Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 325, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS FOR A SCALAR CONSERVATION LAW WITH A FLUX OF LOW REGULARITY

MARTIN LAZAR, DARKO MITROVIĆ

Abstract. We prove existence of solutions to Cauchy problem for scalar conservation laws with non-degenerate discontinuous flux

$$\partial_t u + \operatorname{div} f(t,\mathbf{x},u) = s(t,\mathbf{x},u), \quad t \geq 0, \mathbf{x} \in \mathbb{R}^d,$$

where for every $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}$, the flux $f(t,\mathbf{x},\cdot) \in \operatorname{Lip}(\mathbb{R};\mathbb{R}^d)$ and $\partial_\lambda f \in L^r(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$, additionally satisfying $\max_{|\lambda|\leq M} f(\cdot,\cdot,\lambda) \in L^r(\mathbb{R}^+ \times \mathbb{R}^d)$, for some $r > 1$ and every $M > 0$, and, for every $\lambda \in \mathbb{R}$, $\operatorname{div}_{(t,\mathbf{x})} f(\cdot,\cdot,\lambda) \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ where $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ is the space of Radon measures. Moreover, the function s is measurable and both f and s satisfy certain growth rate assumptions with respect to λ. The result is obtained by means of the H-measures.

1. Introduction

In this article, we consider the Cauchy problem for the non-linear transport equation

$$\partial_t u + \operatorname{div}_{\mathbf{x}} f(t,\mathbf{x},u) = s(t,\mathbf{x},u) \tag{1.1}$$

$$u|_{t=0} = u_0(\mathbf{x}) \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), \quad p > 1. \tag{1.2}$$

The given equation describes many natural phenomena. We mention some of them: flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, etc.

While the local Lipschitz assumption on $\lambda \mapsto f(t,\mathbf{x},\lambda)$ is natural in many applications, the assumption of regular dependence on the spatial variable $\mathbf{x}$ is very restrictive. Indeed, even in the simplified situation in which the diffusion is neglected such as road traffic with variable number of lanes [9], Buckley-Leverett equation in a layered porous medium [2, 24], sedimentation processes [10, 15, 16], etc. It may appear in models with discontinuous (in $\mathbf{x}$) flux functions.

Scalar conservation laws with discontinuous flux attracted significant amount of attention in recent years. It appears that neither existence nor uniqueness for (1.1), (1.2) can be resolved applying the methods used when the flux is regular. More precisely, the well-posedness of the Cauchy problem for (1.1) with a regular flux is completely settled in [27] by means of the vanishing viscosity method and the shifting of variables (existence) and the doubling of variables (uniqueness). The

2010 *Mathematics Subject Classification.* 35L65.
Key words and phrases. Scalar conservation law; discontinuous coefficient; existence; velocity averaging; H-measure.

Submitted May 18, 2016. Published December 22, 2016.

mentioned techniques cannot be extended on (1.1) without substantial improvements. This fact generated development of various methods and concepts during attempts to solve and understand equation (1.1). Different uniqueness concepts are thoroughly described in [3] and they are further developed in [4, 11, 12]. As for the existence of solution to (1.1), (1.2), which is the main objective of this article, the first general result is obtained in one-dimensional case using the compensated compactness method [26]. This result is later extended on two-dimensional case [1, 25] under a non-degeneracy conditions (see a similar requirement (4.4) below). The first multidimensional existence result is obtained in [35] through an extension of H-measures [22, 38].

The strategy of the proof based on H-measures requires rewriting (1.1) in the kinetic formulation [13, 32] (see Theorem 4.2) and then applying the velocity averaging results (see Theorem 3.4). We remark that Theorem 3.4 is the most general velocity averaging result from the viewpoint of regularity of coefficients of the underlying transport equation. We dedicated Section 3 to this issue where we also provided a short overview of the related results.

The aim of this article is to generalize results from [35] when the flux f and the source s satisfy:

(i) For each fixed $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$ the flux $f(t,\mathbf{x},\cdot) \in \operatorname{Lip}(\mathbb{R};\mathbb{R}^d)$ and $\partial_\lambda f \in L^r(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$, $r>1$;

(ii) For every $M>0$, it holds $\sup_{|\lambda|<M} |f(t,\mathbf{x},\lambda)| \in L^r(\mathbb{R}^+ \times \mathbb{R}^d)$, $r>1$.

(iii) For every $\lambda \in \mathbb{R}$, the divergence of the flux is a Radon measure,

$$\operatorname{div}_{(t,\mathbf{x})} f(\cdot,\cdot,\lambda) \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d);$$

(iv) There exists a non-negative convex function $S:\mathbb{R}\to\mathbb{R}$ such that $S(0)=0$ and there exists $M,C,c>0$ such that for every $|\lambda|<M$,

$$c \le \frac{S(\lambda)}{|\lambda|^p} \le C, \tag{1.3}$$

$$\int_{\mathbb{R}^+\times\mathbb{R}^d} \sup_{\lambda\in\mathbb{R}} |S''(\lambda) \operatorname{div}_{(t,\mathbf{x})} f(t,\mathbf{x},\lambda)| d\lambda dt d\mathbf{x} < C_1 < \infty \tag{1.4}$$

$$\max\Big\{ \int_{\mathbb{R}^+\times\mathbb{R}^d} \sup_{\lambda\in\mathbb{R}} |s(t,\mathbf{x},\lambda)| d\lambda, \int_{\mathbb{R}^+\times\mathbb{R}^d} \sup_{\lambda\in\mathbb{R}} |s(t,\mathbf{x},\lambda) S'(\lambda)| d\lambda \Big\} < C_2 < \infty. \tag{1.5}$$

In [35], it was assumed that the flux $f(t,\mathbf{x},\lambda)$ satisfies (ii), and (iii), but for $r>2$. The reason for such an assumption is in the tools used – the H-measures. Original H-measures were adapted for continuous coefficients (see [22, 38]). In [35], the notion of H-measures is generalized for application in the framework of L^q, $q>2$, coefficients. After that, we made generalizations in this direction [29] but we were not able to decrease regularity of the coefficients under L^2.

To achieve such a low regularity, one can apply H-distributions which were introduced in [5] and further generalized in [30, 33]. H-distributions are an extension of H-measures to the L^p, $p>1$, setting. Another approach, followed in this paper, relies on an appropriate generalization of H-measures which takes into account additional regularity of the defining sequence that compensates for a low regularity of test functions (coefficients).

In the last section, we shall introduce the notion of the quasi-solution and prove strong precompactness of the sequence of quasi-solutions to (1.1) under non-degeneracy conditions (4.4). As a corollary of the precompactness, we prove existence of weak solutions to (1.1).

As for the assumption (i), in [35] it was assumed that for each fixed $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$ the flux $f(t,\mathbf{x},\cdot) \in C^1(\mathbb{R};\mathbb{R}^d)$, but there was no assumption $\partial_\lambda f \in L^r(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$, $r > 1$.

Let us also remark that a usual assumption for the scalar conservation laws with discontinuous flux (in particular, such an assumption is used in [35] also) is that there exists $a, b \in \mathbb{R}$ such that

$$f(t,\mathbf{x},a) = f(t,\mathbf{x},b) = 0 \quad \text{and} \quad a \le u_0 \le b.$$

This provides boundedness of the sequence of approximate solutions obtained with the vanishing viscosity perturbation of (1.1). We replace such assumptions by iv) given above which provides only L^q-control of the approximate solutions. Details are given in the last section.

Let us shortly remind on the contents of the paper. Section 2 provides results on H-measures that we are going to use. In Section 3 we prove a velocity averaging result which will be in the essence of the existence proof given in Section 4.

2. H-measures

H-measures were introduced by Tartar [38] and, independently, by Gerard [22] as an object which measures defect of strong L^2-precompactness for bounded sequences in $L^2(\mathbb{R}^d)$. More precisely, a sequence converging weakly in $L^2_{\rm loc}(\mathbb{R}^d)$ converges strongly as well, if and only if the corresponding H-measure is zero. H-measures appeared to be very powerful tool in the analysis of PDEs and have been successfully applied in many mathematical fields - let us here mention the generalization of compensated compactness results to equations with variable coefficients [22, 38], applications in the control theory [14, 31], the velocity averaging results [22, 29], as well as explicit formulae and bounds in homogenisation [6, 7, 38].

Notation. Throughout this article, by $C_0(\mathbb{R}^d)$ we denote the closure of $C_c(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$ topology, while $\mathcal{M}_b(\Omega)$ stands for the space of bounded Radon measures on a set $\Omega \subseteq \mathbb{R}^d$.

For a fixed $p \ge 1$, we shall denote by p' its conjugate i.e. the number such that $1/p + 1/p' = 1$.

S^{d-1} stands for the unit sphere in $\mathbb{R}^d$ centered at the origin.

By $L^2_{w^*}(\mathbb{R}^{2m};\mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1}))$ we denote the dual of $L^2(\mathbb{R}^{2m};\mathrm{C}_0(\mathbb{R}^d \times \mathrm{S}^{d-1}))$, which is a Banach space of weakly $*$ measurable functions $\mu : \mathbb{R}^{2m} \to \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1})$ such that $\int_{\mathbb{R}^{2m}} \|\mu(\mathbf{y},\tilde{\mathbf{y}})\|^2 d\mathbf{y} d\tilde{\mathbf{y}} < \infty$.

The Fourier transform is defined as $\hat{\mathsf{u}}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \mathsf{u}(\mathbf{x})\, d\mathbf{x}$, and its inverse as $(\mathsf{u})^\vee(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \mathsf{u}(\mathbf{x})\, d\mathbf{x}$.

A (Fourier) multiplier operator $\mathcal{A}_\psi$ on $\mathbb{R}^d$ with a symbol ψ is defined as

$$(\mathcal{A}_\psi u)(x) = (\psi(\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi}))^\vee (x).$$

In this article, where there is no fear of ambiguity, we shall not distinguish the symbols for a function ψ defined on the unit sphere S^{d-1} and its extension to $\mathbb{R}^d$ given by $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi})$.

Throughout this paper $\langle\cdot,\cdot\rangle$ stands for a sesquilinear dual product, taken to be antilinear in the first, while linear in the second variable. By $\otimes$ we denote the tensor product of functions in different variables.

A sequence (u_n) of functions from $L^1(\Omega)$ is called equi-integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set $E \subset \Omega$ satisfying $\mathrm{meas}(E) < \delta$ it holds $\sup_n \int_E |u_n(\mathbf{x})| d\mathbf{x} < \varepsilon$.

Let us now recall the extension of H-measures introduced in [29] whose existence and properties are restated in the next theorem.

Theorem 2.1. *Assume that a sequence (u_n) converges weakly to zero in $L^2(\mathbb{R}^{d+m}) \cap L^2(\mathbb{R}^m; L^p(\mathbb{R}^d))$, $p \geq 2$. Then, after passing to a subsequence (not relabeled), there exists a measure $\mu \in L^2_{w^*}(\mathbb{R}^{2m}; \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1}))$ such that for all $\phi_1 \in L^2(\mathbb{R}^m; L^{\tilde{p}'}(\mathbb{R}^d))$, $\frac{1}{\tilde{p}'} + \frac{2}{p} = 1$ (with $L^\infty(\mathbb{R}^d)$ being replaced by $C_0(\mathbb{R}^d)$ if $p = 2$), $\phi_2 \in L^2_c(\mathbb{R}^m; C_0(\mathbb{R}^d))$, and $\psi \in C^d(\mathrm{S}^{d-1})$ it holds*

$$\begin{aligned} &\lim_n \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d} (\phi_1 u_n)(\mathbf{x},\mathbf{y}) \Big(\overline{\mathcal{A}_\psi\, \phi_2 u_n(\cdot,\tilde{\mathbf{y}})}\Big)(\mathbf{x}) d\mathbf{x} d\mathbf{y} d\tilde{\mathbf{y}} \\ &= \int_{\mathbb{R}^{2m}} \langle \mu(\mathbf{y},\tilde{\mathbf{y}},\cdot,\cdot), \phi_1(\cdot,\mathbf{y})\overline{\phi_2}(\cdot,\tilde{\mathbf{y}}) \otimes \overline{\psi}\rangle d\mathbf{y} d\tilde{\mathbf{y}}\,, \end{aligned}$$

where $\mathcal{A}_\psi$ is the (Fourier) multiplier operator on $\mathbb{R}^d$ associated to $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$.

Furthermore, the operator μ has the form

$$\mu(\mathbf{y},\tilde{\mathbf{y}},\mathbf{x},\boldsymbol{\xi}) = f(\mathbf{y},\tilde{\mathbf{y}},\mathbf{x},\boldsymbol{\xi})\nu(\mathbf{x},\boldsymbol{\xi}), \tag{2.1}$$

where $\nu \in \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1})$ is a non-negative scalar Radon measure whose $\mathbb{R}^d$ projection $\int_{\mathrm{S}^{d-1}} d\nu(\mathbf{x},\boldsymbol{\xi})$ can be extended to a bounded functional on $L^{\tilde{p}'}(\mathbb{R}^d)$ in the case $p > 2$, while f is a function from $L^2(\mathbb{R}^{2m}; L^1(\mathbb{R}^d \times \mathrm{S}^{d-1} : \nu))$.

An H-measure defined above is an object associated to a single L^2 sequence. However, there are no obstacles to adjoin a similar object to two different sequences. This can be done by forming a vector sequence, and consider non-diagonal elements of corresponding (matrix) H-measure (e.g. [38]). Another way is to joint two sequences in a single one by means of a dummy variable, as it is done in the next theorem.

Theorem 2.2. *Let (u_n) be a bounded sequence in $L^2(\mathbb{R}^{d+m}) \cap L^2(\mathbb{R}^d; L^p(\mathbb{R}^m)) \cap L^p(\mathbb{R}^{d+m})$ and let (v_n) be a sequence weakly converging to zero in $L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $p, q \geq 2$ such that there exists $r > 1$ satisfying $1/r + 1/p + 1/q = 1$. Then, after passing to a subsequences (not relabeled), there exists a measure $\mu \in L^2_{w^*}(\mathbb{R}^m; \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1}))$ such that for all $\phi_1 \in L^{p'}(\mathbb{R}^m; L^r(\mathbb{R}^d))$ (with $L^\infty(\mathbb{R}^d)$ being replaced by $C_0(\mathbb{R}^d)$ if $p = q = 2$), $\phi_2 \in C_0(\mathbb{R}^d)$, $\psi \in C^d(S^{d-1})$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^m} \langle \mu(\mathbf{y},\cdot,\cdot), \phi_1(\cdot,\mathbf{y})\overline{\phi_2} \otimes \overline{\psi}\rangle dy \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{m+d}} \phi_1(\mathbf{x},\mathbf{y}) u_n(\mathbf{x},\mathbf{y}) \overline{\mathcal{A}_\psi\big(\phi_2 v_n\big)(\mathbf{x})} d\mathbf{x} d\mathbf{y}\,. \end{aligned} \tag{2.2}$$

Furthermore, the measure μ is of the form

$$\mu(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) = f(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) d\nu(\mathbf{x},\boldsymbol{\xi}), \tag{2.3}$$

where f belongs to $L^2(\mathbb{R}^m; L^1(\mathbb{R}^d \times \mathrm{S}^{d-1} : \nu))$, while $\nu \in \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1})$ is a non-negative, bounded, scalar Radon measure. In the case $\min\{p, q\} > 2$ its $\mathbb{R}^d$

projection $\int_{\mathrm{S}^{d-1}} d\nu(\mathbf{x},\boldsymbol{\xi})$ *can be extended to a bounded functional on* $L^s(\mathbb{R}^d)$*, where* s *is the dual index of* $\min\{p,q\}/2$*, i.e. there exists an* $h \in L^{s'}(\mathbb{R}^d)$ *such that* $\int_{\mathrm{S}^{d-1}} d\nu(\mathbf{x},\boldsymbol{\xi}) = h(\mathbf{x})d\mathbf{x}$*.*

We call μ *the generalized H-measure corresponding to (sub)sequences (of)* (u_n) *and* (v_n)*.*

Proof. Let us first prove that the relation (2.2) holds for smooth (with respect to $\mathbf{x}$) test functions ϕ_1 and ϕ_2.

To this effect denote by u an L^2 weak limit of the sequence (u_n) along a (non-relabeled) subsequence. Fix an arbitrary non-negative compactly supported $\rho \in C_{\rm c}(\mathbb{R}^m)$ with the total mass equal to one. Let

$$W_n(\mathbf{x},\mathbf{y},\lambda) = \begin{cases} (u_n - u)(\mathbf{x},\mathbf{y}), & \lambda \in \langle 0,1\rangle \\ \rho(\mathbf{y})v_n(\mathbf{x}), & \lambda \in \langle -1,0\rangle \\ 0, & \text{else.} \end{cases}$$

Clearly, we have that $W_n \rightharpoonup 0$ in $L^2(\mathbb{R}^{d+m+1}) \cap L^2(\mathbb{R}^{m+1}; L^{\min\{p,q\}}(\mathbb{R}^d))$, and by Theorem 2.1 it admits a measure $\tilde\mu \in L^2_{w^*}(\mathbb{R}^{2(m+1)}; \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1}))$ such that for any $\tilde\phi_1 \in L^2(\mathbb{R}^{m+1}; C_0(\mathbb{R}^d))$, $\tilde\phi_2 \in L^2_c(\mathbb{R}^{m+1}; C_0(\mathbb{R}^d))$, and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^{2m}} \langle \tilde\mu(\mathbf{y},\tilde{\mathbf{y}},\cdot,\cdot), \tilde\phi_1(\cdot,\mathbf{y})\overline{\tilde\phi_2}(\cdot,\tilde{\mathbf{y}}) \otimes \overline{\psi}\rangle d\mathbf{y}d\tilde{\mathbf{y}} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{d+2(m+1)}} (\tilde\phi_1 W_n)(\mathbf{x},\mathbf{y},\lambda)\overline{\mathcal{A}_\psi\big(\tilde\phi_2 W_n(\cdot,\tilde{\mathbf{y}},\tilde\lambda)\big)(\mathbf{x})}d\mathbf{x}d\mathbf{w}, \end{aligned} \tag{2.4}$$

where $\mathbf{w} = (\mathbf{y},\tilde{\mathbf{y}},\lambda,\tilde\lambda) \in \mathbb{R}^{2m+2}$.

According to the representation (2.1), the measure $\tilde\mu$ is of the form

$$\tilde\mu = \tilde f(\mathbf{y},\tilde{\mathbf{y}},\lambda,\tilde\lambda,\mathbf{x},\boldsymbol{\xi})d\nu(\mathbf{x},\boldsymbol{\xi}), \quad \mathbf{y},\tilde{\mathbf{y}} \in \mathbb{R}^m,\ \lambda,\tilde\lambda \in \mathbb{R},$$

where $\tilde f$ is a function from $L^2(\mathbb{R}^{2(m+1)}; L^1(\mathbb{R}^d \times \mathrm{S}^{d-1} : \nu))$, while $\nu \in \mathcal{M}_b(\mathbb{R}^d \times \mathrm{S}^{d-1})$ is a non-negative scalar Radon measure, whose $\mathbb{R}^d$ projection $\int_{\mathrm{S}^{d-1}} d\nu(\mathbf{x},\boldsymbol{\xi})$can be extended to a bounded functional on $L^s(\mathbb{R}^d)$, where s is the dual index of $\min\{p,q\}/2$ in the case $\min\{p,q\} > 2$.

By taking in (2.4) $\tilde\phi_1(\mathbf{x},\mathbf{y},\lambda) = \phi_1(\mathbf{x},\mathbf{y}) \otimes \theta_1(\lambda)$, and $\tilde\phi_2(\mathbf{x},\tilde{\mathbf{y}},\tilde\lambda) = \phi_2(\mathbf{x}) \otimes \rho_2(\tilde{\mathbf{y}}) \otimes \theta_2(\tilde\lambda)$, where $\phi_1 \in L^2(\mathbb{R}^m; C_0(\mathbb{R}^d))$ and $\phi_2 \in C_0(\mathbb{R}^d)$ are arbitrary test functions, while $\theta_1 = \chi_{[0,1]}, \theta_2 = \chi_{[-1,0]}$, and $\rho_2(\tilde{\mathbf{y}}) = 1$ for $\tilde{\mathbf{y}} \in \operatorname{supp}\rho$, we see that the measure

$$d\mu(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) = \Big(\int_{-1}^0 \int_0^1 \int_{\mathbb{R}^m} \tilde f(\mathbf{y},\tilde{\mathbf{y}},\lambda,\tilde\lambda,\mathbf{x},\boldsymbol{\xi})\, d\tilde{\mathbf{y}}d\lambda d\tilde\lambda\Big) d\nu(\mathbf{x},\boldsymbol{\xi})d\mathbf{y},$$

satisfies (2.2) and (2.3) for all $\phi_1 \in L^2(\mathbb{R}^m; C_0(\mathbb{R}^d))$, $\phi_2 \in C_0(\mathbb{R}^d)$, and $\psi \in C^d(S^{d-1})$.

In the second part of the proof we show that the relation (2.2) extends to test functions ϕ_1 taken from the space $L^{p'}(\mathbb{R}^m; L^r(\mathbb{R}^d))$. To this effect, take an arbitrary such function and denote by (ϕ_1^k) a sequence of compactly supported continuous functions such that $\phi_1^k \to \phi_1$ strongly in $L^{p'}(\mathbb{R}^m; L^r(\mathbb{R}^d))$. For ϕ_2 and ψ as in (2.2), we define

$$\begin{aligned}
&\int_{\mathbb{R}^{2m}} \langle \mu(\mathbf{y},\cdot,\cdot), \varphi_1(\cdot,\mathbf{y})\bar{\varphi}_2 \otimes \bar{\psi}\rangle d\mathbf{p}d\mathbf{q} \\
&:= \lim_{k\to\infty} \int_{\mathbb{R}^{2m}} \langle \mu(\mathbf{y},\cdot,\cdot), \varphi_1^k \bar{\varphi}_2 \otimes \bar{\psi}\rangle d\mathbf{p}d\mathbf{q} \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d} (\varphi_1^k u_n)(\mathbf{x},\mathbf{p}) \Big(\overline{\mathcal{A}_\psi\, \varphi_2 u_n}\Big)(\mathbf{x}) d\mathbf{x}d\mathbf{p}d\mathbf{q}.
\end{aligned} \tag{2.5}$$

The above limit (with respect to k) exists, since by means of the Hölder inequality and the Marcinkiewicz multiplier theorem (see Remark 2.6 below) for $k_1, k_2 \in \mathbb{N}$ it holds

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{2m}} \langle \mu(\mathbf{y},\cdot,\cdot), (\varphi_1^{k_1} - \varphi_1^{k_2})(\cdot,\mathbf{y})\bar{\varphi}_2 \otimes \bar{\psi}\rangle d\mathbf{y}\Big| \\
&\le \limsup_n \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d} \Big|(\varphi_1^{k_1} - \varphi_1^{k_2}) u_n(\mathbf{x},\mathbf{y}) \Big(\overline{\mathcal{A}_\psi\, \varphi_2 v_n}\Big)(\mathbf{x})\Big| d\mathbf{x}d\mathbf{y} \\
&\le \limsup_n C \,\|\psi\|_{C^d(S^{d-1})} \|\varphi_2 v_n\|_{L^q(\mathbb{R}^d)} \int_{\mathbb{R}^m} \|(\varphi_1^{k_1} - \varphi_1^{k_2})(\cdot,\mathbf{y})\|_{L^r(\mathbb{R}^d)} \\
&\quad \times \|(u_n(\cdot,\mathbf{y})\|_{L^p(\mathbb{R}^d)} d\mathbf{y} \\
&\le \limsup_n C \|\psi\|_{C^d(S^{d-1})} \|(\varphi_1^{k_1} - \varphi_1^{k_2})\|_{L^{p'}(\mathbb{R}^m; L^r(\mathbb{R}^d))} \|\varphi_2\|_{L^\infty(\mathbb{R}^d)} \\
&\quad \times \|u_n\|_{L^p(\mathbb{R}^{d+m})} \|v_n\|_{L^q(\mathbb{R}^d)}\,,
\end{aligned}$$

where C depends on p and d only. Since (φ_1^k) is a convergent sequence, hence the Cauchy one, the limit in (2.5) exists, and it does not depend on the approximating sequence (φ_1^k).

The same analysis as above implies

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d} \Big|(\varphi_1^k - \varphi_1) u_n(\mathbf{x},\mathbf{p}) \,\Big(\overline{\mathcal{A}_\psi\, \varphi_2 u_n(\cdot,\mathbf{q})}\Big)(\mathbf{x})\Big| d\mathbf{x}d\mathbf{p}d\mathbf{q} = 0$$

and the convergence is uniform with respect to n. Thus we can exchange limits in the second line of (2.5), which proves (2.2). □

The last theorem takes into account additional regularity of sequences (u_n) and (v_n). If the first one is bounded in L^p and the latter one in L^∞, test functions can be taken from $L^{p'}$ almost (more precisely, from $L^{p'+\varepsilon}$, $\varepsilon > 0$ arbitrary), which is maximal regularity one can expect in order for the right hand side of (2.2) to make sense.

The following statement on the measure ν now follows from results on slicing measures [21, Theorem 1.10].

Lemma 2.3. *Under the assumptions of the above theorem, in the case $s > 1$ for a.e. $\mathbf{x} \in \mathbb{R}^d$ there exists a Radon probability measure $\nu_{\mathbf{x}}$ such that $d\nu(\mathbf{x},\boldsymbol{\xi}) = d\nu_{\mathbf{x}}(\boldsymbol{\xi})h(\mathbf{x})d\mathbf{x}$, where h is a $L^{s'}$ function from Theorem 2.2. More precisely, for each $\phi \in \mathrm{C}_0(\mathbb{R}^d \times \mathrm{S}^{d-1})$,*

$$\int_{\mathbb{R}^d\times \mathrm{S}^{d-1}} \phi(\mathbf{x},\boldsymbol{\xi}) d\nu(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \Big(\int_{\mathrm{S}^{d-1}} \phi(\mathbf{x},\boldsymbol{\xi}) d\nu_{\mathbf{x}}(\boldsymbol{\xi})\Big) h(\mathbf{x}) d\mathbf{x}\,.$$

The above result is also valid if we take a test function $\phi \in L^s(\mathbb{R}^d; \mathrm{C}(\mathrm{S}^{d-1}))$.

We end this section with the analysis of the Fourier multipliers which form the basis in construction and application of H-measures.

Definition 2.4. For a given $p \in [1\infty\rangle$, let the multiplier operator $\mathcal{A}_\psi$ satisfy

$$\|\mathcal{A}_\psi(u)\|_{L^p} \le C\|u\|_{L^p}, \quad u \in \mathcal{S},$$

where C is a positive constant, while $\mathcal{S}$ stands for a Schwartz space. Then its symbol ψ is called an L^p (Fourier) multiplier.

There are many criteria on a symbol ψ providing it to be an L^p multiplier. In the paper, we shall need the Marcinkiewicz multiplier theorem [23, Theorem 5.2.4.], more precisely its corollary which we provide here:

Corollary 2.5. *Suppose that $\psi \in C^d(\mathbb{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\})$ is a bounded function such that for some constant $C > 0$ it holds*

$$|\boldsymbol{\xi}^{\tilde{\boldsymbol{\alpha}}} \partial^{\tilde{\boldsymbol{\alpha}}} \psi(\boldsymbol{\xi})| \le C, \quad \boldsymbol{\xi} \in \mathbb{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\} \tag{2.6}$$

for every multi-index $\tilde{\boldsymbol{\alpha}} = (\tilde\alpha_1, \dots, \tilde\alpha_d) \in \mathbf{N}_0^d$ such that $|\tilde{\boldsymbol{\alpha}}| = \tilde\alpha_1 + \tilde\alpha_2 + \dots + \tilde\alpha_d \le d$. Then, the function ψ is an L^p-multiplier for $p \in \langle 1, +\infty\rangle$, and the operator norm of $\mathcal{A}_\psi$ depends only on C, p and d.

Remark 2.6. Using this corollary, it is proved in [23] that for a bounded function ψ defined on the unit sphere S^{d-1} and smooth outside coordinate hyperplanes, its extension (not relabelled) $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$ is an L^p multiplier (see also [29, Lemma 5]). If in addition we assume that ψ is smooth on the whole manifold, i.e. $\psi \in C^d(\mathrm{S}^{d-1})$, then the corresponding operator satisfies

$$\|\mathcal{A}_\psi\|_{L^p \to L^p} \le C\|\psi\|_{C^d(\mathrm{S}^{d-1})},$$

with a constant C depending only on $p \in \langle 1, \infty\rangle$ and d.

Here, we shall need a similar statement.

Lemma 2.7. *Let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a smooth compactly supported function equal to one on the unit ball centered at the origin. Then for any $\gamma > 0$ the multiplier operator $\mathcal{T}^\gamma$ with the symbol*

$$T^\gamma(\boldsymbol{\xi})(1 - \theta(\boldsymbol{\xi})) = \frac{1}{|\boldsymbol{\xi}|^\gamma}(1 - \theta(\boldsymbol{\xi}))$$

is a continuous $L^p(\mathbb{R}^d) \to W^{\gamma,p}(\mathbb{R}^d)$ operator for any $p \in \langle 1, +\infty\rangle$. Specially, due to the Rellich theorem it is a compact $L^p(\mathbb{R}^d) \to L^p_{\mathrm{loc}}(\mathbb{R}^d)$ operator.

Proof. We shall first prove that the operator $\mathcal{T}^\gamma$ is a continuous operator on $L^p(\mathbb{R}^d)$. To this effect, remark that it is sufficient to prove that T^γ satisfies condition of Corollary 2.5 away from the origin. Around the origin, the operator $\mathcal{T}^\gamma$ is controlled by the term $(1 - \theta)$ (which is equal to zero on $B(0, 1)$ and obviously satisfies conditions of Corollary 2.5). We use the induction argument with respect to the order of derivative in (2.6).

- $n = 1$: In this case, we compute

$$\partial_k T^\gamma(\boldsymbol{\xi}) = C_k \frac{1}{\xi_k} T^\gamma(\boldsymbol{\xi}) \big(\frac{\xi_k}{|\boldsymbol{\xi}|})\big)^2$$

for some constant C_k. From here, it obviously follows $|\xi_k \partial_k T^\gamma(\boldsymbol{\xi})| \le C$ for $\boldsymbol{\xi} \in \mathbb{R}^d$ away from the origin.

• $n = m$: Our inductive hypothesis is that a $\boldsymbol{\alpha}$-order derivatives of $T^\gamma(\boldsymbol{\xi})$ can be represented as

$$\partial^{\boldsymbol{\alpha}} T^\gamma(\boldsymbol{\xi}) = \frac{1}{\boldsymbol{\xi}^{\boldsymbol{\alpha}}} T^\gamma(\boldsymbol{\xi}) P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{2.7}$$

where $P_{\boldsymbol{\alpha}}$ is a bounded function satisfying (2.6) for $|\tilde{\boldsymbol{\alpha}}| \le d - |\boldsymbol{\alpha}|$.

• $n = m+1$: To prove that (2.7) holds for $|\boldsymbol{\alpha}| = m+1$ it is enough to notice that $\boldsymbol{\alpha} = \mathsf{e}_k + \boldsymbol{\alpha}'$, where $|\boldsymbol{\alpha}'| = m$, and that according to the induction hypothesis we have

$$\partial^{\boldsymbol{\alpha}} T^\gamma = \partial_k \partial^{\boldsymbol{\alpha}'} T^\gamma = \partial_k \Big(\frac{1}{\boldsymbol{\xi}^{\boldsymbol{\alpha}'}} T^\gamma(\boldsymbol{\xi}) P_{\boldsymbol{\alpha}'}(\boldsymbol{\xi}) \Big) = \frac{1}{\boldsymbol{\xi}^{\boldsymbol{\alpha}}} T^\gamma(\boldsymbol{\xi}) P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}),$$

where

$$P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = (P_{\mathsf{e}_k} P_{\boldsymbol{\alpha}'} + \xi_k \partial_k P_{\boldsymbol{\alpha}'} - \alpha_k P_{\boldsymbol{\alpha}'})(\boldsymbol{\xi}),$$

thus satisfying conditions (2.6) as well.

From these, (2.6) immediately follows for T^γ away from the origin, thus proving that the operator $\mathcal{T}^\gamma$ is a continuous operator on $L^p(\mathbb{R}^d)$.

It remains to prove that for any j the multiplier operator $\partial^\gamma_{x_j} \mathcal{T}^\gamma$ is a continuous $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ operator. To accomplish this, notice that its symbol is $(1 - \theta(\boldsymbol{\xi}))(2\pi i \xi_j / |\boldsymbol{\xi}|)^\gamma$. Thus, away from the origin, it is a composition of a smooth function and the projection $\boldsymbol{\xi} \to \boldsymbol{\xi}/|\boldsymbol{\xi}|$, and by Remark 2.6 it satisfies conditions of Corollary 2.5. □

3. Velocity averaging

In this section, we provide a velocity averaging result for a linear transport equation with a low regularity assumptions on the coefficients.

More precisely, we consider a sequence of functions (u_n) weakly converging to zero in $L^p(\mathbb{R}^{d+m})$ for $p \ge 2$, and satisfying the following sequence of equations

$$\mathcal{P} u_n(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^d \partial_{x_k} \left(a_k(\mathbf{x}, \mathbf{y}) u_n(\mathbf{x}, \mathbf{y}) \right) = \partial^{\boldsymbol{\kappa}}_{\mathbf{y}} G_n(\mathbf{x}, \mathbf{y}), \tag{3.1}$$

where $\partial^{\boldsymbol{\kappa}}_{\mathbf{y}} = \partial^{\kappa_1}_{y_1} \dots \partial^{\kappa_m}_{y_m}$ for a multi-index $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_m) \in \mathbb{N}^m$.

By A we denote the principal symbol of the differential operator $\mathcal{P}$, which is of the form

$$A(\mathbf{y}, \mathbf{x}, \boldsymbol{\xi}) = \sum_{k=1}^d 2\pi i \boldsymbol{\xi}_k a_k(\mathbf{x}, \mathbf{y}). \tag{3.2}$$

We assume it satisfies the classical non-degeneracy conditions

$$\forall (\mathbf{x}, \boldsymbol{\xi}) \in D \times \mathrm{S}^{d-1} \quad A(\mathbf{y}, \mathbf{x}, \boldsymbol{\xi}) \neq 0 \quad (\text{a.e. } \mathbf{y} \in \mathbb{R}^m), \tag{3.3}$$

where $D \subseteq \mathbb{R}^d$ is a full measure set, while S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

As for the coefficients from (3.1), we assume that

(a) $a_k \in L^{\bar{p}'}(\mathbb{R}^{m+d})$ for some $\bar{p} \in \langle 1, p \rangle$ $k = 1, \dots, d$;
(b) The sequence (G_n) is strongly precompact in the space $W^{-1,q}_{\rm loc}(\mathbb{R}^{m+d})$, where $q > 1$ is determined by the relation $1 + \frac{1}{p} = \frac{1}{\bar{p}} + \frac{1}{q}$.

The idea behind the extension of the velocity averaging result to coefficients of lower regularity (as given in (a)) is to consider the H-measure corresponding to the sequence (u_n) and the truncated sequence $(T_l(u_n))$ for the truncation operator T_l given as follows: for $l \in \mathbb{N}$,

$$T_l(u) = \begin{cases} \operatorname{sign}(u)l, & |u| > l \\ u, & u \in [-l, l]\,. \end{cases} \tag{3.4}$$

The operator T_l and its variants have been widely used [8, 19] where it was noticed that convergence of $(T_l(u_n))$, for every $l \in \mathbb{N}$, in $L^1_{\rm loc}(\mathbb{R}^d)$ implies the strong convergence of (u_n) in $L^1_{\rm loc}(\mathbb{R}^d)$. This property, beside being explored in the proof of Theorem 3.4, will be also used for proving an existence of a solution to non-linear transport equation (1.1) in the next section.

Let us now prove necessary properties of the truncation operator.

Lemma 3.1. *Let (u_n) be an equi-integrable sequence, bounded in $L^1(\Omega)$, where Ω is an open set in $\mathbb{R}^d$. Then for the sequence of truncated functions it holds*

$$\lim_l \sup_n \|T_l(u_n) - u_n\|_{L^1(\Omega)} \to 0\,. \tag{3.5}$$

Proof. Denote by $\Omega_n^l = \{\mathbf{x} \in \Omega :\, u_n(\mathbf{x}) > l\}$. Since (u_n) is bounded in $L^1(\Omega)$ we have

$$\sup_{k\in\mathbb{N}} \int_\Omega |u_n(\mathbf{x})| d\mathbf{x} \geq \sup_{k\in\mathbb{N}} \int_{\Omega_n^l} l d\mathbf{x} \implies \frac{1}{l} \sup_{k\in\mathbb{N}} \int_\Omega |u_n(\mathbf{x})| d\mathbf{x} \geq \sup_{k\in\mathbb{N}} \operatorname{meas}(\Omega_n^l),$$

implying that

$$\lim_{l\to\infty} \sup_{n\in\mathbb{N}} \operatorname{meas}(\Omega_n^l) = 0. \tag{3.6}$$

Now

$$\int_\Omega |u_n - T_l(u_n)| dx \leq \int_{\Omega_n^l} |u_n| dx \underset{l\to\infty}{\to} 0$$

uniformly with respect to n according to (3.6) and equi-integrability of (u_n). Thus, (3.5) is proved. □

Lemma 3.2. *Let (u_n) be an equi-integrable sequence, bounded in $L^1(\Omega)$, where Ω is an open set in $\mathbb{R}^d$. Suppose that for each $l \in \mathbb{N}$ the sequence of truncated functions $(T_l(u_n))$ is precompact in $L^1(\Omega)$. Then there exists a subsequence (u_{n_k}) and function $u \in L^1(\Omega)$ such that*

$$u_{n_k} \to u \quad \text{in } L^1(\Omega).$$

Proof. By the strong precompactness assumptions on truncated sequences, there exists a subsequence (u_{n_k}) such that for every $l \in \mathbb{N}$ the sequence $(T_l(u_{n_k}))$ is convergent in $L^1(\Omega)$, with a limit denoted by u^l. We prove that the obtained sequence (u^l) converges strongly in $L^1(\Omega)$ as well. To this end, note that

$$\begin{aligned} \|u^{l_1} - u^{l_2}\|_{L^1(\Omega)} &\leq \|u^{l_1} - T_{l_1}(u_{n_k})\|_{L^1(\Omega)} + \|T_{l_1}(u_{n_k}) - u_{n_k}\|_{L^1(\Omega)} \\ &\quad + \|T_{l_2}(u_{n_k}) - u_{n_k}\|_{L^1(\Omega)} + \|T_{l_2}(u_{n_k}) - u^{l_2}\|_{L^1(\Omega)}\,, \end{aligned}$$

which together with Lemma 3.1 implies that (u^l) is a Cauchy sequence. Thus, there exists $u \in L^1(\Omega)$ such that

$$u^l \to u \quad \text{in } L^1(\Omega). \tag{3.7}$$

Now it is not difficult to see that entire (u_{n_k}) converges toward u in $L^1(\Omega)$ as well. Namely, it holds

$$\|u_{n_k} - u\|_{L^1(\Omega)} \leq \|u_{n_k} - T_l(u_{n_k})\|_{L^1(\Omega)} + \|T_l(u_{n_k}) - u^l\|_{L^1(\Omega)} + \|u^l - u\|_{L^1(\Omega)},$$

which by the definition of functions u^l, and convergences (3.5) and (3.7) imply the statement. □

An elementary corollary of the previous lemma is:

Corollary 3.3. *Let (u_n) be a bounded sequence in $L^p(\Omega) \cap L^1(\Omega)$ for some $p > 1$, where Ω is an open set in $\mathbb{R}^d$. Suppose that for each $l \in \mathbb{N}$ the sequence of truncated functions $(T_l(u_n))$ is precompact in $L^1(\Omega)$. Then there exists a subsequence (u_{n_k}) and function $u \in L^p(\Omega)$ such that*

$$u_{n_k} \to u \quad \text{in } L^1(\Omega).$$

Proof. It is sufficinet to notice that every sequence (u_n) which is bounded in $L^p(\Omega)$, for $p > 1$, is equi-integrable. □

The result given in Theorem 3.4 below is usually called a velocity averaging lemma. Its importance is demonstrated in many works, but we shall mention only very famous [32] (kinetic formulation of conservation laws) and [17] (existence of weak solution to the Boltzman equation). Concerning the averaging lemma itself, one can consult e.g. [18, 20, 36, 37]. We remark that almost all the velocity averaging results were given for homogeneous equations (i.e. the ones where coefficients do not depend on $\mathbf{x} \in \mathbb{R}^d$). The reason for this one can search in the fact that, in the homogeneous situation, one can separate the solutions u_n from the coefficients (e.g. by applying the Fourier transform with respect to $\mathbf{x}$), and this is basis of most of the methods (see e.g. [37] and references therein). We remark that more detailed observations on this issue one can find in the introduction of [29]. In order to deal with the heterogeneous situation that we have here, we need microlocal defect tools, in particular H-measures. This brings obstacles of their own which forced us to use the restrictive non-degeneracy condition in [30]:

$$\frac{|A|^2}{|A|^2 + \delta} \to 1 \quad \text{in } L^{\bar{p}'}_{\rm loc}(\mathbb{R}^{d+m}; C^d(\mathrm{S}^{d-1}))$$

strongly as $\delta \to 0$. By using the characterization of H-measures provided by Theorem 2.2 we are able to generalize that result by assuming merely the classical non-degeneracy condition (3.3).

In addition, by allowing for lower regularity assumptions on the coefficients a_k the following theorem also generalizes the velocity averaging results provided in [29].

Theorem 3.4. *Assume that $u_n \rightharpoonup 0$ weakly in $L^p_{\rm loc}(\mathbb{R}^{d+m})$, $p \geq 2$, where u_n represent weak solutions to* (3.1) *with conditions (a) and (b) being fulfilled. Furthermore, we assume that the classical non-degeneracy conditions* (3.3) *are satisfied.*

Then, for any $\rho \in L^2_c(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} \rho(\mathbf{y}) u_n(\mathbf{x}, \mathbf{y}) d\mathbf{y} \to 0 \quad \text{strongly in } L^1_{\rm loc}(\mathbb{R}^d). \tag{3.8}$$

Proof. Fix $\rho \in C_{\rm c}(\mathbb{R}^m)$, $\varphi \in L^\infty_c(\mathbb{R}^d)$, and $l \in \mathbb{N}$. Denote by V_l a weak $*$ $L^\infty(\mathbb{R}^d)$ limit along a subsequence of truncated averages defined by

$$V_n^l = \varphi T_l\Big(\int_{\mathbb{R}^m} \rho(\tilde{\mathbf{y}}) u_n(\cdot, \tilde{\mathbf{y}}) d\tilde{\mathbf{y}}\Big),$$

where T_l is the truncation operator introduced in (3.4) below. Denote $v_n^l = V_n^l - V_l$ and remark that $v_n^l \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R}^d)$ with respect to n.

Next, let μ_l be the generalized H-measure defined in Theorem 2.2 corresponding to (sub)sequences (of) φu_n and v_n^l.

Take a dual product of (3.1) with the test functions

$$g_n(\mathbf{x},\mathbf{y}) = \varphi_1(\mathbf{x})\rho_1(\mathbf{y})(\mathcal{T}^1 \circ \mathcal{A}_{\psi_{\mathrm{S}^{d-1}}})(v_n)(\mathbf{x}),$$

where $\psi \in C^d(S^{d-1})$, $\varphi_1 \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, and $\rho_1 \in \mathrm{C}_c^{|\boldsymbol{\kappa}|}(\mathbb{R}^m)$ are arbitrary test functions, while $\boldsymbol{\kappa}$ is the multi-index appearing in (3.1), while $\mathcal{T}^1$ is the multiplier operator defined in Lemma 2.7 with the symbol equal to $\frac{1}{|\boldsymbol{\xi}|}$ outside the unit ball centered at the origin.

Letting $n \to \infty$ in such obtained expression, and taking into account Theorem 2.1, we obtain

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d \times \mathrm{P}} A(\mathbf{x},\boldsymbol{\xi},\mathbf{p}) \overline{\rho_1(\mathbf{p})\rho_2(\mathbf{q})\varphi(x)\psi(\boldsymbol{\xi})} d\mu_l(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi}) d\mathbf{p} d\mathbf{q} = 0.$$

The right-hand side of the last expression equals zero as, by assumption (b), the sequence (G_n) of functions on the right hand side of (3.1) converges strongly to zero in $L^1(\mathbb{R}^m; W^{-1,q}(\mathbb{R}^d))$, while, according to Lemma 2.7, the multiplier operator $\mathcal{T}^1 \circ \mathcal{A}_{\psi_{\mathrm{P}}} : L^q(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded.

As the test functions ρ_i, φ, and ψ are taken from dense subsets in appropriate spaces, one gets that μ_l satisfies localization principle

$$A\mu_l = 0, \tag{3.9}$$

where A is the principal symbol given by (3.2).

We aim to prove that from here, under condition (3.3), it follows that $\mu_l \equiv 0$. To this effect, we take an arbitrary $\delta > 0$, and for $\rho \in L^2_c(\mathbb{R}^m)$ and $\phi \in C_{\mathrm{c}}(\mathbb{R}^d); C^d(S^{d-1})$ we consider the test function

$$\frac{\rho(\mathbf{y})\phi(\mathbf{x},\boldsymbol{\xi})\overline{A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})}}{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})|^2 + \delta}.$$

The localization principle (3.9) implies

$$\big\langle \mu_l, \rho(\mathbf{y})\phi(\mathbf{x},\boldsymbol{\xi}) \frac{|A(\mathbf{x},\mathbf{y},\boldsymbol{\xi})|^2}{|A(\mathbf{x},\mathbf{y},\boldsymbol{\xi})|^2 + \delta} \big\rangle = 0\,,$$

which by means of representation (2.3) and Fubini's theorem takes the form

$$\int_{\mathbb{R}^d \times \mathrm{S}^{d-1}} \int_{\mathbb{R}^m} \frac{\rho(\mathbf{y})\phi(\mathbf{x},\boldsymbol{\xi})|A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})|^2}{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})|^2 + \delta} f_l(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) d\mathbf{y} d\nu_l(\mathbf{x},\boldsymbol{\xi}) = 0. \tag{3.10}$$

Let us denote

$$I_\delta(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^m} \rho(\mathbf{y}) \frac{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})|^2}{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{y})|^2 + \delta} f_l(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) d\mathbf{y}\,.$$

According to the non-degeneracy condition (3.3) and the representation of the measure ν_l given in Lemma 2.3, we have

$$I_\delta(\mathbf{x},\boldsymbol{\xi}) \to \int_{\mathbb{R}^m} \rho(\mathbf{y}) f_l(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) d\mathbf{y},$$

as $\delta \to 0$ for ν_l - a.e. $(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^d \times \mathrm{S}^{d-1}$. By using the Lebesgue dominated convergence theorem, it follows from (3.10) after letting $\delta \to 0$:

$$\langle \mu_l, \rho \otimes \phi \rangle = \int_{\mathbb{R}^d \times \mathrm{S}^{d-1}} \int_{\mathbb{R}^m} \rho(\mathbf{y})\phi(\mathbf{x},\boldsymbol{\xi}) f_l(\mathbf{y},\mathbf{x},\boldsymbol{\xi}) d\mathbf{y} d\nu_l(\mathbf{x},\boldsymbol{\xi}) = 0\,,$$

i.e. $\mu_l = 0$ for every l.

From the definitions of the generalized H-measures and the truncation operator T_l, we conclude by taking in (2.2) test functions $\psi = 1$ and $\phi_1\phi_2 = \chi_{\mathrm{supp}\varphi} \otimes \rho$ for the previously chosen φ and ρ (see the beginning of the proof):

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi^2(x) \Big| T_l \int_{\mathbb{R}^m} \rho(\mathbf{y}) u_n(\mathbf{x}, \mathbf{y}) d\mathbf{y} \Big|^2 d\mathbf{x}, \quad l \in \mathbb{N}. \tag{3.11}$$

Now, using Corollary 3.3, we obtain the desired convergence (3.8). □

4. Quasi-solutions and kinetic formulation

In this section, we shall introduce the notion of quasi-solution to (1.1). A similar notion was first introduced in [34]. In a special situation, e.g. in the case when the flux is independent of $(t, \mathbf{x})$ and when the measure ζ from below is non-negative, then the quasi-solution is an entropy admissible solution that singles out a physically relevant solutions to the equation (1.1) (see [27]). The notion of quasi-solution will lead to an appropriate kinetic formulation of the equation under consideration which will enable us to use H-measures.

Definition 4.1. A measurable function u defined on $\mathbb{R}^+ \times \mathbb{R}$ is called a quasi-solution to (1.1), accompanied by assumptions (i) and (ii), if the following Kruzhkov type entropy equality holds

$$\partial_t |u - \lambda| + \mathrm{div}[\mathrm{sgn}(u - \lambda)(f(t, \mathbf{x}, u) - f(t, \mathbf{x}, \lambda))] = -\zeta(t, \mathbf{x}, \lambda), \tag{4.1}$$

for some $\zeta \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ that we call the quasi-entropy defect measure.

From the above entropy condition, the following kinetic formulation can be derived.

Theorem 4.2. *Denote $F = \partial_\lambda f$. If the function u is a quasi-solution to* (1.1) *then the function*

$$h(t, \mathbf{x}, \lambda) = \mathrm{sgn}(u(t, \mathbf{x}) - \lambda) = -\partial_\lambda |u(t, x) - \lambda| \tag{4.2}$$

is a weak solution to the linear equation:

$$\partial_t h + \mathrm{div}\,(F(t, \mathbf{x}, \lambda) h) = \partial_\lambda \zeta(t, \mathbf{x}, \lambda) \tag{4.3}$$

Proof. It is sufficient to find the derivative of (4.1) with respect to $\lambda \in \mathbb{R}$ to obtain (4.3). □

To prove existence of a solution to the Cauchy problem (1.1), (1.2), we need the following general statement.

Theorem 4.3. *Assume that the function $F = \partial_\lambda f$, where f is the flux appearing in* (1.1)*, is such that for almost every $(t, \mathbf{x}) \in \mathbb{R}^d_+$ and every $\boldsymbol{\xi} \in S^d$ the mapping*

$$\lambda \mapsto \Big(\xi_0 + \sum_{k=1}^d F_k(t, \mathbf{x}, \lambda)\xi_k\Big), \tag{4.4}$$

is not zero on any set of positive measure.

Let (u_n) be a sequence of quasi-solutions to (1.1) *bounded in $L^p(\mathbb{R}^+ \times \mathbb{R}^d), p > 1$. Then it is strongly precompact in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ if the sequence of corresponding entropy defect measures (ζ_n) is strongly precompact in $W^{-1,q}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ for some $q > 1$.*

Proof. Denote by $h_\infty \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ a weak-$\star$ limit along a subsequence of the sequence $h_n(t,x,\lambda) := sign(u_n(t,x)-\lambda)$. By the boundedness assumption, the sequence (ζ_n) of quasi-entropy defect measures corresponding to (u_n) weakly converges to $\zeta \in L^1(\mathbb{R}; W^{-1,q}(\mathbb{R}^+ \times \mathbb{R}^d))$ (possibly passing to a subsequence). Put $v_n(t,\mathbf{x},\lambda) = h(t,\mathbf{x},\lambda) - h_n(t,\mathbf{x},\lambda)$ and $\sigma_n = \zeta - \zeta_n$. The sequence (v_n) satisfies

$$\partial_t v_n + \operatorname{div}\left(F(t,\mathbf{x},\lambda)v_n\right) = \partial_\lambda \sigma_n(t,\mathbf{x},\lambda) \tag{4.5}$$

Applying Theorem 3.4 we obtain

$$\int \rho(\lambda) v_n(t,\mathbf{x},\lambda)d\lambda \to 0 \quad \text{in } L^1_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d),$$

for any test function $\rho \in L^2_c(\mathbb{R}^m)$. Since $v_n = h_n - h$, and $h_n(t,\mathbf{x},\lambda) = \operatorname{sgn}(u_n - \lambda)$, we conclude that for every $l > 0$,

$$\int_{-l}^{l} h_n(t,\mathbf{x},\lambda)d\lambda = 2T_l(u_n)(t,\mathbf{x}),$$

is strongly $L^1_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ precompact itself. Applying Corollary 3.3, the statement follows directly. □

Theorem 4.4. *Assume that the coefficients of* (1.1) *satisfy the non-degeneracy condition* (4.4)*. Then there exists a weak solution to* (1.1) *augmented with the initial condition* $u|_{t=0} = u_0 \in L^1 \cap L^p(\mathbb{R}^d)$*,* $p > 1$*. The weak solution is at the same time the quasi-solution to* (1.1)*.*

Proof. It is sufficient to consider the regularization of problem (1.1):

$$\partial_t u_n + \operatorname{div}_x f^n(t,\mathbf{x},u_n) = s^n(t,\mathbf{x},u_n) + \frac{1}{n}\Delta u_n \tag{4.6}$$

$$u_n|_{t=0} = u_0(x) \in L^1 \cap L^p(\mathbb{R}^d), \tag{4.7}$$

where f^n and s^n are smooth regularization in variables t,x of the functions f and s respectively given by the convolution with a smooth, compactly supported, non-negative kernel ω with total mass one.

$$f^n(t,\mathbf{x},\lambda) = f(\cdot,\cdot,\lambda) \star_{t,\mathbf{x}} n^{d+1}\omega_n(t,\mathbf{x}), \quad s^n(t,\mathbf{x},\lambda) = s(\cdot,\cdot,\lambda) \star_{t,\mathbf{x}} n^{d+1}\omega_n(t,\mathbf{x}),$$

where $\omega_n(t,\mathbf{x}) = \omega(nt,n\mathbf{x})$.

The existence of smooth L^1-solution to (4.6), (4.7) can be found in [28]. Let us prove that the sequence of such solutions (u_n) is bounded in $L^p_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ (locally with respect to $t \in \mathbb{R}^+$).

Take the function S given in (iv) and multiply (4.6) by $S'(u)$. After standard manipulations, we reach

$$\begin{aligned} &\partial_t S(u_n) + \operatorname{div}\Big(\int_0^{u_n} S'(\lambda)\partial_\lambda f^n(t,\mathbf{x},\lambda)d\lambda\Big) + \int_0^{u_n} S''(\lambda)\operatorname{div} f^n(t,\mathbf{x},\lambda)d\lambda \\ &= s^n(t,\mathbf{x},u_n)S'(u_n) + \frac{1}{n}\Delta S(u_n) - \frac{1}{n}S''(u_n)|\nabla u_n|^2. \end{aligned}$$

Integrating this over $[0,t] \times \mathbb{R}^d$ and using (1.4) and (1.5), we see that

$$\begin{aligned} &\int_{\mathbb{R}^d} S(u_n(t,\mathbf{x}))d\mathbf{x} + \frac{1}{n}\int_0^t \int_{\mathbb{R}^d} S''(u_n)|\nabla u_n|^2 d\mathbf{x}dt' \\ &\le \int_{\mathbb{R}^d} S(u_0(\mathbf{x}))d\mathbf{x} + C_1 + C_2, \end{aligned} \tag{4.8}$$

and the L^p-bound follows from the condition (1.3). We remark that we also have $\frac{1}{n}\int_0^t\int_{\mathbb{R}^d}|\nabla u_n|^2 d\mathbf{x}dt' \leq C < \infty$ since S is convex.

Next, remark that for any $M>0$ and $K\subset\subset \mathbb{R}^+\times\mathbb{R}^d$, and any $q<r$, we have

$$\|\sup_{\lambda\in[-M,M]}|f^n(\cdot,\lambda)-f(\cdot,\lambda)|\|_{L^q(K)}\to 0, \tag{4.9}$$

as $n\to\infty$. Indeed, since f is continuous with respect to λ, for any fixed $(t,\mathbf{x})\in K$ there exists $\lambda(t,\mathbf{x})\in[-M,M]$ such that

$$\sup_{\lambda\in[-M,M]}|f^n(t,\mathbf{x},\lambda)-f(t,\mathbf{x},\lambda)| = |f^n(t,\mathbf{x},\lambda(t,\mathbf{x}))-f(t,\mathbf{x},\lambda(t,\mathbf{x}))|.$$

Then, according to the definition of the convolution, we have

$$\begin{aligned}
&\|\sup_{\lambda\in[-M,M]}|f^n(\cdot,\lambda)-f(\cdot,\lambda)|\|^q_{L^q(K)}\\
&=\int_K\Big|\int_{\mathbb{R}^{d+1}} f(\tau,\mathbf{y},\lambda(t,\mathbf{x}))\omega_n(t-\tau,\mathbf{x}-\mathbf{y})d\tau d\mathbf{y}-f(t,\mathbf{x},\lambda(t,\mathbf{x}))\Big|^q dtd\mathbf{x}\\
&=\int_K\Big|\int_{\mathbb{R}^{d+1}}(f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t,\mathbf{x}))-f(t,\mathbf{x},\lambda(t,\mathbf{x})))\omega(\eta,\mathbf{z})d\tau d\mathbf{z}\Big|^p dtd\mathbf{x}\\
&\leq\int_K\Big|\int_{\mathbb{R}^{d+1}}(f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z}))\\
&\quad -f(t,\mathbf{x},\lambda(t,\mathbf{x})))\omega(\eta,\mathbf{z})d\tau d\mathbf{z}\Big|^q dtd\mathbf{x}\\
&\quad +\int_K\Big|\int_{\mathbb{R}^{d+1}}(f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z}))\\
&\quad -f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t,\mathbf{x})))\omega(\eta,\mathbf{z})d\tau d\mathbf{z}\Big|^q dtd\mathbf{x}.
\end{aligned} \tag{4.10}$$

Now, since $f(t,\mathbf{x},\lambda(t,\mathbf{x}))\in L^r(K)$, it follows that for almost every $(t,\mathbf{x})\in K$ and $(\tau,\mathbf{z})\in\operatorname{supp}\omega$ it holds $f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z}))\to f(t,\mathbf{x},\lambda(t,\mathbf{x}))$ pointwise (since almost every point of $L^1_{\rm loc}$ function is the Lebesgue one). The same holds for the bounded function $\lambda(t,\mathbf{x})$, such implying the following almost everywhere convergence as $n\to\infty$:

$$\begin{aligned}
&|f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z}))-f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda(t,\mathbf{x}))|\\
&\leq\sup_{|\lambda|<M}|\partial_\lambda f(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z},\lambda)|\,|\lambda(t+\frac{1}{n}\eta,\mathbf{x}+\frac{1}{n}\mathbf{z})-\lambda(t,\mathbf{x})|\to 0.
\end{aligned}$$

Therefore, the Lebesgue dominated convergence theorem applied in (4.10) provides (4.9).

In the rest of the proof, we shall show that conditions of Theorem 4.3 are satisfied for (u_n). By multiplying (4.6) by $\operatorname{sgn}(u_n-\lambda)$, it is easy to see that the sequence (u_n) satisfies the entropy inequality

$$\begin{aligned}
&\partial_t|u_n-\lambda|+\sum_{k=1}^d\partial_{x_k}\Big(\operatorname{sgn}(u_n-\lambda)(f_k^n(t,\mathbf{x},u_n)-f_k^n(t,\mathbf{x},\lambda))\Big)\\
&+\frac{1}{n}\sum_k^d\partial^2_{x_k}|u_n-\lambda|-\operatorname{sgn}(u_n-\lambda)s^n(t,\mathbf{x},u_n)+\sum_{k=1}^d\operatorname{sgn}(u_n-\lambda)f^n_{k,x_k}(t,\mathbf{x},u_n)\leq 0.
\end{aligned}$$

By the Schwartz lemma on non-negative distributions,

$$\begin{aligned}
&\partial_t|u_n-\lambda|+\sum_{k=1}^d\partial_{x_k}\Big(\operatorname{sgn}(u_n-\lambda)(f_k(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},\lambda))\Big)\\
&=\gamma_n(t,\mathbf{x},\lambda)-\sum_{k=1}^d\partial_{x_k}\Big(\operatorname{sgn}(u-\lambda)(f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n)\\
&\quad+f_k(t,\mathbf{x},\lambda)-f_k^n(t,\mathbf{x},\lambda))\Big)-\frac{1}{n}\sum_k^d\partial_{x_k}^2|u_n-\lambda|\\
&\quad-\sum_{k=1}^d\operatorname{sgn}(u_n-\lambda)f^n_{k,x_k}(t,\mathbf{x},u_n)+\operatorname{sgn}(u_n-\lambda)s^n(t,\mathbf{x},u_n),
\end{aligned}\tag{4.11}$$

where (γ_n) is a bounded sequence in $\mathcal{M}(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$. In the sequel we shall prove that every term of the right-hand side of (4.11) is strongly precompact in $W^{-1,q}_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$ for every $q\in[1,\frac{d+1}{d})$. Let us fix here such a q and use it throughout this proof.

Indeed, (γ_n) is bounded in $\mathcal{M}(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}))$, and same holds for the penultimate term due to conditions from (iii) on $\operatorname{div} f$. The compactness for measures result (e.g. [21, Theorem 1.6]) provides that both terms are strongly precompact in $W^{-1,q}_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$.

To proceed, fix relatively compact $K\Subset\mathbb{R}^+\times\mathbb{R}^d$. Then, for an arbitrary fixed $\varepsilon>0$ find $M>0$ such that

$$\operatorname{meas}(\{|u_n|>M\}\cap K)\le\varepsilon\quad\text{for any } n\in\mathbb{N}.\tag{4.12}$$

Such a choice is possible because of Lemma 3.1, more precisely relation (3.6) and the fact that every L^p, $p>1$, bounded sequence is equi-integrable.

Now, we can prove that for every $k=1,\dots,d$,

$$\|f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n)\|_{L^q(K)}\to0\quad\text{as } n\to\infty.\tag{4.13}$$

First notice that

$$\begin{aligned}
&\|f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n)\|_{L^q(K)}\\
&\le\|(f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n))\|_{L^q(K\cap\{|u_n|>M\})}\\
&\quad+\|\sup_{\lambda\in[-M,M]}(f_k^n(t,\mathbf{x},\lambda)-f_k(t,\mathbf{x},\lambda))\|_{L^q(K)}.
\end{aligned}\tag{4.14}$$

Then, according to (4.12), by the Hölder inequality we have

$$\begin{aligned}
&\|(f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n))\|_{L^q(K\cap\{|u_n|>M\})}\\
&\le\|(f_k^n(t,\mathbf{x},u_n)-f_k(t,\mathbf{x},u_n))\|_{L^r(K\cap\{|u_n|>M\})}\operatorname{meas}(\{|u_n|>M\}\cap K)^{\frac{r-q}{rq}}\\
&\le2\varepsilon^{\frac{r-q}{rq}}\|\sup_{\lambda\in\mathbb{R}}f(\cdot,\cdot,\lambda)\|_{L^r(\mathbb{R}^+\times\mathbb{R}^d)}.
\end{aligned}$$

Now, according to (4.9)

$$\|\sup_{\lambda\in[-M,M]}(f_k^n(t,\mathbf{x},\lambda)-f_k(t,\mathbf{x},\lambda))\|_{L^p(K)}\to0\quad\text{as } n\to\infty.$$

From the last two estimates, (4.14), and arbitrariness of ε, we conclude that (4.13) holds. This implies

$$\sum_{k=1}^{d} \partial_{x_k} \left(\operatorname{sgn}(u-\lambda)(f_k^n(t,\mathbf{x},u_n) - f_k(t,\mathbf{x},u_n) + f_k(t,\mathbf{x},\lambda) - f_k^n(t,\mathbf{x},\lambda))\right) \to 0$$

as $n \to \infty$ in $W^{-1,q}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$.

Also, according to (1.3), for any relatively compact $K \subset\subset \mathbb{R}^+ \times \mathbb{R}^d$, the sequence of functions $(\operatorname{sgn}(u_n - \lambda) s^n(t,x,u_n))$ satisfies

$$\sup_n \|s^n(t,\mathbf{x},u_n)\|_{L^1(K)} < \infty,$$

i.e. $s^n(t,\mathbf{x},u_n) \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$, and thus it is strongly precompact in $W^{-1,q}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$.

Next, according to (iii) the term

$$\sum_{k=1}^{d} \operatorname{sgn}(u_n - \lambda) f^n_{k,x_k}(t,\mathbf{x},u_n) = \operatorname{sgn}(u-\lambda) \operatorname{div} f^n(t,\mathbf{x},\lambda)|_{\lambda=u_n}$$

s bounded in $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ and thus precompact in $W^{-1,q}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$.

Finally, the sequence $(\frac{1}{n}\Delta|u_n - \lambda|)$ strongly converges to zero as $n \to \infty$ in $W^{-1,2}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ since, By the energy inequality $\sup_n \|\frac{1}{\sqrt{n}} \nabla u_n\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^d)} < \infty$ (see (4.8)) implying that $\frac{1}{\sqrt{n}}\big(\operatorname{sgn}(u_n - \lambda)\frac{1}{\sqrt{n}}\nabla u_n\big) \to 0$ in $L^2_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$. Therefore

$$\frac{1}{n}\Delta|u_n - \lambda| = \operatorname{div}\Big(\frac{1}{\sqrt{n}}\big(\operatorname{sgn}(u_n - \lambda)\frac{1}{\sqrt{n}}\nabla u_n\big)\Big)$$

converges strongly to zero in $W^{-1,2}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ and thus it is precompact in $W^{-1,q}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$ as well (recall that $q < 2$).

Consequently, (u_n) is a sequence of quasi-solutions to (1.1) satisfying assumptions of Theorem 4.3, which implies its strong precompactness in $L^1_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d)$. The strong $L^1_{\rm loc}$-limit of (u_n) along a subsequence represents a weak solution to (1.1). □

Acknowledgements. This research was partially developed while Martin Lazar was a Postdoctoral Fellow on the Basque Center for Applied Mathematics (Bilbao, Spain) within the NUMERIWAVES FP7-246776 project. The work is partly supported by Croatian Science Foundation under Project WeConMApp/HRZZ-9780, by the bilateral Croatian-Montenegro project *Multiscale methods and calculus of variations*, as well as by the DAAD project *Center of Excellence for Applications of Mathematics*.

References

[1] J. Aleksic, D. Mitrovic; *On the compactness for two dimensional scalar conservation law with discontinuous flux*, Commun. Math. Sci., **7** (2009), 963–971.

[2] B. Andreianov, C. Cancès; *Vanishing capillarity solutions of Buckley-Leverett equation with gravity in two-rocks medium*, Comput. Geosci., **17**(3) (2013), 551–572.

[3] B. Andreianov, K. H. Karlsen, N. H. Risebro; *A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux*, Arch. Ration. Mech. Anal., **201** (2011), 27–86.

[4] B. Andreianov, D. Mitrovic; *Entropy conditions for scalar conservation laws with discontinuous flux revisited*, Annales Institut Henri Poincare C, Analyse Non Lineaire Vol. **6** (2015), 1307–1335

[5] N. Antonić, D. Mitrović; *H-distributions - an extension of the H-measures in $L^p - L^q$ setting*, Abstr. Appl. Anal., **2011** (2011), 12 pp.

[6] N. Antonić, M. Lazar; *H-measures and variants applied to parbolic equations*, J. Math. Anal. Appl., **343** (2008), 207–225.

[7] N. Antonić, M. Lazar; *Parabolic variant of H-measures in homogenisation of a model problem based on Navier-Stokes equation*, *Nonlinear Anal. B: Real World Appl.*, **11** (2010), 4500–4512.

[8] M. Bendahmane, K. H. Karlsen; *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*, Siam J. Math. Anal., **36** (2004), 405-422.

[9] R. Burger, K. H. Karlsen, J. Towers; *A conservation law with discontinuous flux modelling traffic flow with abruptly changing road surface conditions.* in Hyperbolic problems: theory, numerics and applications, 455-464, Proc. Sympos. Appl. Math. 67, Part 2, Amer. Math. Soc. Providence, 2009.

[10] R. Burger, K. H. Karlsen, J. Towers; *A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units*, SIAM J. Appl. Math. ,**65**(3) (2005), 882–940.

[11] G. Crasta, V. De Cicco, G. De Philippis; *Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux*, Comm. in PDEs, **40** (2015), 694–726,

[12] G. Crasta, V. De Cicco, G. De Philippis, F. Ghiraldin; *Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness*, to appear in Arch. Ration. Mech. Anal.

[13] A. L. D'Alibard; *Kinetic formulation for heterogeneous scalar conservation laws*, Annales Institut Henri Poincare C, Analyse Non Lineaire Vol. **23** (2006), 475–498.

[14] B. Dehman, M. Léautaud, J. Le Rousseau; *Controllability of two coupled wave equations on a compact manifold.* Arch. Rational Mech. Anal., **211(1)** (2014) 113–187.

[15] S. Diehl; *On scalar conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Math. Anal., **6** (1995), 1425–1451.

[16] S. Diehl; *A conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Appl. Anal., **2** (1996), 388–419.

[17] R. J. Diperna, P. L. Lions; *Global Solutions of Boltzmann Equations and the Entropy Inequality*, Arch. Rat. Mech. Anal., **114** (1991), 47–55.

[18] R. J. DiPerna, P. L. Lions, Y. Meyer; *L^p regularity of velocity averages*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **8** (1991), 271–287.

[19] G. Dolzmann, N. Hungerbuhler, S. Müller; *Nonlinear elliptic systems with measure valued right-hand side*, Math. Zeitschrift, **226**, (1997) 545–574.

[20] F. Golse, L. Saint-Raymond; *Velocity averaging in L^1 for the transport equation*, C. R. Acad. Sci. Paris Sér. I Math., **334** (2002), 557–562.

[21] L. C. Evans; *Weak Convergence Methods for Nonlinear Partial Differential Equations*, Regional Conference Series in Mathematics, No. 74. Conference Board of the Mathematical Sciences, 1990.

[22] P. Gérard; *Microlocal Defect Measures*, Comm. Partial Differential Equations **16** (1991), 1761–1794.

[23] L. Grafakos; *Classical Fourier Analysis*, Graduate Text in Mathematics 249, Springer Science and Business Media, 2008

[24] E. Kaasschieter; *Solving the Buckley-Leverret equation with gravity in a heterogeneous porous media*, Comput. Geosci. **3** (1999), 23–48.

[25] K. H. Karlsen, M. Rascle, E. Tadmor; *On the existence and compactness of a two- dimensional resonant system of conservation laws*, Commun. Math. Sci. **5** (2007), 253–265.

[26] K. H. Karlsen, N. H. Risebro, J. Towers; *L^1-stability for entropy solutions of nonlinear degenerate parabolic connection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid. Selsk, **3** (2003), 1–49.

[27] S. N. Kruzhkov; *First order quasilinear equations in several independent variables*, Mat. Sb. **81** (1970), 217-243.

[28] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Ural'ceva, (1968); *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs 23, Providence, RI: American Mathematical Society, pp. XI+648

[29] M. Lazar, D. Mitrovic; *Velocity averaging – a general framework*, Dynamics of PDEs, **9** (2012), 239-260.

[30] M. Lazar, D. Mitrović; *On an extension of a bilinear functional on $L^p(\mathbb{R}^d) \times E$ to Bochner spaces with an application on velocity averaging*, C. R. Acad. Sci. Paris Sér. I Math. **351** (2013), 261–264.

[31] M. Lazar, E. Zuazua; *Averaged control and observation of parameter-depending wave equations*, C. R. Acad. Sci. Paris, Ser. I, **352** (2014) 497–502.

[32] P. L. Lions, B. Perthame, E. Tadmor; *A kinetic formulation of multidimensional scalar conservation law and related equations*, J. Amer. Math. Soc., **7**, (1994), 169–191.

[33] M. Mišur, D. Mitrović; *On a generalization of compensated compactness in the $L^p - L^q$ setting*, Journal of Functional Analysis, **268** (2015), 1904–1927.

[34] E. Yu. Panov; *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws.*, J. Hyperb. Differ. Equ., **2** (2005), 885–908.

[35] E. Yu. Panov; *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux*, Arch. Rational Mech. Anal. **195** (2010) 643-673.

[36] B. Perthame, P. Souganidis; *A limiting case for velocity averaging*, Ann. Sci. Ec. Norm. Sup. **4** (1998), 591–598.

[37] T. Tao, E. Tadmor; *Velocity Averaging, Kinetic Formulations, and Regularizing Effects in Quasi-Linear Partial Differential Equations*, Comm. Pure Appl. Math. **60** (2007), 1488–1521.

[38] L. Tartar; *H-measures, a new approach for studying homogenisation, oscillation and concentration effects in PDEs*, Proc. Roy. Soc. Edinburgh Sect. A, **115** (1990) 193–230.

Martin Lazar
University of Dubrovnik, Croatia
E-mail address: `mlazar@unidu.hr`

Darko Mitrović
Faculty of Mathematics, University of Montenegro, Montenegro
E-mail address: `darkom@ac.me`