Delta_c f(z) \equiv C f(z) + \Delta_c a(z) - a(z)$ and the proof is complete. $\square$

4. Open Problem

It has been proved in [6] that

Theorem 4.1 ([6, Corollary 1.1])**.** *Let $f(z)$ be a non-periodic entire function of finite order, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^3 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

It is an open question to see under what conditions Theorem 4.1 holds for entire functions share a small function with $\Delta_c^n f(z)$ and $\Delta_c^{n+2} f(z)$ ($n \geq 1$). We believe that:

> Let $f(z)$ be a nonconstant entire function of finite order such that $\Delta_c^n f(z) \not\equiv 0$, and let $a(z) \in S(f)$ ($\not\equiv 0$) be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+2} f(z)$ ($n \geq 1$) share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.

Unfortunately, we have not succeed in proving this.

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