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MORREY ESTIMATES FOR SUBELLIPTIC p-LAPLACE TYPE SYSTEMS WITH VMO COEFFICIENTS IN CARNOT GROUPS

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Abstract. In this article, we study estimates in Morrey spaces to the horizontal gradient of weak solutions for a class of quasilinear sub-elliptic systems of p-Laplace type with VMO coefficients under the controllable growth over Carnot group if p is not too far from 2. We also show a local Hölder continuity with an optimal exponent to the solutions.

1. Introduction

Let $\mathcal{G}$ be a Carnot group of step $r \geq 2$, that is, a simply connected Lie group with Lie algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \oplus_{i=1}^r V_j$ such that $[V_1, V_j] = V_{j+1}$ for $1 \leq j \leq r-1$ and $[V_1, V_r] = 0$. The homogeneous dimension of $\mathcal{G}$ is defined as $Q = \sum_{i=1}^r i m_i$, where $m_i = \dim V_i$ is the topological dimension with $m_1 = m$. For a family of vector fields $X = (X_1, X_2, \dots, X_m)$ satisfying the Hörmander's finite rank condition rank Lie$[X_1, X_2, \dots, X_m] = r$, we assume that each component $b_{ij}(x)$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ of vector-field $X_i = \sum_{j=1}^n b_{ij}(x)\partial_j$ is a smooth function defined in Carnot group $\mathcal{G}$ for $i = 1, 2, \dots, m$. Therefore, $Xu = (X_1u, X_2u, \dots, X_mu)$ may be called the horizontal gradient of u, which is understood as $X_iu = \langle X_i, \nabla u\rangle = \Sigma_{j=1}^n b_{ij}(x)\partial_j u$ for $i = 1, 2, \dots, m$ if $u \in C^1(\mathcal{G})$, see [11, 15, 21, 7, 29]. To describe our assumptions and main results better, we first recall some relevant notations and basic facts.

Definition 1.1. An absolutely continuous path $\gamma : [0, T] \to \mathcal{G}$ is said X-subunit if
$$\dot{\gamma}(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t))$$
with $\sum_{i=1}^m c_i(t) \leq 1$, for almost every $t \in [0, T]$.

With X-subunit in hand, we can define the Carnot-Caratheodory's metric (the C-C distance) $d_X(x, y)$ as follows, see [19, 27].
$$d_X(x, y) = \inf\{T > 0 : \exists\gamma : [0, T] \to \mathcal{G}\ X\text{-subunit with } \ \gamma(0) = x,\ \gamma(T) = y\}.$$

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Note that these vector-fields $(X_1, \dots, X_m)$ are free up to the order r, then there exists a positive constant $C > 0$ satisfying the following relation between the C-C distance and the Euclidean metric, see [27, 10];

$$C^{-1}|x-y| \leq d_X(x,y) \leq C|x-y|^{1/r}.$$

In this context, all balls centered at x of radius R with respect to $d_X(x,y)$ are called metric balls and denoted still by $B_R(x)$. The distance function $d_X(\cdot,\cdot)$ satisfies the local doubling property, that is, for $B_{2R}(x) \Subset \mathcal{G}$ there exists a positive constant R_0 depending on vector fields X and $\mathcal{G}$ such that for all $0 < R \leq R_0$ there holds

$$|B_{2R}(x)| \leq C_d |B_R(x)|, \tag{1.1}$$

where the number $Q = \log_2 C_d$ is called the local homogeneous dimension of $\mathcal{G}$ with respect to the vector fields $X_1, X_2, \dots, X_m$. In fact, Q will play a role of the dimension in the local analysis involving what we are considering problems.

Let Ω be a bounded open subset in Carnot group $\mathcal{G}$. Let us recall the following horizontal Sobolev space with respect to given a family of vector fields $X = (X_1, X_2, \dots, X_m)$. For any $1 < p < \infty$ and $k \in \mathbb{N}$, we define

$$HW^{k,p}(\Omega) := \{u \in L^p(\Omega) : X_{i_1} \dots X_{i_k} u \in L^p(\Omega) \text{ for all } \{i_1, \dots, i_k\} \subset \{1, \dots, m\}\}$$

with the norm $\|u\|_{HW^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|X^k u\|_{L^p(\Omega)}$. Furthermore, the closure of $C_0^\infty(\Omega)$ in $HW^{k,p}(\Omega)$ is denoted by $HW_0^{k,p}(\Omega)$.

In this article, we consider the estimates in Morrey spaces to the horizontal gradient of weak solutions in $HW_0^{1,p}(\Omega)$ for the following degenerate subelliptic systems of p-Laplace type.

$$-\sum_{i=1}^m X_i \Big(\langle A(x) X_i u, X_i u \rangle^{\frac{p-2}{2}} A(x) X_i u \Big) = B(x, u, Xu), \quad \text{a. e. } x \in \Omega, \tag{1.2}$$

where $A(x) \in VMO \bigcap L^\infty(\Omega)$ and $B(x,u,Xu)$ satisfies a controllable growth. In order to more precisely impose structural assumptions on $A(x), B(x,u,Xu)$ and state our main results, we need recall two useful notations (see[2, 17, 28]).

Definition 1.2 (BMO functions)**.** Let $\Omega(x,r) = \Omega \cap B_r(x)$. For any $0 < s < +\infty$, we say $u \in L^1_{\rm loc}(\Omega)$ belongs to $BMO(\Omega)$ if

$$M_u(s) := \sup_{x \in \Omega, 0 < r < s} \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} |u(y) - \bar{u}_{\Omega(x,r)}| dy < +\infty,$$

where

$$\bar{u}_{\Omega(x,r)} = \fint_{\Omega(x,r)} u(y) dy = \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} u(y) dy.$$

Definition 1.3 (VMO functions)**.** Let $M_u(s)$ be defined as above. We say $u \in BMO(\Omega)$ belongs to $VMO(\Omega)$ if

$$\lim_{s \to 0} M_u(s) = 0,$$

where $M_u(s)$ is called the VMO modulus of u.

Definition 1.4 (Morrey space)**.** Let $p \geq 1, \lambda > 0$. For $u \in L^p_{\rm loc}(\Omega)$, if

$$\|u\|_{L_X^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega,\, 0 < r \leq d_0} \Big(\frac{r^\lambda}{|\Omega(x_0,r)|} \int_{\Omega(x_0,r)} |u|^p \, dx \Big)^{1/p} < +\infty, \tag{1.3}$$

then $u \in L_X^{p,\lambda}(\Omega)$, where $d_0 = \min\{\mathrm{diam}(\Omega), R_D\}$, and the norm of u is $\|u\|_{L_X^{p,\lambda}(\Omega)}$.

Definition 1.5 (Campanato space)**.** Let $p \geq 1, \lambda > -p$. If $u \in L^p_{\rm loc}(\Omega)$ satisfies

$$|u|_{\mathcal{L}_X^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega,\, 0<r\leq d_0} \Big(\frac{r^\lambda}{|\Omega(x_0,r)|}\int_{\Omega(x_0,r)} |u-u_{x_0,r}|^p\,dx\Big)^{1/p} < +\infty, \tag{1.4}$$

then $u \in \mathcal{L}_X^{p,\lambda}(\Omega)$, and has norm $\|u\|_{\mathcal{L}_X^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + |u|_{\mathcal{L}_X^{p,\lambda}(\Omega)}$.

On the basis of the above notation, we are now in a position to impose some structure assumptions on $A(x)$ and $B(x,u,Xu)$ as follows.

(H1) (Uniform ellipticity) For $A(x) = (a_{ij}^{\alpha\beta}(x))$, there exist L and ν, $L \geq \nu > 0$, such that for a.e. $x \in \Omega$ and for any $\xi \in \mathbb{R}^{nN}$ we have

$$\nu|\xi|^2 \leq a_{ij}^{\alpha\beta}(x)\xi_i^\alpha\xi_j^\beta \leq L|\xi|^2;$$

(H2) $a_{ij}^{\alpha\beta}(x) \in L^\infty(\Omega)\bigcap VMO_X$;

(H3) (Controllable growth) The inhomogeneity $B(x,u,Xu)$ satisfies

$$|B(x,u,Xu)| \leq \mu(|Xu|^{p(1-\frac{1}{\gamma})} + g(x)),$$

where

$$\gamma = \begin{cases} \frac{pQ}{Q-p}, & 1<p<Q \\ \text{any } \gamma \geq p, & p \geq Q, \end{cases}$$

$g(x) \in L^{q,\mu}(\Omega, \mathbb{R}^{nN})$ with $q > \frac{\gamma}{\gamma-1}$, and Q is the homogenous dimension.

We say that $u \in HW^{1,p}(\Omega, \mathbb{R}^{nN})$ is a weak solution of (1.2), if

$$\int_\Omega \big\langle \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu, X\varphi\big\rangle\,dx = \int_\Omega B(x,u,Xu)\varphi\,dx, \tag{1.5}$$

for all $\varphi \in HW_0^{1,p}(\Omega)$.

Recently several studies on subelliptic PDEs arising from non-commuting vector fields have been well developed based on the Hörmander's fundamental work [18]; see [4, 3, 10, 11, 15, 23, 24, 27, 22, 26, 29, 30]. Many important results about the fundamental solution to subelliptic operators and the Harmonic analysis theory on stratified nilpotent Lie groups have been obtained by Folland [15], Rothschild-Stein [24] and Nagel-Stein-Wainger [23]. These results laid a solid foundation for further investigation of subelliptic Partial Differiential Equations theory. Up to the 90s, the function theory and harmonic analysis tools on Carnot groups, such as the Sobolev embedding inequality of X-gradient and the isoperimetric inequality, become increasingly mature, cf. [3, 15, 16, 23, 24]. In fact, such subelliptic problems have received continuous attention due to their significant applications in geometry and physics [1, 24]. In the case of Euclidean spaces (i.e. $m = n$, $X_i = \frac{\partial}{\partial x_i}$), it was an important observation by Uhlenbeck [25] that there exists the interior $C^{1,\alpha}$-regularity by using the classical De Giorgi-Moser-Nash iteration to the homogeneous p-harmonic systems as a prototype. However, it is not true for subelliptic systems of p-Laplacian with $p > 1$. Actually, Domokos-Manfredi in [11, 12] and Domokos in [13, 14] have derived $\Gamma^{1,\alpha}$ regularity for p-harmonic systems only while p is in a neighborhood of 2 in Heisenberg group and in Carnot group, respectively. Very recently, Zheng-Feng [29, 30] also got the estimates and an application of the Green functions for subelliptic A-harmonic operators, and $\Gamma^{1,\alpha}$ regularity for weak solutions to subelliptic p-harmonic systems under the subcritical growth with p

close to 2, respectively. Notice that Fazio-Fanciullo [8] and Dong-Niu [9] recently established the estimates of the gradient in Morrey spaces to nonlinear subelliptic systems for $p = 2$. Therefore, this is a natural thought what happens if one consider a regularity of the gradient in Morrey spaces to subelliptic A-harmonic systems. In this article, we are devoted to local Morrey regularity of the horizontal gradient to a class of subelliptic A-harmonic systems with VMO coefficients under the controllable growth. More precisely, we have the following result.

Theorem 1.6. *Let $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.2) *with p close to 2. Suppose $A(x)$ and $B(x, u, Xu)$ satisfy* (H1)–(H3). *Then $Xu \in L_X^{p,\lambda}(\Omega, \mathbb{R}^{nN})$; moreover, there exists a constant $C > 0$ such that for any $\Omega' \Subset \Omega$ we have*

$$\|Xu\|_{L_X^{p,\lambda}(\Omega', \mathbb{R}^{nN})} \le C(\|Xu\|_{L^p(\Omega, \mathbb{R}^{nN})} + \|g\|_{L_X^{q,\mu}(\Omega, \mathbb{R}^{nN})}^{\frac{1}{p-1}}), \tag{1.6}$$

where

$$\lambda = \begin{cases} \frac{p}{p-1} \frac{\mu - q}{q}, & \frac{\gamma}{\gamma - 1} < q < \mu, \\ \text{any } \lambda \in (0, Q), & q \ge \mu. \end{cases}$$

Here, we employ a classical disturbance argument which is compared with subelliptic Laplacian due to the lack of regularity to subelliptic systems of p-Laplace. Indeed, our approach on the singularity ($1 < p < 2$) and degeneracy ($p > 2$) for A-harmonic systems (1.2) was essentially influenced by way of a comparison with sub-Laplacian while p is close to 2 because of the Cordes conditions. This is an important technique to consider PDEs with wild coefficients, also see [6].

With Theorem 1.6 in hand, as a direct consequence we can obtain an interior Hölder continuity of weak solutions of subelliptic systems (1.2) while $Q - n < \lambda < p$. Let us first recall the concept of Hölder space under the Carnot-Caratheodory metric.

Definition 1.7 (Hölder space)**.** Let $\Omega \Subset \mathcal{G}$ and $0 < \alpha < 1$. We say that $u \in \Gamma_X^{0,\alpha}(\Omega)$ has norm $\|u\|_{\Gamma_X^{0,\alpha}(\Omega)}$, if

$$\|u\|_{\Gamma_X^{0,\alpha}(\Omega)} := \sup_{\Omega} |u| + \sup_{\Omega} \frac{|u(x) - u(y)|}{[d_X(x, y)]^\alpha} < \infty. \tag{1.7}$$

Now we state the interior Hölder continuity of weak solutions with a sharp index to subelliptic systems (1.2) as follows.

Theorem 1.8. *Let $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.2) *with p close to 2. Suppose $A(x)$ and $B(x, u, Xu)$ satisfy H1-H3. If $Q - n < \lambda < p$, we have*

$$u \in \Gamma^{0,\alpha}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$$

with $\alpha = 1 - \frac{\lambda}{p}$.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries concerning the sub-elliptic setting and some several technical lemmas. In Section 3, we are devoted to proving our main results.

2. Preliminaries

We adopt the usual convention of denoting by C a general constant, which may vary from line to line in the same chain of inequalities. Now let us first recall the Sobolev embedding inequality with respect to the horizontal vector fields, see [3, 5, 16].

Lemma 2.1. *Let $1 \le p < Q$ and $1 \le q \le \frac{Qp}{Q-p}$, where Q is the homogeneous dimension of X in $\mathcal{G}$. Then we have:*

(1) *If $u(x) \in HW^{1,p}(B_{R_0}, X)$, then there exists a positive constant $C = C(p, q, Q, X)$ such that for any $0 < R < R_0$,*
$$\|u - \bar{u}_R\|_{L^q(B_R)} \le CR^{1+Q(\frac{1}{q}-\frac{1}{p})}\|Xu\|_{L^p(B_R)}, \tag{2.1}$$
where $\bar{u}_R = \frac{1}{|B_R|}\int_{B_R} u dx$.

(2) *If $u \in HW_0^{1,q}(B_{R_0}, X)$, then there exists a positive constant $C = C(p, q, Q, X)$ such that for any $0 < R < R_0$,*
$$\Big(\fint_{B_R} |u|^q\Big)^{1/q} \le CR\Big(\fint_{B_R} |Xu|^p\Big)^{1/p}. \tag{2.2}$$

To obtain the higher integrability of the horizontal gradients of solutions for (1.2) we recall the following reverse Hölder inequality originated from Gehring's celebrating work on quasiconformal mappings, see [17, Theorem 2.3 of Chapter 5].

Lemma 2.2. *Suppose that $h(x)$ and $f(x)$ are nonnegative measurable functions satisfying $h(x) \in L^t(\Omega)$ and $f(x) \in L^s(\Omega)$ with $t > s > 1$. If for all $x_0 \in \Omega$ and all $R : 0 < R < R_0 \le \operatorname{dist}(x_0, \partial\Omega)$ there holds*
$$\fint_{B_{\frac{R}{2}}(x_0)} f^s\,dx \le b\Big(\Big\{\fint_{B_R(x_0)} f\,dx\Big\}^s + \fint_{B_R(x_0)} h^s\,dx\Big) + \theta\fint_{B_R(x_0)} f^s\,dx, \tag{2.3}$$
with constants $b > 1$ and $0 \le \theta < 1$, then there exist positive constants $\delta = \delta(b, Q, q, s)$ and $C = C(b, Q, q, r)$ such that $f \in L^t_{\rm loc}(\Omega)$ for any $t \in [s, s+\delta)$ and
$$\Big\{\fint_{B_{\frac{R}{2}}(x_0)} f^t\,dx\Big\}^{1/t} \le C\Big\{\fint_{B_R(x_0)} f^s\,dx\Big\}^{1/s} + C\Big\{\fint_{B_R(x_0)} h^t\,dx\Big\}^{1/t}. \tag{2.4}$$

With the reverse Hölder inequality above in hand, We can obtain the following higher integrability of the horizontal gradients to systems (1.2).

Lemma 2.3 (Higher integrability)**.** *Let $u \in HW^{1,p}(\Omega)$ be any weak solution of quasilinear subelliptic systems (1.2) with $A(x), B(x, u, Xu)$ satisfying assumptions (H1) and (H3). Then, there exists a higher exponent $r : p < r < p + \delta$ such that for $\Omega' \Subset \Omega$, we have $Xu \in HW^{1,r}(\Omega')$. Moreover, there exists a positive constant $C = C(Q, L, \nu, p)$ such that for any $B_R(x_0) \Subset \Omega$ with the estimate*
$$\begin{aligned} &\Big(\fint_{B_{\frac{R}{2}}(x_0)} |Xu|^r)dx\Big)^{1/r} \\ &\le C\Big(\fint_{B_R(x_0)} |Xu|^p dx\Big)^{1/p} + C\Big(R\Big(\fint_{B_R(x_0)} |g(x)|^q dx\Big)^{1/q}\Big)^{\frac{1}{p-1}}. \end{aligned} \tag{2.5}$$

Proof. Given any $x_0 \in \Omega$, we take $R > 0$ such that $B_R := B_R(x_0) \Subset \Omega$. Let η be a cutting-off function with $\eta \in C_0^\infty(B_R)$ such that $0 \le \eta(x) \le 1$, $\eta = 1$ for $x \in B_{R/2}$, $\eta = 0$ for $x \in \mathbb{R}^n \setminus \overline{B}_R$ and $|X\eta| \le \frac{C}{R}$. Let us take a test function $\varphi = \eta^p(u - \bar{u}_R)$ in (1.2), it follows from (1.5) that
$$\begin{aligned} &\int_\Omega \Big\langle \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu, \eta^p Xu + p\eta^{p-1}(u - \bar{u}_R)X\eta\Big\rangle dx \\ &= \int_\Omega B(x, u, Xu)\eta^p(u - \bar{u}_R)dx, \end{aligned}$$

By considering the uniformly ellipticity (H1) and the controllable growth, it yields

$$
\begin{aligned}
&\nu^{p/2}\int_\Omega \eta^p|Xu|^p dx\\
&\le \int_\Omega \eta^p\langle A(x)Xu,Xu\rangle^{p/2}dx\\
&= -\int_\Omega \langle\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}A(x)Xu, p\eta^{p-1}(u-\bar u_R)X\eta\rangle dx\\
&\quad+\int_\Omega B(x,u,Xu)\eta^p(u-\bar u_R)dx\\
&\le pL^{p/2}\int_\Omega |\eta Xu|^{p-1}|(u-\bar u_R)X\eta|dx+\mu\int_\Omega \eta^p|u-\bar u_R||Xu|^{p(1-\frac{1}{\gamma})}dx\\
&\quad+\mu\int_\Omega \eta^p|u-\bar u_R||g(x)|dx\\
&:= pL^{p/2}I_1+\mu I_2+\mu I_3.
\end{aligned}
\tag{2.6}
$$

Next, we estimate I_1, I_2 and I_3. For I_1, using Young inequality with $\varepsilon_1>0$ and Sobolev inequality we have

$$
\begin{aligned}
I_1&=\int_\Omega |\eta Xu|^{p-1}|(u-\bar u_R)X\eta|dx\\
&\le \varepsilon_1\int_\Omega |\eta Xu|^p dx+C(\varepsilon_1)\int_\Omega |(u-\bar u_R)X\eta|^p dx\\
&\le \varepsilon_1\int_{B_R}|Xu|^p dx+\frac{C(\varepsilon_1)}{R^p}\int_{B_R}|u-\bar u_R|^p dx\\
&\le \varepsilon_1\int_{B_R}|Xu|^p dx+\frac{C(\varepsilon_1)}{R^p}\Big(\int_{B_R}|Xu|^{\frac{Qp}{Q+p}}dx\Big)^{\frac{Q+p}{Q}}.
\end{aligned}
\tag{2.7}
$$

For estimating I_2, by Sobolev inequality and Hölder inequality, it follows that

$$
\begin{aligned}
I_2&=\int_\Omega \eta^p|u-\bar u_R||Xu|^{p(1-\frac{1}{\gamma})}dx\\
&\le \Big(\int_{B_R}|u-u_R|^\gamma dx\Big)^{1/\gamma}\Big(\int_{B_R}|Xu|^p dx\Big)^{1-\frac{1}{\gamma}}\\
&\le CR^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\Big(\int_{B_R}|Xu|^p dx\Big)^{\frac{1}{p}-\frac{1}{\gamma}}\Big(\int_{B_R}|Xu|^p dx\Big),
\end{aligned}
\tag{2.8}
$$

where $\gamma\ge p$ is defined as the assumption H3 with $1+Q(\frac{1}{p}-\frac{1}{\gamma})\ge 0$.

Similarly, to estimate I_3 we use Hölder inequality, Young inequality with $\varepsilon_2>0$ and Sobolev inequality; it yields

$$
\begin{aligned}
I_3&=\int_\Omega \eta^p|u-\bar u_R||g(x)|dx\\
&\le \Big(\int_{B_R}|u-\bar u_R|^\gamma dx\Big)^{1/\gamma}\Big(\int_{B_R}|g(x)|^{\frac{\gamma}{\gamma-1}}dx\Big)^{\frac{\gamma-1}{\gamma}}\\
&\le CR^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\Big(\int_{B_R}|Xu|^p dx\Big)^{1/p}\Big(\int_{B_R}|g(x)|^{\frac{\gamma}{\gamma-1}}dx\Big)^{\frac{\gamma-1}{\gamma}}\\
&\le \varepsilon_2\int_{B_R}|Xu|^p dx+C(\varepsilon_2)R^{\frac{p}{p-1}(1+Q(\frac{1}{\gamma}-\frac{1}{p}))}\Big(\int_{B_R}|g(x)|^{\frac{\gamma}{\gamma-1}}dx\Big)^{\frac{\gamma-1}{\gamma}\cdot\frac{p}{p-1}}.
\end{aligned}
$$

Now let us put the estimates of I_1, I_2, I_3 together into (2.6), we obtain

$$
\begin{aligned}
&\int_{B_R} |\eta Xu|^p dx \\
&\le \frac{C(L,p,\varepsilon_1)}{R^p}\Big(\int_{B_R} |Xu|^{\frac{Qp}{Q+p}} dx\Big)^{\frac{Q+p}{Q}} \\
&\quad + C(\mu,\varepsilon_2) R^{\frac{p}{p-1}\left(1+Q(\frac{1}{\gamma}-\frac{1}{p})\right)}\Big(\int_{B_R} |g(x)|^{\frac{\gamma}{\gamma-1}} dx\Big)^{\frac{\gamma-1}{\gamma}\cdot\frac{p}{p-1}} \\
&\quad + \Big\{\mu\varepsilon_2 + pL^{p/2}\varepsilon_1 + \mu C R^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\Big(\int_{B_R}|Xu|^p dx\Big)^{\frac{1}{p}-\frac{1}{\gamma}}\Big\}\Big(\int_{B_R}|Xu|^p dx\Big).
\end{aligned}
$$

Let us write

$$
\vartheta = \mu\varepsilon_2 + pL^{p/2}\varepsilon_1 + \mu C R^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\Big(\int_{B_R}|Xu|^p dx\Big)^{\frac{1}{p}-\frac{1}{\gamma}}.
$$

Notice that from the absolute continuity of the Lebesgue integral, we have that $R^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\int_{B_R}|Xu|^p \to 0$ as $R \to 0$. Consequently we can take small $R > 0$ such that $0 < \vartheta < 1$, and

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}} |Xu|^p dx &\le \frac{C}{R^p}\Big(\int_{B_R}|Xu|^{\frac{Qp}{Q+p}} dx\Big)^{\frac{Q+p}{Q}} \\
&\quad + C R^{\frac{p}{p-1}\left(1+Q(\frac{1}{\gamma}-\frac{1}{p})\right)}\Big(\int_{B_R}|g(x)|^{\frac{\gamma}{\gamma-1}} dx\Big)^{\frac{\gamma-1}{\gamma}\cdot\frac{p}{p-1}} + \vartheta\int_{B_R}|Xu|^p dx,
\end{aligned}
$$

which implies

$$
⨍_{B_{\frac{R}{2}}} |Xu|^p dx \le C\Big(⨍_{B_R}|Xu|^{\frac{Qp}{Q+p}} dx\Big)^{\frac{Q+p}{Q}} + C\Big(⨍_{B_R}|Rg(x)|^{\gamma'} dx\Big)^{\frac{p'}{\gamma'}} + \vartheta ⨍_{B_R}|Xu|^p dx,
$$

with $p' = \frac{p}{p-1}$ and $\gamma' = \frac{\gamma}{\gamma-1}$. Therefore, we obtain

$$
\begin{aligned}
\Big(⨍_{B_{\frac{R}{2}}}|Xu|^p dx\Big)^{1/p} &\le C\Big(⨍_{B_R}|Xu|^{\frac{Qp}{Q+p}} dx\Big)^{\frac{Q+p}{pQ}} + C\Big(⨍_{B_R}(|Rg(x)|^{\gamma'/p})^p dx\Big)^{\frac{1}{p}\frac{p'}{\gamma'}} \\
&\quad + \vartheta^{1/p}\Big(⨍_{B_R}|Xu|^p dx\Big)^{1/p}.
\end{aligned}
$$

Using the reverse Hölder inequality of Lemma 2.2, it yields

$$
\Big(⨍_{B_{\frac{R}{2}}}|Xu|^r dx\Big)^{1/r} \le C\Big(⨍_{B_R}|Xu|^p dx\Big)^{1/p} + C\Big(⨍_{B_R}(|Rg(x)|^{\gamma'/p})^r dx\Big)^{\frac{1}{r}\frac{p'}{\gamma'}}, \tag{2.9}
$$

for some $p < r \le \frac{pq(\gamma-1)}{\gamma}$ due to $q > \frac{\gamma}{\gamma-1}$. Note that

$$
\begin{aligned}
\Big(⨍_{B_R}(|Rg(x)|^{\gamma'/p})^r dx\Big)^{\frac{1}{r}\frac{p'}{\gamma'}} &= R^{\frac{1}{p-1}}\Big(⨍_{B_R}|g(x)|^{\frac{\gamma}{\gamma-1}\cdot\frac{r}{p}} dx\Big)^{\frac{p}{r}\cdot\frac{\gamma-1}{\gamma}\cdot\frac{1}{p-1}} \\
&\le R^{\frac{1}{p-1}}\Big(⨍_{B_R}|g(x)|^q dx\Big)^{\frac{1}{q}\cdot\frac{1}{p-1}},
\end{aligned}
$$

because $r \le \frac{pq(\gamma-1)}{\gamma}$, then we obtain (2.5) which completes the proof. $\square$

The following elementary inequalities concerning $A(x)$ are useful to our main proof, see [20].

Lemma 2.4. *Suppose that $A = (A_{ij})$ is a symmetric matrix and satisfies uniform ellipticity* (H1). *Then there exists a positive constant $C = C(p, \nu, L)$ such that for $1 < p < \infty$ we have*

$$\big\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta\rangle^{\frac{p-2}{2}} A\eta, \xi - \eta\big\rangle \geq C(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2. \tag{2.10}$$

In addition, for $p \geq 2$ there holds

$$\big\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta\rangle^{\frac{p-2}{2}} A\eta, \xi - \eta\big\rangle \geq \nu^{p/2} |\xi - \eta|^p; \tag{2.11}$$

and for $1 < p < 2$, there exists $C = C(p, \nu, L)$ such that for every $0 < \varepsilon < 1$ we have

$$|\xi - \eta|^p \leq C\varepsilon^{\frac{p-2}{p}} \big\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta\rangle^{\frac{p-2}{2}} A\eta, \xi - \eta\big\rangle + \varepsilon |\eta|^p. \tag{2.12}$$

To use Campanato's freezing argument, we obverse the following local Dirichlet problems of homogeneous elliptic systems with constant coefficients.

$$\begin{gathered} -X^*(\langle A_R Xv, Xv\rangle^{\frac{p-2}{2}} A_R Xv) = 0, \quad x \in B_R \\ v = u, \quad x \in \partial B_R, \end{gathered} \tag{2.13}$$

where $A_R = \frac{1}{|B_R|}\int_{B_R} A(x)dx$ is the integral average of $A(x)$. Now we recall [30, Lemma 3.4] while p is close to 2, and have the following perturbation estimates.

Lemma 2.5. *Let $v \in HW^{1,p}(\Omega)$ be a weak solution to the Dirichlet problems* (2.13) *with p close to 2. Then for any $u \in HW^{1,p}(\Omega)$, there exists $C > 0$ such that for any $x_0 \in \Omega$ we have*

$$\int_{B_\rho(x_0)} |Xu|^p dx \leq C\big(\frac{\rho}{R}\big)^Q \int_{B_R(x_0)} |Xu|^p dx + C\int_{B_R(x_0)} |Xu - Xv|^p dx, \tag{2.14}$$

for all $0 < \rho < R \leq R_0$.

Moreover, by a direct calculation we obtain the following conclusion, see [30, Lemma 6].

Lemma 2.6. *Let $v \in HW^{1,p}(\Omega)$ be any weak solution to the Dirichlet problem* (2.13) *with any $x_0 \in \Omega$ and $0 < R \leq R_0$. Then we have*

$$\int_{B_R(x_0)} |Xv|^p dx \leq C\int_{B_R(x_0)} |Xu|^p dx.$$

In addition, we need the following iteration lemma from [17] in the proof of our main theorem.

Lemma 2.7. *Let $\Phi(\rho)$ be a non-negative and non-decreasing function on $(0, R)$. Suppose that*

$$\Phi(\rho) \leq A\big\{\Big(\frac{\rho}{R}\Big)^\alpha + \epsilon\big\}\Phi(R) + BR^\beta, \quad \forall\, 0 < \rho < R \leq R_0 = \operatorname{dist}(x_0, \partial\Omega),$$

with non-negative constants A, B, α and β, and $\alpha > \beta$. Then there exist two constants $\epsilon_0 = \epsilon_0(A, \alpha, \beta)$ and $C = C(A, \alpha, \beta)$ such that for any $0 < \epsilon < \epsilon_0$ we have

$$\Phi(\rho) \leq C\big\{\Big(\frac{\rho}{R}\Big)^\beta \Phi(R) + B\rho^\beta\big\},$$

for any $0 < \rho < R \leq R_0 = \operatorname{dist}(x_0, \partial\Omega)$.

Finally, the following equivalence of spaces is useful to prove a local Hölder continuity of the weak solutions based on the main Theorem, see [9, 8].

Lemma 2.8. *If $0 < \lambda < Q$, the Campanato space $\mathcal{L}_X^{p,\lambda}(\Omega)$ is isomorphic to the Morrey space $L_X^{p,\lambda}(\Omega)$. If $-p < \lambda < 0$, then the Campanato space $\mathcal{L}_X^{p,\lambda}(\Omega)$ is isomorphic to the Hölder space $\Gamma_X^{0,\alpha}(\Omega)$ with $\alpha = -\lambda/p$.*

3. Proof of the main results

Proof of theorem 1.6. Let $u(x) \in HW^{1,p}(\Omega)$ be any weak solution of (1.2). To obtain an interior estimate for the horizontal gradients to the solutions, for any fixed point $x_0 \in \Omega$ let us take a ball $B_R(x_0) \Subset \Omega$, and write $B_R =: B_R(x_0)$ in the context. Suppose that $v(x)$ is a weak solution of the local Dirichlet problem (2.13). Computing the difference between (1.2) and (2.13) yields

$$\begin{aligned}
&- X^*\Big(\langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu - \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu\Big)\\
&- X^*\Big(\langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A_R Xv, Xv\rangle^{\frac{p-2}{2}} A_R Xv\Big)\\
&= -B(x, u, Xu).
\end{aligned}$$

Let us take $\varphi = u - v$ as a test function in the weak sense to the equations above, then we have

$$\begin{aligned}
&\int_{B_R} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A_R Xv, Xv\rangle^{\frac{p-2}{2}} A_R Xv, Xu - Xv\big\rangle dx\\
&= \int_{B_R} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu, Xu - Xv\big\rangle dx \quad (3.1)\\
&\quad + \int_{B_R} \langle B(x, u, Xu), u - v\rangle dx.
\end{aligned}$$

Note that (2.14) implies that if p is close to 2, for any $0 < \rho < R$ there holds

$$\int_{B_\rho(x_0)} |Xu|^p dx \le C\big(\frac{\rho}{R}\big)^Q \int_{B_R(x_0)} |Xu|^p dx + C \int_{B_R(x_0)} |Xu - Xv|^p dx. \quad (3.2)$$

Next, we focus on the estimate of $\int_{B_R(x_0)} |Xu - Xv|^p dx$. To that end, we will estimate it by dividing into two cases.

Case 1: $p \ge 2$. Let us put an elementary inequality (2.11) and (H3) into the equation (3.1) above it follows that

$$\begin{aligned}
&\nu^{p/2} \int_{B_R} |Xu - Xv|^p dx\\
&\le \int_{B_R} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A_R Xv, Xv\rangle^{\frac{p-2}{2}} A_R Xv, Xu - Xv\big\rangle dx\\
&= \int_{B_R} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu, Xu - Xv\big\rangle dx\\
&\quad + \int_{B_R} \langle B(x, u, Xu), u - v\rangle dx\\
&\le C \int_{B_R} |A_R - A(x)||Xu|^{p-1}|Xu - Xv| dx + \mu \int_{B_R} |Xu|^{p(1-\frac{1}{\gamma})} \cdot |u - v| dx
\end{aligned}$$

$$
\begin{aligned}
&+\mu\int_{B_R}|g|\cdot|u-v|dx\\
&:=J_1+J_2+J_3. \qquad (3.3)
\end{aligned}
$$

For estimating J_1, the Hölder inequality and Young inequalities with any $\varepsilon>0$ yield

$$
\begin{aligned}
&\int_{B_R}|A_R-A(x)||Xu|^{p-1}|Xu-Xv|dx\\
&\le\Big(\int_{B_R}|A_R-A(x)|^{\frac{p}{p-1}}|Xu|^pdx\Big)^{1-\frac{1}{p}}\Big(\int_{B_R}|Xu-Xv|^pdx\Big)^{1/p}\\
&\le C(\varepsilon_4)\int_{B_R}|A_R-A(x)|^{\frac{p}{p-1}}|Xu|^pdx+\varepsilon_4\int_{B_R}|Xu-Xv|^pdx\\
&\le C(\varepsilon_4)|B_R|\Big(\fint_{B_R}|A_R-A(x)|^{\frac{p}{p-1}\cdot\frac{r}{r-p}}dx\Big)^{1-\frac{p}{r}}\Big(\fint_{B_R}|Xu|^rdx\Big)^{p/r}\\
&\quad+\varepsilon_4\int_{B_R}|Xu-Xv|^pdx,
\end{aligned}
$$

where an $r>p$ is the same integrable index as that in Lemma 2.3. Setting $t=\frac{pr}{(p-1)(r-p)}$, and by a higher integrability from Lemma 2.3 we obtain

$$
\begin{aligned}
J_1&\le C(\varepsilon_4)|B_R|\Big(\fint_{B_R}|A_R-A(x)|^tdx\Big)^{1-\frac{p}{r}}\Big(\fint_{B_R}|Xu|^pdx+R^{\frac{p}{p-1}}\Big(\fint_{B_R}|g|^qdx\Big)^{\frac{1}{q}\cdot\frac{p}{p-1}}\Big)\\
&\quad+\varepsilon_4\int_{B_R}|Xu-Xv|^pdx\\
&\le C(\varepsilon_4)M_A(R)^{1-\frac{p}{r}}\int_{B_R}|Xu|^pdx+CR^{Q+\frac{p}{p-1}\cdot\frac{q-Q}{q}}\|g\|_{L^q}^{\frac{p}{p-1}}\\
&\quad+\varepsilon_4\int_{B_R}|Xu-Xv|^pdx.
\end{aligned}
$$

To estimate J_2, we employ Höder inequality and Young inequality again, and obtain

$$
\begin{aligned}
J_2&\le\mu\Big(\int_{B_R}|Xu|^pdx\Big)^{1-\frac{1}{\gamma}}\Big(\int_{B_R}|u-v|^\gamma dx\Big)^{1/\gamma}\\
&\le\varepsilon_5\int_{B_R}|Xu|^pdx+C(\mu,Q,p,\varepsilon_5)\int_{B_R}|u-v|^\gamma dx\\
&\le\varepsilon_5\int_{B_R}|Xu|^pdx+CR^{\gamma+Q(1-\frac{\gamma}{p})}\Big(\int_{B_R}|Xu-Xv|^pdx\Big)^{\frac{\gamma-p}{p}}\int_{B_R}|Xu-Xv|^pdx.
\end{aligned}
$$

Observing $\delta(R):=R^{\gamma+Q(1-\frac{\gamma}{p})}\Big(\int_{B_R}|Xu-Xv|^pdx\Big)^{\frac{\gamma-p}{p}}\to0$ as $R\to0$, then there holds

$$
J_2\le C\delta(R)\int_{B_R}|Xu-Xv|^pdx+\varepsilon_5\int_{B_R}|Xu|^pdx. \qquad (3.4)
$$

To estimate J_3, by using Hölder inequality, Sobolev embedding inequality and Young inequality it follows that

$$
J_3\le\mu\Big(\int_{B_R}|g|^{\frac{\gamma}{\gamma-1}}dx\Big)^{\frac{\gamma-1}{\gamma}}\Big(\int_{B_R}|u-v|^\gamma dx\Big)^{1/\gamma}
$$

$$
\begin{aligned}
&\le \Big(\int_{B_R} |g|^{\frac{\gamma}{\gamma-1}} dx\Big)^{\frac{\gamma-1}{\gamma}} R^{1+Q(\frac{1}{\gamma}-\frac{1}{p})}\Big(\int_{B_R} |Xu-Xv|^p dx\Big)^{1/p} \\
&\le \varepsilon_6 \int_{B_R} |Xu-Xv|^p dx + C(\varepsilon_6) R^{\frac{p}{p-1}[1+Q(\frac{1}{\gamma}-\frac{1}{p})]}\Big(\int_{B_R} |g|^{\frac{\gamma}{\gamma-1}} dx\Big)^{\frac{\gamma-1}{\gamma}\cdot\frac{p}{p-1}} \\
&\le \varepsilon_6 \int_{B_R} |Xu-Xv|^p dx + CR^{Q+\frac{p}{p-1}\cdot\frac{q-Q}{q}} \|g\|_{L^q}^{\frac{p}{p-1}}.
\end{aligned}
$$

Putting estimates of J_1, J_2 and J_3 together in (3.3), one deduces

$$
\begin{aligned}
&\nu^{p/2} \int_{B_R} |Xu-Xv|^p dx \\
&\le C\big(\delta(R)+\varepsilon_4+\varepsilon_6\big) \int_{B_R} |Xu-Xv|^p dx + \Big(C(\varepsilon_4) M_A(R)^{1-\frac{p}{r}} + \varepsilon_5\Big) \int_{B_R} |Xu|^p dx \\
&\quad + CR^{Q+\frac{p}{p-1}\cdot\frac{q-Q}{q}} \|g\|_{L^q}^{\frac{p}{p-1}}.
\end{aligned}
$$

Therefore, by choosing arbitrary positive constants $\varepsilon_4, \varepsilon_6$ and $0 < R < R_0$ small enough that $C(\delta(R)+\varepsilon_4+\varepsilon_6) < \nu^{p/2}$ we obtain

$$
\begin{aligned}
&\int_{B_R} |Xu-Xv|^p dx \\
&\le \Big(C(\varepsilon_4) M_A(R)^{1-\frac{p}{r}} + \varepsilon_5\Big) \int_{B_R} |Xu|^p dx + CR^{Q+\frac{p}{p-1}\cdot\frac{q-Q}{q}} \|g\|_{L^q}^{\frac{p}{p-1}} \\
&\le \Big(C(\varepsilon_4) M_A(R)^{1-\frac{p}{r}} + \varepsilon_5\Big) \int_{B_R} |Xu|^p dx + CR^{Q+\frac{p}{p-1}\cdot\frac{q-\mu}{q}} \|g\|_{L_X^{q,\mu}}^{\frac{p}{p-1}}.
\end{aligned}
\tag{3.5}
$$

Let us set $\varpi = C(\varepsilon_4) M_A(R)^{1-\frac{p}{r}} + \varepsilon_5$ in (3.5), and put the estimate (3.5) into (3.2), then for any $0 < \rho < R$ we have

$$
\int_{B_\rho} |Xu|^p dx \le C\big[\big(\frac{\rho}{R}\big)^Q + \varpi\big] \int_{B_R} |Xu|^p dx + CR^{Q+\frac{p}{p-1}\cdot\frac{q-\mu}{q}} \|g\|_{L_X^{q,\mu}}^{\frac{p}{p-1}}. \tag{3.6}
$$

While $q \ge \mu$ such that $Q + \frac{p}{p-1}\cdot\frac{q-\mu}{q} \ge Q$, it follows from Lemma 2.7 that

$$
\int_{B_\rho} |Xu|^p dx \le C\big(\frac{\rho}{R}\big)^{Q-\lambda} \int_{B_R} |Xu|^p dx + C\rho^{Q-\lambda} \|g\|_{L_X^{q,\mu}(B_R)}^{\frac{p}{p-1}}, \tag{3.7}
$$

for any $0 < \lambda < Q$. This implies $Xu \in L_{X,\mathrm{loc}}^{p,\lambda}(\Omega')$ for $\Omega' \Subset \Omega$ with the estimate

$$
\|Xu\|_{L_X^{p,\lambda}(\Omega')} \le C\Big(\|Xu\|_{L^p(\Omega)} + \|g\|_{L_X^{q,\mu}(\Omega)}^{\frac{1}{p-1}}\Big).
$$

While $\frac{\gamma}{\gamma-1} < q < \mu$ such that $Q + \frac{p}{p-1}\cdot\frac{q-\mu}{q} < Q$, then we deduce from Lemma 2.7 that

$$
\int_{B_\rho} |Xu|^p dx \le C\big(\frac{\rho}{R}\big)^{Q-\frac{p}{p-1}\frac{\mu-q}{q}} \int_{B_R} |Xu|^p dx + C\rho^{Q-\frac{p}{p-1}\cdot\frac{\mu-q}{q}} \|g\|_{L_X^{q,\mu}}^{\frac{p}{p-1}}, \tag{3.8}
$$

which implies $Xu \in L_{X,\mathrm{loc}}^{p,\frac{p}{p-1}\frac{\mu-q}{q}}(\Omega')$ with the estimate

$$
\|Xu\|_{L_X^{p,\frac{p}{p-1}\frac{\mu-q}{q}}(\Omega')} \le C\Big(\|Xu\|_{L^p(\Omega)} + \|g\|_{L_X^{q,\mu}(\Omega)}^{\frac{1}{p-1}}\Big).
$$

Case 2: $1 < p < 2$**.** Using inequality (2.12) yields

$$
\begin{aligned}
&\int_{B_\rho} |Xu - Xv|^p dx \\
&\le C\varepsilon^{\frac{p-2}{2}} \int_{B_\rho} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A_R Xv, Xv\rangle^{\frac{p-2}{2}} A_R Xv, Xu - Xv\big\rangle dx \\
&\quad + \varepsilon \int_{B_\rho} |Xu|^p dx \\
&= C(\varepsilon) \int_{B_\rho} \big\langle \langle A_R Xu, Xu\rangle^{\frac{p-2}{2}} A_R Xu - \langle A(x) Xu, Xu\rangle^{\frac{p-2}{2}} A(x) Xu, Xu - Xv\big\rangle dx \\
&\quad + B(x, u, Xu) + \varepsilon \int_{B_\rho} |Xu|^p dx \\
&\le C(p, \nu, L) \int_{B_R} |A_R - A(x)||Xu|^{p-1}|Xu - Xv| dx + \mu \int_{B_R} |Xu|^{p(1-\frac{1}{\gamma})} \cdot |u - v| dx \\
&\quad + \mu \int_{B_R} |g| \cdot |u - v| dx + \varepsilon \int_{B_R} |Xu|^p dx \\
&:= J_1 + J_2 + J_3 + \varepsilon \int_{B_R} |Xu|^p dx.
\end{aligned}
$$

Considering the estimates of J_1, J_2, J_3 in Case 1, and for any $\varepsilon > 0$ we obtain

$$
\int_{B_\rho} |Xu|^p dx \le C\big[\big(\frac{\rho}{R}\big)^Q + \varpi'\big] \int_{B_R} |Xu|^p dx + CR^{Q+\frac{p}{p-1}\cdot\frac{q-\mu}{q}} \|g\|^{\frac{p}{p-1}}_{L_X^{q,\mu}}, \tag{3.9}
$$

with $\varpi' = C(\varepsilon_4) M_A(R)^{1-\frac{p}{r}} + \varepsilon_5 + \varepsilon$. While $q \ge \mu$, it follows form Lemma 2.7 as the same as Case 1 that

$$
\int_{B_\rho} |Xu|^p dx \le C\big(\frac{\rho}{R}\big)^{Q-\lambda} \int_{B_R} |Xu|^p dx + C\rho^{Q-\lambda} \|g\|^{\frac{p}{p-1}}_{L^q(B_R)},
$$

for any $0 < \lambda < Q$. It yields $Xu \in L^{p,\lambda}_{X,\mathrm{loc}}(\Omega')$ for any $\Omega' \Subset \Omega$ with the estimate

$$
\|Xu\|_{L_X^{p,\lambda}(\Omega')} \le C\Big(\|Xu\|_{L^p(\Omega)} + \|g\|^{\frac{1}{p-1}}_{L_X^{q,\mu}(\Omega)}\Big).
$$

While $\frac{\gamma}{\gamma-1} < q < \mu$, by Lemma 2.7 we obtain

$$
\int_{B_\rho} |Xu|^p dx \le C\big(\frac{\rho}{R}\big)^{Q-\frac{p}{p-1}\frac{\mu-q}{q}} \int_{B_R} |Xu|^p dx + C\rho^{Q-\frac{p}{p-1}\cdot\frac{\mu-q}{q}} \|g\|^{\frac{p}{p-1}}_{L_X^{q,\mu}}.
$$

which implies $Xu \in L^{p,\frac{p}{p-1}\frac{\mu-q}{q}}_{X,\mathrm{loc}}(\Omega')$ with the estimate

$$
\|Xu\|_{L_X^{p,\frac{p}{p-1}\frac{\mu-q}{q}}(\Omega')} \le C\Big(\|Xu\|_{L^p(\Omega)} + \|g\|^{\frac{1}{p-1}}_{L_X^{q,\mu}(\Omega)}\Big).
$$

This completes the proof. □

Proof of Theorem 1.8. For any $0 < \rho < R$ with $B_R(x_0) \Subset \Omega$, by Poincare inequality we have

$$
\int_{B_\rho} |u - u_\rho|^p dx \le C\rho^p \int_{B_\rho} |Xu|^p dx \le C\rho^p \rho^{Q-\lambda} \|Xu\|^p_{L_X^{p,\lambda}(B_\rho)}. \tag{3.10}
$$

Note that by Theorem 1.6,

$$\|Xu\|^p_{L^{p,\lambda}_X(B_\rho)} \le C\big(\frac{1}{R}\big)^{Q-\lambda}\int_{B_R}|Xu|^p dx + C\|g\|^{\frac{p}{p-1}}_{L^{q,\mu}_X(B_R)}. \tag{3.11}$$

Therefore, combining (3.10) and (3.11) yields

$$\frac{\rho^{\lambda-p}}{|B_\rho|}\int_{B_\rho}|u-u_\rho|^p dx \le C\big(\frac{1}{R}\big)^{Q-\lambda}\int_{B_R}|Xu|^p dx + C\|g\|^{\frac{p}{p-1}}_{L^{q,\mu}_X(B_R)}, \tag{3.12}$$

which implies

$$u \in \mathcal{L}^{p,\lambda-p}_{X,\mathrm{loc}}(\Omega).$$

Note that $Q-n<\lambda<p$ implies $-p<Q-n-p<\lambda-p<0$, it follows from Lemma 2.8 that

$$u \in \Gamma^{0,\alpha}_{X,\mathrm{loc}}(\Omega),$$

with $\alpha = -\frac{\lambda-p}{p} = 1-\frac{\lambda}{p}$. □

Remark 3.1. We would like to point out that Theorems 1.6 and 1.8 are valid only under the assumption of p close to 2 when we consider subelliptic (1.2) in Carnot group. Indeed, we employ the perturbation inequality (2.14) in our main proof, which is attained by a local Lipschitz boundedness of subelliptic p-harmonic only if p close to 2. However, it is not necessary to limit p close to 2 if $X = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\big)$ is a classical usual gradient in the Euclidian spaces $\mathbb{R}^n$.

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