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DECAY ESTIMATES FOR SOLUTIONS OF ABSTRACT WAVE EQUATIONS WITH GENERAL DAMPING FUNCTION

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ABSTRACT. In this article we prove convergence to equilibrium and decay estimates for a class of damped abstract wave equations. We focus on the damping term to be as general as possible, including functions that oscillate between two positive functions in a neighborhood of the origin and/or behave differently in each direction.

1. Introduction

In this article, we prove convergence to equilibrium and show decay estimates for solutions of the second-order equation

$$\ddot{u} + g(\dot{u}) + M(u) = 0 \tag{1.1}$$

on a Hilbert space H for a broad class of damping functions g and (unbounded) nonlinear operators $M = E'$ satisfying Kurdyka-Łojasiewicz-Simon estimates.

There are many convergence results for second-order equations with linear damping and various operators M, see [10, 14, 11] for M in the form $-\Delta u + f(x,u)$ and [9] for a more general theory. Some decay estimates were shown in [12] for $-\Delta u + f(x,u)$, and in [8] for a general nonlinear operator $M = E'$ satisfying the Łojasiewicz gradient inequality. Convergence and decay estimates for nonlinear damping and a linear operator $M = -\Delta u$ and the right-hand side $h(x,t)$ was shown in [13]. An example, where bounded solutions do not converge to equilibrium, can be found in [15] (a nonlinear wave equation on a bounded domain with Dirichlet boundary conditions and linear damping).

Concerning nonlinear damping and a nonlinear operator M, the equation

$$u_{tt} + |u_t|^\alpha u_t - \Delta u = f(x,u) \tag{1.2}$$

was studied by Chergui [6], where convergence to equilibrium was proved. Later, Ben Hassen and Haraux [5] proved convergence to equilibrium and decay estimates in the abstract setting (1.1) with $M = E' \in C^1(V, V^*)$ where $V \hookrightarrow H \hookrightarrow V^*$ are Hilbert spaces, and for damping functions $g : V \to V^*$ satisfying

$$c_1\|v\|^{\alpha+2} \le \langle g(v), v\rangle_{V^*,V} \quad \text{and} \quad \|g(v)\|_* \le c_2\|v\|^{\alpha+1},$$

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which implies

$$c_1\|v\|^{\alpha+1}\frac{\|v\|}{\|v\|_V}\le \|g(v)\|_*\le c_2\|v\|^{\alpha+1}. \tag{1.3}$$

In [3], Fašangová and the author of this paper showed that the upper and lower estimates for g can be independent, they proved convergence to equilibrium (without decay estimates) for pointwise damping operators $g(v)(x)=G(v(x))$ on $V=H_0^1(\Omega)$ with G estimated from below and above by two independent functions.

In this article we combine ideas from [5] and [3] to prove convergence and decay estimates for $g:V\to V^*$ where V is an arbitrary Hilbert space, g satisfying

$$h(\|v\|)\|v\|\le \langle g(v),v\rangle_{V^*,V}\quad\text{and}\quad \|g(v)\|_*\le c_2\|v\|,$$

where h is a positive function (not necessarily a power $s^{\alpha+1}$). We also show that the upper estimate for g can be replaced by $\gamma(\|g(v)\|_*)\le\langle g(v),v\rangle_{V^*,V}$, which is satisfied by a wide class of poinwise damping operators. Moreover, we assume that $M=E'$ satisfies Kurdyka-Łojasiewicz-Simon inequality (see Kurdyka [16])

$$\Theta(E(u))\le\|M(u)\|_*,$$

which is a generalization of the Łojasiewicz gradient inequality (see Łojasiewicz [17]) considered in [5, 6].

This conditions on g allow much more general damping functions than the previous results. In particular, if we focus on the special case $g(v)(x)=G(v(x))$, then the following cases are covered in this article and not in [5]:

- growth of G near zero and near infinity are different, e.g. $G(s)=|s|^a s$ for small s and $G(s)=|s|^b s$ for large s,
- steeper growth of G in infinity than in [5, Example 3.1], e.g. $G(s)=|s|^b s$ for $b\le\frac{4}{N-2}$,
- G with different behavior in every direction around zero, e.g. for a scalar valued v one allows $G(s)=|s|^a s$ for $s>0$ and $G(s)=|s|^b s$ for $s<0$, $a\ne b$,
- G with non-power-like behavior, e.g. $G(s)=|s|^a\ln^b(1/|s|)\ln^c(\ln(1/|s|))s$ for small s.

Moreover, our results

- show that the decay estimates depend on the growth of G near zero only (this is not obvious since $\|v\|<\varepsilon$ does not imply that $|v(x)|$ is small for every $x\in\Omega$),
- yield more delicate decay estimates, e.g. in the logarithmic scales $\|u(t)-\varphi\|\le C|t|^a\ln^b(1/|t|)\ln^c(\ln(1/|t|))$.

In fact, similar decay estimates (based on Kurdyka-Łojasiewicz-Simon inequality) were shown in [4, 2] for second order ordinary differential equations, and in [7] for first-order partial differential equations.

We present two kinds of results. The first kind (Theorems 2.1 and 2.3) applies if we know a-priori that the whole solution (for all $t\ge t_0$) lies in a ball where the Kurdyka-Łojasiewicz-Simon estimates are satisfied. In the second kind (Theorems 2.2 and 2.3) we have Kurdyka-Łojasiewicz-Simon estimates only in a small neighborhood U of an omega-limit point of the solution and we assume that the solution is relatively compact, but we do not know a-priory that it is contained in U for all $t\ge t_0$.

This article is organized as follows. In Section 2 we introduce our settings and assumptions and formulate the main results. Sections 3 and 4 are devoted to proofs of the two main Theorems. In Section 5, the results are applied to some semilinear wave equations. Section 6 is an appendix where we prove some technical lemmas.

2. Assumptions and statement of main results

Let $V \hookrightarrow H \hookrightarrow V^*$ be Hilbert spaces with the embedding being dense, we identify $\langle v, u\rangle_{V^*,V} = \langle v, u\rangle_H$ for $u \in V \subset H$, $v \in H \subset V^*$. The norm and the scalar product on V^* (resp. on H, V) are denoted by $\|\cdot\|_*$ and $\langle\cdot,\cdot\rangle_*$ (resp. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, $\|\cdot\|_V$ and $\langle\cdot,\cdot\rangle_V$). By $B(0,R)$ we denote the ball in H of radius R centered in 0, while $B_V(0,R)$ is the corresponding ball in V. In the whole paper, C denotes a generic constant which may change from line to line or from expression to expression.

Now, we define several properties of real functions. We say that a differentiable function $f : \mathbb{R}_+ \to \mathbb{R}_+$

- is *admissible* if f is nondecreasing and there exists $c_A \geq 1$ such that $f(s) > 0$ and $sf'(s) \leq c_A f(s)$ for all $s > 0$.
- has *property (K)* if for every $K > 0$ there exists $C(K) > 0$ such that $f(Ks) \leq C(K)f(s)$ holds for all $s > 0$.
- is *C-sublinear* if there exists $C > 0$ such that $f(t+s) \leq C(f(t) + f(s))$ holds for all t, $s > 0$.

It is shown in the Appendix that the first property implies the other two. It is easy to see that any nonnegative increasing concave function is admissible with $c_A = 1$ provided it is everywhere differentiable (otherwise $sf'_\pm(s) \leq f(s)$ holds, which would be also sufficient for our purpose).

Let us introduce our assumptions on the operator E.

(A1) Let $E \in C^2(V)$, $M = E' \in C^1(V, V^*)$ and let B be a fixed ball in V. Assume that:

(e1) E is nonnegative on B and there exists an admissible function Θ such that $\Theta(s) \leq C_\Theta\sqrt{s}$ for all $s \geq 0$ and some $C_\Theta > 0$, $\frac{1}{\Theta}$ is integrable in a neighbourhood of zero and

$$\|M(u)\|_* \geq \Theta(E(u)), \quad \text{for all } u \in B, \tag{2.1}$$

i.e., E satisfies the Kurdyka-Łojasiewicz-Simon gradient inequality with function Θ on B.

(e2) There exists $C_M \geq 0$ such that

$$|\langle M'(u)v, v\rangle_*| \leq C_M\|v\|^2 \quad \text{for all } u \in B,\ v \in V,$$

(e3) There exists a nondecreasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|M(u)\|_* \leq G(E(u)), \quad \text{for all } u \in B. \tag{2.2}$$

Let us comment on the above assumptions. Chergui [6] worked with $H = L^2(\Omega)$, $V = H^1_0(\Omega)$, $E'(u) = \Delta u + f(x,u)$ which corresponds to $E(u) = \int_\Omega \frac{1}{2}|\nabla u(x)|^2 + F(x,u)\,\mathrm{d}x$, where $F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s$. By [6, Corollary 1.2], this function E satisfies the Łojasiewicz gradient inequality

$$\|E'(u)\|_* \geq C|E(u) - E(\varphi)|^{1-\theta} \tag{2.3}$$

with some $\theta \in [0, 1/2)$ in a neighbourhood of stationary points, provided f satisfies certain assumptions. The Łojasiewicz inequality (2.3) is a special case of the Kurdyka-Łojasiewicz-Simon inequality (2.1) with the function $\Theta(s) = s^{1-\theta}$, θ being the Łojasiewicz exponent. It is easy to see that Chergui's operator satisfies (e2) as well. The conditions (e1) and (e2) (with (2.3) instead of (2.1)) appear also in [8], where linear damping is considered.

Concerning assumption (e3), there is one more condition (g4) below, which connects functions G and Θ with a function h defined below. Let us mention that (e3) is often satisfied with $G(s) = C\sqrt{s}$, in particular in all applications in [5] and in finite-dimensional case for any $E \in C^{1,1}_{\rm loc}(\mathbb{R}^n)$ satisfying that $E(u) = 0$ for all critical points u (see [4, Lemma 2.7]).

We now formulate the assumptions on the damping function.

- (A2) The function $g : V \to V^*$ is continuous and there exists an admissible function h such that
  - (g1) there exists $C_2 > 0$ such that $\|g(v)\|_* \le C_2\|v\|$ on $V \cap B(0, R)$ for any $R > 0$ with C_2 depending on R,
  - (g2) $\langle g(v), v\rangle_{V^*,V} \ge h(\|v\|)\|v\|^2$ on V,
  - (g3) the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0, 1))$,
  - (g4) there exists $C_G > 0$ such that $G(s) \le C_G\frac{\sqrt{s}}{h(\Theta(s))}$ on $(0, K]$ for any $K > 0$ with C_G depending on K,
  - (g5) the function $\psi : s \mapsto sh(\sqrt{s})$ is convex for all $s > 0$.

Let us comment on these assumptions. If we take $(g(v))(x) = |v(x)|^\alpha$, we obtain equation (1.2) studied by Chergui [6], and (g2) holds with $h(s) = s^\alpha$. Chergui's condition $\alpha < \frac{4}{N-2}$ (and also condition (g3) in [3]) implies $g(v) \in V^*$. Moreover, taking $\Theta(s) = s^{1-\theta}$ (e.g. the Łojasiewicz inequality instead of (2.1)), then (g3) corresponds to condition $0 < \alpha < \frac{\theta}{1-\theta}$ in [6] and [5]. Condition (g3) is a condition coupling the damping function g with the operator E. Another condition coupling g and E is (g4). But (as was said above) in many applications $G(s) = C\sqrt{s}$, and in this case (g4) holds for any h and Θ since $h(\Theta(s))$ is bounded on $(0, 1)$.

In [5] the authors work with (g2) for $h(s) = s^\alpha$ and (g1) replaced by $\|g(v)\|_* \le C_2\|v\|^{1+\alpha}$. It is easy to modify the proof in [5] in such a way that the upper bound for $\|g(v)\|_*$ can be relaxed to (g1) (it is easy to show that $\|v\| \to 0$, so $\|v\|^{1+\alpha} < \|v\|$). After doing this, one can apply the result in [5] e.g. to

$$g(v)(x) = |v(x)|^\alpha \ln(1/|v(x)|)v(x)$$

with $h(s) = s^{1+\alpha}$. However, applying Theorem 2.1 below one can take $h(s) = s^{1+\alpha}\ln(1/s)$ in (g2) and get better convergence rates.

One can show (by differentiating), that functions

$$h(s) = s^a \ln^{r_1}(1/s)\ln^{r_2}(\ln(1/s))\cdots\ln^{r_k}(\ln\cdots\ln(1/s))$$

are positive increasing and concave on $(0, \varepsilon)$ for $a \in (0, 1)$, $r_i \in \mathbb{R}$. So, they become admissible with $c_A = 1$ after redefining them appropriately on $(\varepsilon, +\infty)$. In Section 5 we give some examples of decay estimates in these scales of functions.

Our main results are formulated for solutions in the following sense. We say that $u \in W^{1,1}_{\rm loc}([0, +\infty), V) \cap W^{2,1}_{\rm loc}([0, +\infty), H)$ is *a strong solution* to (1.1) if (1.1) holds in V^* for almost every $t > 0$.

Theorem 2.1. *Let E and G satisfy* (A1) *and* (A2)*. Let u be a strong solution to* (1.1) *and there exists $t_1 > 0$ such that $u(t) \in B$ for all $t \ge t_1$. Then there exist $\varphi \in B$ and $t_0 \ge 0$ such that*

$$E(u(t)) \le 2\Psi^{-1}(t - t_0), \tag{2.4}$$

$$\|u(t) - \varphi\| \le \Phi(\Psi^{-1}(t - t_0)), \tag{2.5}$$

$$\|\dot{u}(t)\| \le \sqrt{\Psi^{-1}(t-t_0))} \tag{2.6}$$

hold for all $t > t_0$*, some* C_Φ*,* $C_\Psi > 0$ *and*

$$\Phi(t) = C_\Phi \int_0^t \frac{1}{\Theta(s)h(\Theta(s))}ds \quad \text{and} \quad \Psi(t) = C_\Psi \int_t^{1/2} \frac{1}{\Theta^2(s)h(\Theta(s))}ds. \tag{2.7}$$

If we take $\Theta(s) = s^{1-\theta}$ and $h(s) = s^\alpha$ in Theorem 2.1, we obtain the same convergence rate as in [5, Theorem 2.2].

The next result combines the method from [6] (resp. [3]) and [5] to obtain decay estimates for relatively compact solutions with (2.1) satisfied only on a small neighborhood of some $\varphi \in \omega_V(u)$, where

$$\omega_V(u) = \{\varphi \in V : \exists\, t_n \nearrow +\infty, \text{ s.t. } \|u(t_n) - \varphi\|_V \to 0\}.$$

Theorem 2.2. *Let* u *be a strong solution to* (1.1) *with* $U_T := \{(u(t), \dot{u}(t)), t \ge T\}$ *relatively compact in* $V \times H$ *and* $\varphi \in \omega_V(u)$ *with* $E(\varphi) = 0$*. Let* (A1) *and* (A2) *hold with the following changes:*

- (2.1) *and* (2.2) *hold with* B *replaced by* $B_V(\varphi, \delta)$ *for some* $\delta > 0$*,*
- *(e2) holds with* B *replaced by 'any compact subset of* V *with* C_M *depending on the subset',*
- h *is admissible with* $c_A = 1$*,*

Then $\lim_{t\to+\infty} \|u(t)-\varphi\|_V = 0$ *and there exists* $t_0 \ge 0$ *such that the decay estimates* (2.4)*,* (2.5) *and* (2.6) *hold for all* $t > t_0$*, some* C_Φ*,* $C_\Psi > 0$ *and* Φ*,* Ψ *defined in* (2.7)*.*

Theorem 2.3. *Theorems 2.1 and 2.2 remain valid if we replace* (g1) *by*

(g1') *for every* $R > 0$ *there exists a convex function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *with property (K) and such that* $\gamma(0) = 0$*,* $\lim_{s\to+\infty} \gamma(s) = +\infty$*,* $\gamma(s) \ge cs^2$ *for some* $c > 0$ *and all* s *small enough, and* $\gamma(\|g(v)\|_*) \le \langle g(v), v\rangle_{V^*,V}$ *on* $V \cap B(0, R)$*.*

Let us mention, that condition (g1) implies boundedness of $\|g(v(t))\|_*$, while condition (g1') does not. We show in Section 5 that (g1') is useful in many examples.

It was mentioned in [2] and also in [5] that estimating $\|u(t) - \varphi\|$ by the lenght of the trajectory $\int_t^{+\infty} \|\dot{u}(s)\|ds$ often does not yield an optimal result. In fact, the trajectory can be much longer than the distance $\|u(t) - \varphi\|$ if it has a shape of a spiral (which is typically the case for second order equations with weak damping). In many applications, one can obtain a better estimate by estimating $\|u - \varphi\|$ by $E(u)$ directly.

Corollary 2.4. *Let the assumptions of Theorems 2.1, 2.2 or 2.3 are satisfied and* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing function such that* $\alpha(E(u) - E(\varphi)) \ge \|u - \varphi\|$ *on a neighborhood of* φ*. Then*

$$\|u(t) - \varphi\| \le \alpha(2\Psi^{-1}(t - t_0))$$

holds for some t_0 *and all* $t > t_0$*.*

The above corollary follows from $\|u(t) - \varphi\| \le \alpha(E(u(t)) - E(\varphi)) \le \alpha(2\Psi^{-1}(t - t_0))$.

3. Proof of Theorem 2.1

For the strong solution u from the Theorem let us denote $v(t) := \dot{u}(t)$ and

$$E_1(t) := \frac{1}{2}\|v(t)\|^2 + E(u(t)).$$

Then

$$E_1'(t) = \langle v(t), \dot{v}(t)\rangle_{V,V^*} + \langle M(u(t)), \dot{u}(t)\rangle_{V^*,V} = -\langle v(t), g(v(t))\rangle_{V,V^*} \tag{3.1}$$

It follows from (g2) that E_1 is nonincreasing, so it is either positive for all $t \geq 0$ or $v(t) = 0$ for all $t \geq t_0$. In the latter case, $u(t) = \varphi$ for $t \geq t_0$ and there is nothing to prove. So, we may assume that $E_1(t) > 0$ for all $t \geq 0$. Moreover, it follows that $\|v(t)\|$ and $E(u(t))$ are bounded and by (e3) also $\|M(u)\|_*$ is bounded.

Further, for s and $t \geq 0$ we define

$$B(s) := h(\Theta(s)), \quad H(t) = E_1(t) + \varepsilon B(E_1(t))\langle M(u(t)), v(t)\rangle_*,$$

where $\varepsilon > 0$ will be specified later. We first show that for all $t \geq t_1$ the inequality

$$\frac{1}{2}E_1(t) \leq H(t) \leq 2E_1(t) \tag{3.2}$$

holds if $\varepsilon > 0$ is small enough. Both inequalities follow immediately from the estimate

$$\begin{aligned}
|\varepsilon B(E_1(t))\langle M(u(t)), v(t)\rangle_*| &\leq \varepsilon C B(E_1(t)) G(E_1(t))\sqrt{2E_1(t)} \leq \varepsilon C E_1(t) \\
&\leq \frac{1}{2}E_1(t),
\end{aligned} \tag{3.3}$$

where the first inequality is a consequence of definition of E_1 and (e3) if applied the Cauchy-Schwarz inequality and $H \hookrightarrow V^*$, the second inequality is due to (g4) and definition of $B(\cdot)$ and in the third inequality we take $\varepsilon < 1/(2C)$.

We now derive some estimates for $H'(t)$. Let us fix $t > t_1$ and write (u, v) instead of $(u(t), v(t))$ and also E, E_1 instead of $E(t)$, $E_1(t)$. We start with

$$\begin{aligned}
&H'(t) \\
&= E_1' + \varepsilon B'(E_1)E_1'\langle M(u), v\rangle_* + \varepsilon B(E_1)\langle M'(u)v, v\rangle_* + \varepsilon B(E_1)\langle M(u), \dot{v}\rangle_* \\
&= -\langle g(v), v\rangle_{V^*,V} - \varepsilon B'(E_1)\langle g(v), v\rangle_{V^*,V}\langle M(u), v\rangle_* + \varepsilon B(E_1)\langle M'(u)v, v\rangle_* \\
&\quad - \varepsilon B(E_1)\langle M(u), g(v)\rangle_* - \varepsilon B(E_1)\langle M(u), M(u)\rangle_* \\
&= -\langle g(v), v\rangle_{V^*,V} - \varepsilon B(E_1)\|M(u)\|_*^2 + \varepsilon B(E_1)\langle M'(u)v, v\rangle_* \\
&\quad - \varepsilon B'(E_1)\langle v, g(v)\rangle_{V,V^*}\langle M(u), v\rangle_* - \varepsilon B(E_1)\langle M(u), g(v)\rangle_*
\end{aligned} \tag{3.4}$$

In the above expression we keep the first two terms and estimate the other terms from above. By admissibility of h and Θ we have

$$B'(s) = h'(\Theta(s))\Theta'(s) \leq C\frac{h(\Theta(s))}{\Theta(s)} \cdot \frac{\Theta(s)}{s} = C\frac{B(s)}{s}.$$

So, $B(\cdot)$ is admissible. Then the fourth term on the right-hand side in (3.4) can be estimated (with help of (3.3)) by

$$\begin{aligned}
|\varepsilon B'(E_1)\langle v, g(v)\rangle_{V,V^*}\langle M(u), v\rangle_*| &\leq \frac{1}{E_1}|\varepsilon B(E_1)\langle v, g(v)\rangle_{V,V^*}\langle M(u), v\rangle_*| \\
&\leq \frac{1}{2}\langle v, g(v)\rangle_{V,V^*}.
\end{aligned}$$

The third term on the right-hand side in (3.4) is estimated as follows (ψ^* being the convex conjugate to the function ψ from condition (g5))

$$
\begin{aligned}
&|\varepsilon B(E_1)\langle M'(u)v, v\rangle_*| \\
&\le \varepsilon B(E_1)C\|v\|^2 \\
&\le \varepsilon C\Big(\frac{1}{K}\psi^*(B(E_1)) + C(K)\psi(\|v\|^2)\Big) \\
&\le \varepsilon C\Big(\frac{C}{K}\psi(\Theta^2(E_1)) + C(K)\psi(\|v\|^2)\Big) \\
&\le \varepsilon C\Big(\frac{C}{K}\psi(\Theta^2(E)) + \frac{C}{K}\psi(\Theta^2(\|v\|^2)) + C(K)\psi(\|v\|^2)\Big) \\
&\le \varepsilon C\Big(\frac{C}{K}\Theta^2(E)h(\Theta(E)) + 2C(K)\|v\|^2h(\|v\|)\Big) \\
&\le \varepsilon C\Big(\frac{C}{K}\|M(u)\|_*^2h(\Theta(E_1)) + 2C(K)\|v\|^2h(\|v\|)\Big) \\
&\le \frac{1}{4}\varepsilon B(E_1)\|M(u)\|_*^2 + \varepsilon C\langle v, g(v)\rangle_*.
\end{aligned}
\tag{3.5}
$$

Here we used (e2) (first inequality), Young inequality (second), Lemma 6.4 (third), C-sublinearity of $\psi(\Theta^2(\cdot))$ (fourth), definition of ψ and $\Theta(s) \le \sqrt{s}$ (fifth), (2.1) inequality and $E \le E_1$ (sixth) and we have taken $K = \frac{1}{4C^2}$ and used (g2) in the last inequality.

The fifth term on the right-hand side of (3.4) is estimated by

$$
\begin{aligned}
\varepsilon|B(E_1)\langle M(u), g(v)\rangle_*| &\le \varepsilon B(E_1)(\frac{1}{4}\|M(u)\|_*^2 + C\|g(v)\|_*^2) \\
&\le \frac{1}{4}\varepsilon B(E_1)\|M(u)\|_*^2 + \varepsilon C B(E_1)\|v\|^2 \\
&\le \frac{2}{4}\varepsilon B(E_1)\|M(u)\|_*^2 + \varepsilon C\langle v, g(v)\rangle_*,
\end{aligned}
$$

where we used the Cauchy-Schwarz and Young inequalities (first step), (g1) (second step) and (3.5) (last step).

Altogether, we have

$$
\begin{aligned}
H'(t) &\le -(1 - \frac{1}{2} - 2\varepsilon C)\langle v, g(v)\rangle_* - \frac{1}{4}\varepsilon B(E_1)\|M(u)\|_*^2 \\
&\le -c(h(\|v\|))\|v\|^2 + B(E_1)\|M(u)\|_*^2).
\end{aligned}
\tag{3.6}
$$

Denoting $\chi(s) := B(s)\Theta^2(s)$ we obtain

$$
\begin{aligned}
-H'(t) &\ge cB(E)\|M(u)\|_*^2 \\
&\ge cB(E)\Theta(E)^2 \\
&= c\chi(E) \\
&= c\chi(E_1 - \frac{1}{2}\|v\|^2) \\
&\ge C_1\chi(E_1) - C\chi(1/2\|v\|^2)) \\
&= C_1\chi(E_1) - C\Theta^2(1/2\|v\|^2)h(\Theta(1/2\|v\|^2)) \\
&\ge C_1\chi(E_1) - C\|v\|^2h(\|v\|) \\
&\ge C_1\chi(E_1) + CH'(t).
\end{aligned}
$$

Here we used (3.6) (in the first step), (2.1) inequality (second step), definition of χ (third), definition of E_1 (fourth), C-sublinearity of χ (fifth), definition of χ and B (sixth), $\Theta(s) \le C\sqrt{s}$ and property (K) for h (seventh) and (3.6) (last step). It follows that

$$-(C+1)H'(t) \ge C_1\chi(E_1(t)) \ge \frac{1}{2}C_1\chi(H(t)).$$

Take $C_\Psi = 2(C+1)/C_1$. Then

$$\frac{d}{dt}\Psi(H(t)) = C_\Psi \frac{-1}{\chi(H(t))}H'(t) \ge 1$$

and we have

$$\Psi(H(t)) - \Psi(H(t_0))) \ge t - t_0.$$

It follows that $\lim_{t\to+\infty}\Psi(H(t)) = +\infty$, so we can take t_0 such that $\Psi(H(t_0)) \ge 0$ and we obtain $\Psi(H(t)) \ge t - t_0$. Since Ψ is decreasing (by definition) we obtain

$$H(t) \le \Psi^{-1}(t - t_0).$$

Now, (2.4) and (2.6) follow immediately. To show the estimate (2.5), let us compute

$$\begin{aligned} -\frac{1}{C_\Phi}\frac{d}{dt}\Phi(H(t)) &\ge C\frac{h(\|v\|)\|v\|^2 + B(E_1)\|M(u)\|_*^2}{\Theta(H(t))B(H(t))} \\ &\ge C\frac{h(\|v\|)\|v\|^2 + B(E_1)\|M(u)\|_*^2}{(\Theta(\|v\|^2) + \|M(u)\|_*)B(E_1)} \\ &\ge C\|v\|\frac{h(\|v\|)\|v\|^2 + B(E_1)\|M(u)\|_*^2}{B(E_1)\|v\|^2 + B(E_1)\|v\|\|M(u)\|_*}. \end{aligned} \tag{3.7}$$

In the first inequality we used the definition of Φ and (3.6). In the second inequality we used $H \le 2E_1$, C-sublinearity of Θ, (2.1) inequality and C-sublinearity of B. In the last inequality we used $\Theta(s) \le c\sqrt{s}$ only. We estimate the two terms in the last denominator by the nominator. Using (3.5) we obtain

$$B(E_1)\|v\|^2 \le C(B(E_1)\|M(u)\|_*^2 + \|v\|^2 h(\|v\|)) \tag{3.8}$$

and (using Young inequality and (3.8))

$$\begin{aligned} B(E_1(t))\|v\|\|M(u)\|_* &\le B(E_1)\|M(u)\|_*^2 + B(E_1)\|v\|^2 \\ &\le (1+C)B(E_1)\|M(u)\|_*^2 + C\|v\|^2 h(\|v\|). \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9) we obtain $-\frac{d}{dt}\Phi(H(t)) \ge \frac{C}{C_\Phi}\|v\| = \|v\|$ (choosing $C_\Phi = C$) and integrating from t to $+\infty$ we conclude that

$$\int_t^{+\infty} \|v(s)\|ds \le \Phi(H(t)) - \lim_{s\to+\infty}\Phi(H(s)) \le \Phi(\Psi^{-1}(t - t_0)).$$

Hence $\dot{u} \in L^1([0,+\infty))$, so u has a limit φ and (2.5) holds since $\|u(t) - \varphi\| \le \int_t^{+\infty}\|v(s)\|ds$.

4. Proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2. We may assume $\varphi = 0$ and denote $v(t) := \dot{u}(t)$. We show below that $\|u(t)\|_V \to 0$ by the same method as in [3]. So, we know that there exists t_1 such that $u(t) \in B_V(\varphi, \delta)$ for all $t > t_1$ and the assumptions of Theorem 2.1 are satisfied with $B = B_V(\varphi, \delta)$. So, we apply Theorem 2.1 and obtain the desired decay estimates.

So, it only remains to show $\|u(t)\|_V \to 0$. By [1, Theorem 2.6], it is sufficient to find a function $\mathcal{E} \in C(V \times H, \mathbb{R})$, such that $t \mapsto \mathcal{E}(u(t), v(t))$ is nondecreasing for $t \geq 0$ and satisfies

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t), v(t)) \geq c\|\dot{u}(t)\|_* \tag{4.1}$$

whenever $u(t) \in B_V(0, \eta)$ for some fixed $\eta > 0$. We show that these conditions are satisfied by the function

$$\mathcal{E}(u, v) := \Phi(H(u, v)),$$

where

$$H(u, v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon h(\|v\|_*)\langle M(u), v\rangle_*, \quad u \in V,\ v \in H$$

with ε small enough.

Let us write for short $\mathcal{E}(t)$ (resp. $H(t)$) for $\mathcal{E}(u(t), v(t))$ (resp. $H(u(t), v(t))$) and u, v instead of $u(t)$, $v(t)$. By relative compactness of U_T, quantities $\|v\|$ and $\|M(u)\|_*$ are bounded, so we can use (g1), resp. (g1'). We have (in the following, if $v = 0$ then any term containing $\frac{1}{\|v\|_*}$ has to be replaced by 0)

$$\begin{aligned}
H'(t) &= \langle v, \dot{v}\rangle_{V,V^*} + \langle M(u), v\rangle_{V^*,V} + \varepsilon h'(\|v\|_*)\frac{\langle v, v_t\rangle_*}{\|v\|_*}\langle M(u), v\rangle_* \\
&\quad + \varepsilon h(\|v\|_*)\langle M'(u)v, v\rangle_* + \varepsilon h(\|v\|_*)\langle M(u), \dot{v}\rangle_* \\
&= -\langle g(v), v\rangle_{V^*,V} - \varepsilon h'(\|v\|_*)\frac{1}{\|v\|_*}\langle M(u), v\rangle_*^2 \\
&\quad - \varepsilon h'(\|v\|_*)\frac{1}{\|v\|_*}\langle g(v), v\rangle_*\langle M(u), v\rangle_* + \varepsilon h(\|v\|_*)\langle M'(u)v, v\rangle_* \\
&\quad - \varepsilon h(\|v\|_*)\langle M(u), M(u)\rangle_* - \varepsilon h(\|v\|_*)\langle g(v), M(u)\rangle_*
\end{aligned}$$

and by positivity of the second term on the right

$$\begin{aligned}
H'(t) &\leq -\langle g(v), v\rangle_{V^*,V} - \varepsilon h(\|v\|_*)\|M(u)\|_*^2 - \varepsilon h(\|v\|_*)\langle g(v), M(u)\rangle_* \\
&\quad - \varepsilon h'(\|v\|_*)\frac{1}{\|v\|_*}\langle g(v), v\rangle_*\langle M(u), v\rangle_* + \varepsilon h(\|v\|_*)\langle M'(u)v, v\rangle_*.
\end{aligned} \tag{4.2}$$

We show that the third, fourth and fifth terms in the last expression are dominated by the first and second terms.

The last term in (4.2) is estimated (with help of (e2) and (g2)) by

$$|\varepsilon h(\|v\|_*)\langle M'(u)v, v\rangle_*| \leq \varepsilon h(\|v\|_*)C\|v\|^2 \leq \varepsilon C\langle g(v), v\rangle_{V^*,V} \leq \frac{1}{4}\langle g(v), v\rangle_{V^*,V}$$

if ε is small enough. The third term on the right-hand side of (4.2) is estimated by

$$|\varepsilon h(\|v\|_*)\langle g(v), M(u)\rangle_*| \leq \varepsilon h(\|v\|_*)\|M(u)\|_*\|g(v)\|_*.$$

and the fourth term (applying the Cauchy-Schwarz inequality and admissibility of h) by

$$\Big|\varepsilon h'(\|v\|_*)\frac{1}{\|v\|_*}\langle g(v),v\rangle_*\langle M(u),v\rangle_*\Big| \le \varepsilon c_A h(\|v\|_*)\|M(u)\|_*\|g(v)\|_*.$$

By Young's inequality and (g1) we have

$$\|M(u)\|_*\|g(v)\|_* \le \frac{1}{K}\|M(u)\|_*^2 + C(K)\|g(v)\|_*^2 \le \frac{1}{K}\|M(u)\|_*^2 + C(K)\|v\|^2.$$

So, the third and fourth terms from (4.2) are estimated by

$$\begin{aligned}
&\varepsilon(1+c_A)h(\|v\|_*)\Big(\frac{1}{K}\|M(u)\|_*^2 + C(K)\|v\|^2\Big)\\
&\le \frac{1}{2}\varepsilon h(\|v\|_*)\|M(u)\|_*^2 + \varepsilon C h(\|v\|_*)\|v\|^2\\
&\le \frac{1}{2}\varepsilon h(\|v\|_*)\|M(u)\|_*^2 + \frac{1}{4}\langle g(v),v\rangle_{V^*,V}
\end{aligned}$$

(we first took K large enough and then ε small enough). Altogether, we have

$$\begin{aligned}
-H'(t) &\ge \frac{1}{2}\langle g(v),v\rangle_{V^*,V} + \varepsilon\frac{1}{2}h(\|v\|_*)\|M(u)\|_*^2\\
&\ge ch(\|v\|_*)\left(\|v\|^2 + \|M(u)\|_*^2\right)
\end{aligned} \tag{4.3}$$

where we used (g2) in the second inequality. Now we compute

$$\mathcal{E}'(t) = \frac{C_\Phi H'(t)}{\Theta(H(t))h(\Theta(H(t)))} \le -C\frac{h(\|v\|_*)\left(\|v\|^2+\|M(u)\|_*^2\right)}{\Theta(H(t))h(\Theta(H(t)))} \tag{4.4}$$

and see that $\mathcal{E}$ is nonincreasing along solutions for $t>0$.

Now, we assume that $\|u\|_V$ is small and apply (e1) to obtain (4.1). We compute

$$\begin{aligned}
\Theta(H(u,v)) &\le C\Big(\Theta(\frac{1}{2}\|v\|^2) + \Theta(E(u)) + \Theta(\|M(u)\|_*\|v\|_*)\Big)\\
&\le C\left(\Theta(\|v\|^2) + \|M(u)\|_* + \Theta(\|M(u)\|_*^2) + \Theta(\|v\|^2)\right)\\
&\le C(\|v\| + \|M(u)\|_*)\,,
\end{aligned}$$

where we used C-sublinearity and monotonicity of Θ, boundedness of h on compact intervals and property (K) for Θ and the Cauchy–Schwarz inequality (first step), Young's inequality, (2.1), $H \hookrightarrow V^*$ and again C-sublinearity and property (K) (second step), and $\Theta(s) \le C\sqrt{s}$ (third step). Since h is nondecreasing and has property (K) we have

$$\Theta(H(u,v))h(\Theta(H(u,v))) \le C(\|v\|+\|M(u)\|_*)h(\|v\|+\|M(u)\|_*). \tag{4.5}$$

Since h is admissible with $c_A=1$ we have

$$\Big(\frac{s}{h(s)}\Big)' = \frac{h(s)-sh'(s)}{h^2(s)} \ge 0,$$

i. e., $\frac{s}{h(s)}$ is nondecreasing. From $\|v\|+\|M(u)\|_* \ge c^*\|v\|_*$ we obtain

$$\frac{\|v\|+\|M(u)\|_*}{h(\|v\|+\|M(u)\|_*)} \ge \frac{c^*\|v\|_*}{h(c^*\|v\|_*)} \ge \frac{c^*\|v\|_*}{C(c^*)h(\|v\|_*)}. \tag{4.6}$$

Altogether, inserting the estimates (4.5) and (4.6) into (4.4) we obtain

$$-\mathcal{E}'(t) \ge C\cdot\frac{h(\|v\|_*)(\|v\|+\|M(u)\|_*)^2}{(\|v\|+\|M(u)\|_*)h(\|v\|+\|M(u)\|_*)} \ge C\|v(t)\|_*$$

for all t where $\|u(t)\|_V < \eta$ and the proof is complete. $\square$

Proof of Theorem 2.3. The proofs of Theorems 2.1 and 2.2 remain valid except that we have to be more careful by estimating the term $\|M(u)\|_*\|g(v)\|_*$. Take $R > 0$ such that $\|v(t)\| \leq R$ for all $t \geq 0$ and γ corresponding to this R. Let γ^* be the convex conjugate to γ. By [3, Lemma 3.2] we have $\gamma^*(s) \leq Cs^2$ for all s small enough. Then using Young's inequality we obtain

$$\|M(u)\|_*\|g(v)\|_* \leq \gamma^*\Big(\frac{1}{K}\|M(u)\|_*\Big) + \gamma(K\|g(v)\|_*). \tag{4.7}$$

Since we know that $\|M(u)\|_*$ is bounded, taking K large enough yields

$$\|M(u)\|_*\|g(v)\|_* \leq \frac{C}{K^2}\|M(u)\|_*^2 + C(K)\langle g(v), v\rangle_{V^*,V},$$

where we also used property (K) for function γ. The rests of the proofs remain unchanged. $\square$

5. Applications

In this section we show that Theorem 2.3 applies to the damping functions from [3], i.e., we consider a bounded open set $\Omega \subset \mathbb{R}^n$, $H = L^2(\Omega, \mathbb{R}^N)$, $V = H_0^1(\Omega, \mathbb{R}^N)$ (or $V = H^1(\Omega, \mathbb{R}^N)$, Ω with Lipschitz boundary) and a function $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions

- (A3) There exist $\tau > 0$ and an admissible function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (g3), (g4), (g5) such that
  - (gg1) there exists $C_2 > 0$ such that $|G(z)| \leq C_2|z|$ for all $z \in B(0, \tau)$,
  - (gg2) there exists $C_3 > 0$ such that $C_3|z| \leq |G(z)|$ for all $z \in \mathbb{R}^n \setminus B(0, \tau)$,
  - (gg3) if $n = 2$ then there exist $C_4 > 0$, $\alpha > 0$ such that $|G(z)| \leq C_4|z|^{\alpha+1}$ for all $z \in \mathbb{R}^n \setminus B(0, \tau)$; if $n > 2$ then the inequality holds with $\alpha = \frac{4}{n-2}$,
  - (gg4) there exists $C_5 > 0$ such that $\langle G(z), z\rangle \geq C_5|G(z)||z|$ for all $z \in \mathbb{R}^n$.
  - (gg5) $|G(z)| \geq h(|z|)|z|$ for all $z \in B(0, \tau)$.

Proposition 5.1. *Let $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfy* (A3) *and define $(g(v))(x) := G(v(x))$ for $v \in V$. Then $g(V) \subset V^*$ and g satisfies* (A2) *with* (g1) *replaced by* (g1').

Proof. We first show that $g(v) \in V^*$. Since $L^p(\Omega, \mathbb{R}^N) \hookrightarrow V^*$ for $p = \frac{\alpha+2}{\alpha+1}$ it is enough to show that $g(v) \in L^p(\Omega, \mathbb{R}^N)$. We have

$$\begin{aligned}
\int_\Omega |G(v(x))|^p &= \int_{\{|v(x)|\geq\tau\}} |G(v(x))|^p + \int_{\{|v(x)|<\tau\}} |G(v(x))|^p \\
&\leq \int_{\{|v(x)|\geq\tau\}} C_4^p|v(x)|^{p(\alpha+1)} + \int_{\{|v(x)|<\tau\}} C_2^p|v(x)|^p \\
&\leq C_4^p \int_\Omega |v(x)|^{\alpha+2} + |\Omega|C_2^p\tau^p \\
&\leq C\|v\|_V^{\alpha+2} + |\Omega|C_2^p\tau^p,
\end{aligned}$$

where the second inequality follows from (gg3) and (gg1) and the last inequality from $V \hookrightarrow L^{\alpha+2}(\Omega)$.

Now we show (g2). We define

$$\tilde{h}(s) := \begin{cases} \frac{h(s)}{2} & \text{for } s \in [0, \delta) \\ \frac{h(\delta)}{2} + (\frac{1}{\delta} - \frac{1}{s})\frac{h'(\delta)\delta^2}{2} & \text{for } s \in [\delta, +\infty) \end{cases}$$

as in [3, proof of Proposition 3.3]. It is easy to show that $\tilde{h}$ is admissible and $|G(z)| \geq \tilde{h}(|z|)|z|$ holds for all $z \in \mathbb{R}^n$ if $\delta > 0$ is small enough and such that $h'(\delta) > 0$. Moreover, $\tilde{h}$ is bounded and $\tilde{\psi}$ defined by $\tilde{\psi}(s) = s\tilde{h}(\sqrt{s})$ is convex on $\mathbb{R}_+$ (see [3, proof of Proposition 3.3]). Then we have

$$\begin{aligned}
\langle g(v), v\rangle_{V^*,V} &= \int_\Omega \langle G(v(x)), v(x)\rangle \\
&\geq \int_\Omega C_5 \tilde{h}(|v(x)|)|v(x)|^2 \\
&= C_5|\Omega| \int_\Omega \tilde{\psi}(|v(x)|^2)\frac{dx}{|\Omega|} \\
&\geq C_5|\Omega|\tilde{\psi}\Big(\int_\Omega |v(x)|^2 \frac{dx}{|\Omega|}\Big) \\
&\geq C\tilde{\psi}(\|v\|^2) \\
&= C\tilde{h}(\|v\|)\|v\|^2 \\
&\geq Ch(\|v\|)\|v\|^2,
\end{aligned}$$

where we used Jensen's inequality in the fourth step, property (K) in the fifth step and inequality $h(s) \leq C\tilde{h}(s)$ on compact intervals $[0, K]$ in the sixth step.

We show (g1'). By [3, Proposition 3.3] there exists a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(G(s)) \leq CG(s)s$ and $s \mapsto \gamma(s^{1/p})$ is convex for $s \geq 0$ and $\gamma(s) \geq Cs^2$ for small $s \geq 0$. Then we have

$$\begin{aligned}
\gamma(\|g(v)\|_*) &\leq C\gamma\Big(\Big(\int_\Omega |G(v(x))|^p\Big)^{1/p}\Big) \\
&\leq C\int_\Omega \gamma(|G(v(x))|) \\
&\leq C\int_\Omega |G(v(x))||v(x)| \\
&\leq C\int_\Omega \langle G(v(x)), v(x)\rangle \\
&= C\langle g(v), v\rangle_{V^*,V}.
\end{aligned}$$

The first inequality follows from $L^p \hookrightarrow V^*$, monotonicity and property (K) of γ, the second inequality is Jensen's inequality applied to $s \mapsto \gamma(s^{1/p})$ together with property (K), the third follows from $\gamma(G(s)) \leq CG(s)s$ and the fourth from (gg4). $\square$

Let us consider the following examples taken from [5].

A critical semilinear wave equation. Let $\Omega \subset \mathbb{R}^n$ be bounded open and connected. We consider the Dirichlet problem

$$\begin{gathered}
u_{tt} + g(u_t) - \Delta u - \lambda_1 u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\
u(t,x) = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega,
\end{gathered} \tag{5.1}$$

where λ_1 is the first eigenvalue of $-\Delta$ and $p > 1$ with $(N-2)p < N+2$. It corresponds to (1.1) with $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1|u|^2)dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}dx.$$

According to [5], (e1)-(e3) hold with $\Theta(s) = Cs^{1-\theta}$, $\theta = \frac{1}{p+1}$ and $G(s) = C\sqrt{s}$ on any bounded subset of V and any strong solution to (5.1) is bounded in V. Moreover, $E(u) \geq c\|u\|_V^{p+1}$.

A semilinear wave equation with Neumann boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be bounded open and connected. We consider the Neumann problem

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + |u|^{p-1}u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial}{\partial n} u(t,x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.2}$$

where $p > 1$ with $(n-2)p < n+2$. We have $H = L^2(\Omega)$, $V = H^1(\Omega)$ and

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

According to [5], (e1)–(e3) hold with $\Theta(s) = Cs^{1-\theta}$, $\theta = \frac{1}{p+1}$ and $G(s) = C\sqrt{s}$ on any bounded subset of V and any strong solution to (5.1) is bounded in V.

Now, we present some examples of damping functions g and obtain convergence to equilibrium and decay estimates for solutions of (5.1) and (5.2).

Example 5.2. Let us consider $(g(v)) = G(v(x))$ with G having different growth/ decay for $s < 0$, $s > 0$, $|s|$ large, $|s|$ small, e.g.

$$G(s) = \begin{cases} |s|^{b_1}s, & s > 1, \\ |s|^{a_1}s, & s \in [0,1], \\ |s|^{a_2}s, & s \in [-1,0), \\ |s|^{b_2}s, & s < -1, \end{cases}$$

with $0 \leq a_1 < a_2 < \frac{1}{p}$, $b_1, b_2 \leq \frac{4}{n-2}$. Then by Theorem 2.3 we have

$$\|u(t) - \varphi\| \leq Ct^{-\frac{1-a_2 p}{(a_2+1)p-1}},$$

and for equation (5.1) even

$$\|u(t) - \varphi\|_V \leq Ct^{-\frac{1}{(a_2+1)p-1}}$$

by Corollary 2.4.

Example 5.3. In this example we show more delicate decay estimates in the logarithmic scale. Let

$$G(s) = \begin{cases} |s|^a s \ln^r(1/|s|) & |s| \leq 1, \\ c|s|^b s & |s| > 1, \end{cases}$$

with $b < \frac{4}{n-2}$, $0 < a < \frac{1}{p}$, $r \in \mathbb{R}$ or $a = \frac{1}{p}$, $r > 1$.

If $a > \frac{1}{p}$ and $r \geq 0$ then one can apply Theorem 2.3 with $h(s) = s^a$ to obtain

$$\|u(t) - \varphi\| \leq Ct^{-\frac{1-ap}{(a+1)p-1}}$$

as in the previous example. If $a < \frac{1}{p}$, $r < 0$, we can apply Theorem 2.3 with $h(s) = s^{a+\varepsilon}$ (for $\varepsilon > 0$ small enough) to obtain

$$\|u(t) - \varphi\| \leq Ct^{-\frac{1-(a+\varepsilon)p}{(a+\varepsilon+1)p-1}}.$$

If $a = 1/p$, we cannot estimate G by any power such that (g3) holds. However, in all cases, one can take $h(s) = s^a \ln^r(1/s)$ and obtain better decay estimates if $a < \frac{1}{p}$ and obtain some decay estimates even for $a = \frac{1}{p}$. In fact, we have $\Theta^2(s)h(\Theta(s)) = s^{(1-\theta)(2+a)}(1-\theta)^r \ln^r(1/s)$ and by Lemma 6.5

$$\Psi(t) = C\int_t^{1/2} \frac{1}{s^{(1-\theta)(2+a)}\ln^r(1/s)}ds \sim t^{1-(1-\theta)(2+a)}\ln^{-r}(1/t), \quad t \to 0+, \tag{5.3}$$

where $f \sim g$ means $f = O(g)$ and $g = O(f)$. Then by Lemma 6.6

$$\Psi^{-1}(t) \sim t^{\frac{1}{1-(1-\theta)(2+a)}} \ln^{\frac{r}{1-(1-\theta)(2+a)}}(t), \quad t \to +\infty. \tag{5.4}$$

For equation (5.1) by Corollary 2.4 we have

$$\|u(t) - \varphi\|_V \le C\left(\Psi^{-1}(t-t_0)\right)^{\frac{1}{p+1}} \le Ct^{-\frac{1}{(a+1)p-1}} \ln^{-\frac{r}{(a+1)p-1}}(t).$$

For equation (5.2) in the case $a < \frac{1}{p}$ by Lemma 6.5 we have

$$\Phi(t) = C\int_0^t \frac{1}{s^{(1-\theta)(1+a)}\ln^r(1/s)}ds \sim t^{1-(1-\theta)(1+a)}\ln^{-r}(1/t), \quad t \to 0+, \tag{5.5}$$

which for large t yields

$$\|u(t) - \varphi\| \le \Phi(\Psi^{-1}(t-t_0)) \le Ct^{-\frac{1-ap}{(a+1)p-1}} \ln^{-\frac{pr}{(a+1)p-1}}(t). \tag{5.6}$$

If $a = 1/p$, then we have

$$\Phi(t) = C\int_0^t \frac{1}{s^{(1-\theta)(1+a)}\ln^r(1/s)}ds = C\int_0^t \frac{1}{s\ln^r(1/s)}ds \sim \ln^{1-r}(1/t) \tag{5.7}$$

for $t \to 0+$ and therefore for large t,

$$\|u(t) - \varphi\| \le \Phi(\Psi^{-1}(t-t_0)) \le C\ln^{1-r}(t). \tag{5.8}$$

By similar computations as above with the help of Lemmas 6.5, 6.6, we have: if

$$G(s) \ge |s|^a \ln^{r_1}(1/|s|)\cdots\ln^{r_k}(\ln\cdots\ln(1/|s|))$$

on a neighborhood of zero, then for large t we obtain

$$\|u(t)-\varphi\| \le Ct^{-\frac{1-ap}{(a+1)p-1}} \ln^{-\frac{pr_1}{(a+1)p-1}}(t) \ln^{-\frac{pr_2}{(a+1)p-1}}(\ln(t))\cdots\ln^{-\frac{pr_k}{(a+1)p-1}}(\ln\cdots\ln(t))$$

provided $a > 1/p$ and

$$\|u(t) - \varphi\| \le C\ln^{1-r_j}(\ln\cdots\ln(t))\ln^{-r_{j+1}}(\ln\cdots\ln(t))\cdots\ln^{-r_k}(\ln\cdots\ln(t))$$

provided $a = \frac{1}{p}$, $r_1 = \cdots = r_{j-1} = 1$, $r_j > 1$, $r_{j+1}, \dots, r_k \in \mathbb{R}$.

6. Appendix

Lemma 6.1. *If f is admissible, then it has property* (K).

Proof. For $K \le 1$ it is sufficient to take $C(K) = 1$ since f is nondecreasing. Now, let us fix $t \ge 0$. Then for $s > t$ we have $\frac{f'(s)}{f(s)} \le \frac{c_A}{s}$ and integrating from t to $T > t$ we obtain

$$\ln(f(T)) - \ln(f(t)) = \ln\frac{f(T)}{f(t)} \le c_A \ln\frac{T}{t},$$

so $f(T) \le f(t)\left(\frac{T}{t}\right)^{c_A}$ and taking $T = Kt$ for $K > 1$ we have property (K) with $C(K) = K^{c_A}$. ◻

Lemma 6.2. *Let f be nonnegative, nondecreasing and f, g have property (K). Then the composition $f(g(\cdot))$ has property (K).*

Proof. We have $f(g(Kx)) \le f(C(K)g(x)) \le C(C(K))f(g(x))$. ◻

Lemma 6.3. *Let f be nonnegative, nondecreasing and has property* (K). *Then it is C-sublinear, i.e., there exists $C > 0$ such that*

$$f(x+y) \le C(f(x)+f(y)) \quad \text{for all } x,\, y \ge 0.$$

Proof. We have

$$\begin{aligned} f(x+y) &\le f(2\max\{x,y\}) \le C(2)f(\max\{x,y\}) \\ &\le C\max\{f(x), f(y)\} \le C(f(x)+f(y)). \end{aligned}$$

□

Where did you define (h3)?

Lemma 6.4. *Let ψ^* be convex conjugate to the function ψ from* (h3). *Then $\psi^*(h(\sqrt{s})) \le c\psi(s)$ for all $s \ge 0$.*

Proof. It holds that

$$\psi^*(h(\sqrt{s})) = \psi^*(\psi(s)/s) \le \psi^*(\psi'(s)) = s\psi'(s) - \psi(s).$$

Further,

$$\psi(2s) - \psi(s) = \int_s^{2s} \psi'(r)dr \ge s \cdot \psi'(s).$$

So,

$$\psi^*(h(\sqrt{s})) \le \psi(2s) - 2\psi(s) \le (K-2)\psi(s)$$

since ψ has property (K). □

Lemma 6.5. *Let F be a primitive function to*

$$f(t) = t^a \ln^{r_1}(1/t)\ln^{r_2}(\ln(1/t)) \cdots \ln^{r_k}(\ln\cdots\ln(1/t))$$

on $(0,\varepsilon)$, $a \ne -1$. Moreover, if $a > -1$, we assume $\lim_{t\to 0+} F(t) = 0$. Then

$$|F(t)| \sim t^{1+a}\ln^{r_1}(1/t)\ln^{r_2}(\ln(1/t))\cdots\ln^{r_k}(\ln\cdots\ln(1/t)) \quad \text{as } t \to 0+, \tag{6.1}$$

where $F \sim g$ means $F = O(g)$ and $g = O(F)$. If $a = -1$, $r_1 = \cdots = r_{j-1} = -1$, $r_j < -1$, then

$$|F(t)| \sim \ln^{r_j+1}(\ln\cdots\ln(1/t))\ln^{r_{j+1}}(\ln\cdots\ln(1/t))\cdots\ln^{r_k}(\ln\cdots\ln(1/t)) \tag{6.2}$$

as $t \to 0+$.

Proof. Let us denote the right-hand side of (6.1) by $G(t)$ and differentiate

$$G'(t) = (a+1)f(t) + \sum_{i=1}^k tf(t)\frac{r_i}{\ln(\cdots\ln(1/t))\cdots\ln(1/t)\frac{1}{t}} \cdot \frac{-1}{t^2} = f(t)(1+a+o(1)).$$

If $a > -1$, then $\frac{1}{C}G'(s) \le f(s) \le CG'(s)$ on $(0,\varepsilon)$ for some $C > 1$ and

$$F(t) = \int_0^t f(s) \le C\int_0^t G'(s)ds = CG(t)$$

and similarly $F(t) \ge \frac{1}{C}G(t)$. If $a < -1$, then $\frac{1}{C}G'(s) \le f(s) \le CG'(s)$ on $(0,\varepsilon)$ for some $C < -1$.

$$|F(t)| = \int_t^c f(s)ds + d \le C\int_t^c G'(s)ds + d = CG(c) - CG(t) + d \le \tilde{C}G(t),$$

where the last inequality holds since $G(t) \to +\infty$ as $t \to 0+$ and $C < 0$. Analogously we can estimate $|F(t)|$ from below. So, (6.1) is proven and (6.2) can be proven by the same method. □

Lemma 6.6. *Let*

$$f(t) = t^a \ln^{r_1}(1/t) \ln^{r_2}(\ln(1/t)) \cdots \ln^{r_k}(\ln \cdots \ln(1/t))$$

on $(0,\varepsilon)$, $a<0$. *Then*

$$f^{-1}(t) \sim t^{1/a} \ln^{-r_1/a}(t) \ln^{-r_2/a}(\ln(t)) \cdots \ln^{-\frac{r_k}{a}}(\ln \cdots \ln(t)) \quad \text{as } t \to +\infty. \tag{6.3}$$

Proof. Let us denote by $g(t)$ the right-hand side of (6.3) and let us assume that $r_i \geq 0$ for all $i = 1,2,\dots,k$. We show that $f(g(t)) \leq Ct$ for large t. Since

$$\frac{1}{g(t)} = t^{-1/a} o(t^{-1/a}), \quad \text{as } t \to +\infty,$$

for t large enough we have

$$\ln\Big(\frac{1}{g(t)}\Big) \leq \ln\Big(t^{-\frac{2}{a}}\Big) = -\frac{2}{a}\ln(t).$$

Further, if $h(t) \to +\infty$, then for $c>0$ and large t it holds that $\ln(ch(t)) = \ln c + \ln h(t) \leq 2\ln h(t)$. Therefore,

$$\ln^{r_i}\Big(\ln \cdots \ln\Big(\frac{1}{g(t)}\Big)\Big) \leq \ln^{r_i}\Big(\ln \dots \frac{-2}{a}\ln(t)\Big) \leq 2^{r_i}\ln^{r_i}\Big(\ln \cdots \ln(t)\Big).$$

Now, we can compute

$$\begin{aligned}
f(g(t)) &= g(t)^a \prod_{i=1}^k \ln^{r_i}\Big(\ln \cdots \ln\Big(\frac{1}{g(t)}\Big)\Big) \\
&= t \ln^{-r_1}(t) \cdots \ln^{-r_k}(\ln \cdots \ln(t)) \cdot \prod_{i=1}^k \ln^{r_i}\Big(\ln \cdots \ln\Big(\frac{1}{g(t)}\Big)\Big) \\
&\leq t \ln^{-r_1}(t) \cdots \ln^{-r_k}(\ln \cdots \ln(t)) \cdot \big(-\frac{2}{a}\big)^{r_1} \prod_{i=2}^k 2^{r_i}\ln^{r_i}(\ln \cdots \ln(t)) \\
&\leq t \cdot \big(-\frac{1}{a}\big)^{r_1} \prod_{i=1}^k 2^{r_i}.
\end{aligned}$$

We can easily modify the estimates above to obtain $f(g(t)) \geq t(-\frac{1}{a})^{r_1}\prod_{i=1}^k 2^{-r_i}$ and similarly if we omit the assumption that r_i are positive, we obtain

$$\frac{t}{K} \leq f(g(t)) \leq Kt \quad \text{with } K := C^{r_1}\prod_{i=1}^k 2^{|r_i|}, \quad C := \max\{-\frac{1}{a}, -a\}.$$

Applying f^{-1} (which is decreasing for large t) to these inequalities with $s = t/K$, we obtain

$$f^{-1}(s) \geq f^{-1}(f(g(Ks))) = g(sK) \geq \frac{K^{1/a}}{C} g(s),$$

resp. with $s = Kt$

$$f^{-1}(s) \leq f^{-1}(f(g(s/K))) = g(s/K) \leq \frac{C}{K^{1/a}} g(s).$$

$\square$

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