Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 322, pp. 1–4.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

L^p-SUBHARMONIC FUNCTIONS IN $\mathbb{R}^n$

MOUSTAFA DAMLAKHI

Abstract. We prove that if u is an L^p-subharmonic function defined outside a compact set in $\mathbb{R}^n$, it is bounded above near infinity, in particular, if the subharmonic function u is in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then u is non-positive. Some of the consequences of this property are obtained. We discuss the properties of subharmonic functions defined outside a compact set in $\mathbb{R}^n$ if they are also L^p functions.

1. Introduction

An upper semi-continuous function u, taking the value infinity, and not identically $(-\infty)$ is called a subharmonic function in $\mathbb{R}^n$ if it has sub-mean value property. The properties of functions with mean-value properties (called harmonic functions) are given in Axler [2] analogous properties for subharmonic functions are also known. In this note, we derive some properties of subharmonic functions on $\mathbb{R}^n$ when they are also L^p functions. For example, we show that a subharmonic L^p function in $\mathbb{R}^n$ is non-positive.

From Anandam [1] it is easy to see that if $s(x)$ is a subharmonic function defined outside a compact set in $\mathbb{R}^n$, then $s(x) = v(x) + cu(x) + b(x)$ near infinity, where $v(x)$ is subharmonic on $\mathbb{R}^n$, $u(x) = \log|x|$ if $n = 2$ and $u(x) = |x|^{2-n}$ if $n > 2$, c is constant and $b(x)$ is bounded harmonic function. We obtain some properties of $s(x)$ if it is in addition an L^p function also.

In particular, we show that if the subharmonic function outside a compact set is an L^p function, then $s(x)$ tends to 0 at infinity.

2. Subharmonic functions in $L^p(\mathbb{R}^n)$

In this note we consider $p < \infty$ and $n \geq 2$.

Lemma 2.1. *Let $s \geq 0$ be a subharmonic function in $\mathbb{R}^n$. If $s \in L^p(\mathbb{R}^n)$, $p \geq 1$, then $s \equiv 0$.*

Proof. For $x_0 \in \mathbb{R}^n$, let $B_n = \{x : |x - x_0| = 1\}$ and σ_n be the surface area of B_n. Since $s \geq 0$, s^p is subharmonic and using the polar coordinates for $x = (r, w)$,$|x - x_0| = r$.

By using the expression of the sub-mean-value property of s^p we have

$$s^p(x_0) \leq \frac{1}{\sigma_n} \int_{B_n} s^p(r, w) dw.$$

2010 *Mathematics Subject Classification.* 31B05, 30D55.
Key words and phrases. L^p-subharmonic functions; harmonic Hardy class.

Submitted August 8, 2016. Published December 20, 2016.

From this inequality and using that $s \in L^p(\mathbb{R}^n)$ we have

$$\infty > \int_0^\infty \int_{B_n} s^p(r,w) r^{n-1} dr dw \geq \int_0^\infty \sigma_n s^p(x_0) r^{n-1} dr.$$

This is possible if and only if $s^p(x_0) = 0$. Since x_0 is arbitrary, $s^p \equiv 0$ in $\mathbb{R}^n$. □

Theorem 2.2. *If s is subharmonic function in $L^p(\mathbb{R}^n)$ with $p \geq 1$, then $s \leq 0$.*

Proof. $s^+ = \sup(s, 0)$ is subharmonic and is in $L^p(\mathbb{R}^n)$. Hence by the Lemma 2.1 $s^+ \equiv 0$ and consequently $s \leq 0$ in $\mathbb{R}^n$. □

Corollary 2.3. *Let s be a subharmonic function in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{n}{n-1}$. Then $s \equiv 0$.*

Proof. By the Theorem 2.2, $s \leq 0$ for all p, $1 \leq p < \infty$.

(1) Let $n = 2$. Since s is an upper bounded subharmonic function in $\mathbb{R}^2$, it is a constant. If $s \in L^\infty(\mathbb{R}^2)$, even though Theorem 2.2 does not hold, yet s is also a constant in this case.

(2) Let $n \geq 3$. Since $-s$ a is positive supharmonic function, by Riesz decomposition $-s = u + v$ where u is a potential and $h \geq 0$ is harmonic and hence constant. Since $u \leq -s$, $u \in L^p(\mathbb{R}^n)$.

Now, if B is the unit ball in $\mathbb{R}^n$, $n \geq 3$, we define the function

$$\vartheta(x) = \begin{cases} 1 & \text{if } x \in B \\ |x|^{2-n} & \text{if } x \in \mathbb{R}^n - B \end{cases}$$

Then $\vartheta(x)$ is a potential and $u(x) \geq (\inf_{x \in \overline{B}} u(x)) \vartheta(x) \in \mathbb{R}^n$ Consequently, $\vartheta(x) \in L^p(\mathbb{R}^n)$, but this would imply

$$\int_1^\infty (r^{2-n})^p r^{n-1}\, dr < \infty.$$

This is not possible if $p(2-n) + n - 1 \geq -1$ which means that if $p \leq \frac{n}{n-2}, u \equiv 0$ and hence s is constant. Thus, for all $n \geq 2$, $s \equiv A$, a constant. Since $s \in L^p(\mathbb{R}^n)$, $A \equiv 0$ when $p < \infty$. □

Corollary 2.4. *If s is a subharmonic function in $L^p(\mathbb{R}^n)$, $p \geq 1$, which is associated measure μ in a local Riesz representation, μ does not charge points; that is $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^n$.*

Proof. In view of the above corollary, we assume $n \geq 3$. By Theorem 2.2, it follows that $u = -s$ is a potential. Since μ is the measure associated with the subharmonic function s, it is always non-positive. If we Suppose that it is strictly negative at a point, it leads to a contradiction. In fact, we assume that $\mu(\{x_0\}) = \alpha < 0$ for some x_0 in $\mathbb{R}^n$.

Let $B = \{x : |x - x_0| < 1\}$. Since $u(x) \geq -\alpha |x - x_0|^{2-n}$ in B and since u is in $L^p(B)$, we should have $p < n/(n-2)$.

In $\mathbb{R}^n - B$. Since $u(x) \geq (\min_{x \in \overline{B}} |x - x_0|^{2-n})$ and since $u \in L^p(\mathbb{R}^n)$. We should have $p > n/(n-2)$ as in the proof of Corollary 2.3.

Thus, for any choice of $p \geq 1$, $u \notin L^p(\mathbb{R}^n)$. This contradiction shows that $\mu(\{x_0\}) = 0$. □

Recall that a C^∞ function $q(x)$ in an open set in $\mathbb{R}^n$ is called a quasiharmonic function if $\Delta q = -1$.

Corollary 2.5 ([3, pp. 120-122])**.** *Let s be subharmonic in $\mathbb{R}^n$ such that $\Delta s = A$, a constant (with Δ in the sense of distributions). Suppose $s \in L^p(\mathbb{R}^n)$, $p \geq 1$. Then $s \equiv 0$.*

Proof. Since s is subharmonic, $A = \Delta s \geq 0$. Suppose $A > 0$. Note by the Theorem 2.2. $s \leq 0$. Since

$$\Delta s = A, \quad \Delta\Big(s(x) - \frac{A|x|^2}{2n}\Big) = 0$$

and hence there exists a harmonic function $h(x) \in \mathbb{R}^n$ such that $s(x) = \frac{A}{2n}|x|^2 +h(x)$ a.e.

If two subharmonic functions are equal a.e., they are equal every where; hence $s(x) = \frac{A}{2n}|x|^2 +h(x)$. Since $h(x) \leq s(x) \leq 0$, h is a constant which leads to a contradiction since $s \leq 0$ and $A|x|^2$ tends to ∞. Hence $\Delta s = 0$. Thus s is a harmonic function in $L^p(\mathbb{R}^n)$. In this case, the Theorem 2.2. implies that $s \equiv 0$. □

3. L^p Subharmonic function outside a compact set in $\mathbb{R}^n$

Let u be subharmonic function outside a compact set in $\mathbb{R}^n$. We say that u extends subharmonically in $\mathbb{R}^n$ if there exists a subharmonic function V in $\mathbb{R}^n$, such that V is not bounded from above by a harmonic function in $\mathbb{R}^n$ and $V = u$ outside a compact set.

Proposition 3.1. *Let u be an L^p subharmonic function outside a compact set. Then u cannot be extended subharmonically in $\mathbb{R}^n$.*

Proof. Let V be subharmonic function in $\mathbb{R}^n$ not bounded from above by a harmonic function in $\mathbb{R}^n$ such that $V = u$ outside a compact set. Then for large r, the function s defined as

$$s = \begin{cases} u & \text{if } |x| \geq r \\ D_r u & \text{if } |x| < r \end{cases}$$

Where $D_r u$ is the Dirichlet solution in $|x| < r$ with boundary values u, is subharmonic in $\mathbb{R}^n$ and $s \geq V$.

If $u(x) \in L^p$ in $|x| \geq r$, $s(x)$ is in the harmonic Hardy class h^p in $|x| < r$ (see [2, page 103]) and hence there exists a harmonic function $H(x)$ in $|x| < r$ such that $|s|^p < H$. Then

$$\int_{|x|<r} |s(x)|^p \, dx \leq c_n H(0),$$

for a constant c_n. That is $s(x) \in L^p$ in $|x| < r$. Which implies that $s \in L^p(\mathbb{R}^n)$ since $s(x) = u(x)$ in $|x| \geq r$. By Theorem 2.2, $s \leq 0$ and hence $V \leq 0$ in $\mathbb{R}^n$, a contradiction. □

Corollary 3.2. *Let $u(x)$ be subharmonic in an open set w containing $|x| \geq r$ in $\mathbb{R}^n$. Suppose $u \in L^p(w)$ for some $p \geq 1$. Then $u(x)$ is upper bounded in $|x| \geq r$.*

Proof. By hypothesis $u^+(x)$ is an L^p subharmonic function in an open set containing $|x| \geq r$.

(1) In $\mathbb{R}^2$, if u^+ is not upper bounded in $|x| \geq r$, it can be extended subharmonically in $\mathbb{R}^2$ (see [1, Corollary 1]). This is a contradiction with Proposition 3.1, since $u^+ \in L^p$ in $|x| \geq r$. This means that $u^+(x)$ and hence $u(x)$ is upper bounded in $|x| \geq r$.

(2) In $\mathbb{R}^n$, $n \geq 3$, there exists a subharmonic function $s(x)$ in $\mathbb{R}^n$ and some $\alpha \leq 0$ such that $u^+(x) = s(x) - \alpha|x|^{2-n}$ in $|x| \geq r$ (see [1, Theorem 1]). Hence $s(x) \geq \alpha|x|^{2-n}$ in $|x| \geq r$.

Let $M(R, s)$ denote the the mean-value of $s(x)$ in $|x| = R$. If $\lim_{R\to\infty} M(R, s) = \infty$, then $\lim_{R\to\infty} M(R, u^+) = \infty$. Hence u^+ can be extended subharmonically in $\mathbb{R}^n$ (see [1, Theorem 2]), a contradiction; thus $\lim_{R\to\infty} M(R, s) = \infty$.

When $\lim_{R\to\infty} M(R, s)$ is finite, s has a harmonic majorant h in $\mathbb{R}^n$. Since h is lower bounded, it is a constant c and $c \geq 0$. (We remark in passing that c here can be chosen as 0 if $p > \frac{n}{n-2}$, see Corollary 2.3). Hence $u^+(x)$ is bounded in $|x| \geq r$, and consequently $u(x)$ is upper bounded by $c - \alpha|x|^{2-n}$ in $|x| \geq r$. Thus, in all cases $u(x)$ is upper bounded in $|x| \geq r$. □

Remark 3.3. In particular, we deduce that if h is an L^p subharmonic function defined outside a compact set in $\mathbb{R}^n$, then h tends to 0 at at infinity; if h is a harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 3$. Tending to 0 at infinity, then h is in L^p in $|x| \geq r$ for large r if $p > \frac{n}{n-2}$.

References

[1] V. Ananadam; *Subharmonic function outside a compact set in* $\mathbb{R}^n$, Proc. Amer. Math .Soc., **84** (1982), 52–54.
[2] S. Axler, P. Bourdon, W. Ramey; *Harmonic function theory*, Springer Verlag, New York. 1992.
[3] L. Sario, M. Nakai, C. Wang, L. O. Chung; *Classification theory of Riemannian manifolds*, Springer Verlag, L. N. 605, (1977).

Moustafa Damlakhi
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: damlakhi@ksu.edu.sa