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ORBITAL STABILITY OF GAUSSON SOLUTIONS TO LOGARITHMIC SCHRÖDINGER EQUATIONS

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Abstract. In this article we prove of the orbital stability of the ground state for logarithmic Schrödinger equation in any dimension and under nonradial perturbations. This general stability result was announced by Cazenave and Lions [9, Remark II.3], but no details were given there.

1. Introduction

In this article we study the logarithmic Schrödinger equation

$$i\partial_t u + \Delta u + u \log |u|^2 = 0, \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function of $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $N \geq 1$. This equation was proposed by Bialynicki-Birula and Mycielski [2] in 1976 as a model of nonlinear wave mechanics. It laso has several applications in quantum mechanics, quantum optics, nuclear physics, open quantum systems and Bose-Einstein condensation (see e.g. [19] and the references therein). Recently, (1.1) has proved useful for the modeling of several nonlinear phenomena including geophysical applications of magma transport [16] and nuclear physics [13].

The mathematical literature concerning the logarithmic Schrödinger equation does not seem to be very extensive. The Cauchy problem for (1.1) was tread by Cazenave and Haraux [8] in a suitable functional framework. Cazenave [6]; Cazenave and Lions [9]; Blanchard and co. [4, 3]; research the stability properties of standing waves for (1.1). In recent years, the logarithmic NLS equation has attracted some attention both in the theoretical and the applied mathematical literature. Among such works, let us mention [1, 10, 14, 18, 11].

The energy functional E associated with problem (1.1) is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx. \tag{1.2}$$

Unfortunately, because of the singularity of the logarithm at the origin, the functional fails to be finite as well of class C^1 on $H^1(\mathbb{R}^N)$. Because of this loss of smoothness, it is convenient to work in a suitable Banach space endowed with a Luxemburg type norm to make functional E well defined and C^1 smooth. This

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space allow to control the singularity of the logarithmic nonlinearity at infinity and at the origin. Indeed, we consider the reflexive Banach space (see Appendix below)

$$W(\mathbb{R}^N) = \big\{u \in H^1(\mathbb{R}^N) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^N)\big\}, \tag{1.3}$$

then it is well known that the energy functional E is well-defined and of class C^1 on $W(\mathbb{R}^N)$ (see [6]). Moreover, Cazenave [7, Theorem 9.3.4] proved the global well-posedness of the Cauchy problem for (1.1) in the energy space $W(\mathbb{R}^N)$.

Proposition 1.1. *For each $u_0 \in W(\mathbb{R}^N)$, there is a unique maximal solution u of equation (1.1) such that $u \in C(\mathbb{R}, W(\mathbb{R}^N)) \cap C^1(\mathbb{R}, W'(\mathbb{R}^N))$, $u(0) = u_0$ and $\sup_{t\in\mathbb{R}} \|u(t)\|_{W(\mathbb{R}^N)} < \infty$. Furthermore, the conservation of energy and charge hold; that is,*

$$E(u(t)) = E(u_0) \quad \textit{and} \quad \|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2} \quad \textit{for all } t \in \mathbb{R}.$$

Let $\omega \in \mathbb{R}$ and $\varphi \in W(\mathbb{R}^N)$ be solutions of the semilinear elliptic equation

$$-\Delta\varphi + \omega\varphi - \varphi \log |\varphi|^2 = 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

then, $u(x, t) = e^{i\omega t}\varphi(x)$ is a standing wave of (1.1). It is well known (see [2]) that the Gausson

$$\phi_\omega(x) := e^{\frac{\omega+N}{2}} e^{-\frac{1}{2}|x|^2}, \quad x \in \mathbb{R}^N, \tag{1.5}$$

solves (1.4) for any dimension N. Up to translations, (1.5) is the unique strictly positive C^2-solution for (1.4) such that $\varphi(x) \to 0$ as $|x| \to \infty$. Moreover, it is nondegenerate; that is, the dimension of the nullspace of the linearized operator is N, i.e. smallest possible (see [10]).

The orbital stability of the Gausson (1.5) when $N \geq 2$ has been studied in [4, 3, 6]. In particular, Cazenave [6] proved that $e^{i\omega t}\phi_\omega(x)$ is stable in $W(\mathbb{R}^N)$, with respect to radial perturbations, for $N \geq 2$. Their argument is based on the fact that the space of radially symmetric functions in $W(\mathbb{R}^N)$ is compactly embedded into $L^2(\mathbb{R}^N)$ for $N \geq 2$. Other proof, for $N \geq 3$ and under radial perturbations, was given in [4, 3]. This proof relies on application of the Shatah formalism [17].

As we have mentioned, Cazenave and Lions [9, Remark II.3] claimed that the Gausson (1.5) is orbitally stable in the unrestricted space $W(\mathbb{R}^N)$ for all $N \geq 1$, but there the proof is omitted. The main aim of this paper is to give a detailed proof of this fact.

The notions of stability and instability are defined as follows.

Definition 1.2. We say that a standing wave solution $u(x, t) = e^{i\omega t}\phi(x)$ of (1.1) is orbitally stable in $W(\mathbb{R}^N)$ if for any $\epsilon > 0$ there exist $\eta > 0$ such that if $u_0 \in W(\mathbb{R}^N)$ and $\|u_0 - \varphi\|_{W(\mathbb{R}^N)} < \eta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exist for all $t \in \mathbb{R}$ and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|u(t) - e^{i\theta}\varphi(\cdot - y)\|_{W(\mathbb{R}^N)} < \epsilon.$$

Otherwise, the standing wave $e^{i\omega t}\phi(x)$ is said to be unstable in $W(\mathbb{R}^N)$.

Before we state our result, we establish a variational characterization of the Gausson (1.5). For $\omega \in \mathbb{R}$, we define the following functionals of class C^1 on $W(\mathbb{R}^N)$:

$$S_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\omega + 1}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx,$$

$$I_\omega(u)=\int_{\mathbb{R}^N}|\nabla u|^2dx+\omega\int_{\mathbb{R}^N}|u|^2dx-\int_{\mathbb{R}^N}|u|^2\log|u|^2dx.$$

Note that (1.4) is equivalent to $S'_\omega(\varphi)=0$, and $I_\omega(u)=\langle S'_\omega(u),u\rangle$ is the so-called Nehari functional.

Moreover, we consider the minimization problem

$$\begin{aligned}d(\omega)&=\inf\{S_\omega(u):\,u\in W(\mathbb{R}^N)\setminus\{0\},I_\omega(u)=0\}\\&=\frac{1}{2}\inf\{\|u\|^2_{L^2}:u\in W(\mathbb{R}^N)\setminus\{0\},I_\omega(u)=0\},\end{aligned}\tag{1.6}$$

and define the set of ground states by

$$\mathcal{N}_\omega=\big\{\varphi\in W(\mathbb{R}^N)\setminus\{0\}:S_\omega(\varphi)=d(\omega),\ I_\omega(\varphi)=0\big\}.$$

The set $\big\{u\in W(\mathbb{R}^N)\setminus\{0\},I_\omega(u)=0\big\}$ is called the Nehari manifold. Notice that the above set contains all stationary point of S_ω. In Section 2, we show that the quantity $d(\omega)$ is positive for every $\omega\in\mathbb{R}$. Indeed,

$$d(\omega)=\frac{1}{2}\pi^{N/2}e^{\omega+N}.$$

Remark 1.3. Let $u\in\mathcal{N}_\omega$. Then, there exist a Lagrange multiplier $\Lambda\in\mathbb{R}$ such that $S'_\omega(u)=\Lambda I'_\omega(u)$. Thus, we have $\langle S'_\omega(u),u\rangle=\Lambda\langle I'_\omega(u),u\rangle$. The fact that $\langle S'_\omega(u),u\rangle=I_\omega(u)=0$ and $\langle I'_\omega(u),u\rangle=-2\|u\|^2_{L^2}<0$, implies $\Lambda=0$; that is, $S'_\omega(u)=0$. Therefore, u satisfies (1.4).

The existence of minimizers for the minimization problem (1.6) is proved by the standard variational argument. We will show the following proposition in the Section 2.

Proposition 1.4. *There exists a minimizer of $d(\omega)$ for any $\omega\in\mathbb{R}$. Moreover, the set of ground states is given by $\mathcal{N}_\omega=\{e^{i\theta}\phi_\omega(.-y);\theta\in\mathbb{R},y\in\mathbb{R}^N\}$, where ϕ_ω is given in* (1.5).

We remark that Proposition 1.4 was claimed without proof by Cazenave [7, Remark 9.3.8]. It is also important to note that the ground state be unique up to translations and phase shifts. In higher dimensions, it is known that there exist infinitely many weak solutions $u_n\in H^1(\mathbb{R}^N)$ of (1.4) such that $S_\omega(u_n)\to+\infty$ as $n\to+\infty$ (see e.g. [10, Theorem 1.1]). The variational characterization of the Gausson (1.5) as a minimizer of S_ω on the Nehari manifold, contained in Proposition 1.4, will be useful when we will deal with the stability.

Now we state our main result of this paper.

Theorem 1.5. *Let $\omega\in\mathbb{R}$ and $N\geq1$. Then the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable in $W(\mathbb{R}^N)$.*

The rest of the article is organized as follows. In Section 2 we prove, by variational techniques, the existence of a minimizer for $d(\omega)$ (Proposition 1.4). Section 3 is devoted to the proof of Theorem 1.5. In the Appendix we include some information about of the space $W(\mathbb{R}^N)$.

Notation. The space $L^2(\mathbb{R}^N,\mathbb{C})$ will be denoted by $L^2(\mathbb{R}^N)$ and its norm by $\|\cdot\|_{L^2}$. This space will be endowed with the real scalar product

$$(u,v)=\Re\int_{\mathbb{R}^N}u\overline{v}\,dx,\quad\text{for }u,v\in L^2(\mathbb{R}^N).$$

The space $H^1(\mathbb{R}^N,\mathbb{C})$ will be denoted by $H^1(\mathbb{R}^N)$ and its norm by $\|\cdot\|_{H^1(\mathbb{R}^N)}$. $\langle\cdot,\cdot\rangle$ is the duality pairing between X' and X, where X is a Banach space and X' is its dual. Finally, $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or $N = 2$. Throughout this paper, the letter C will denote positive constants.

2. Existence and uniqueness of ground state

Before giving the proof of Proposition 1.4, some preparation is necessary. First, we recall the logarithmic Sobolev inequality. For a proof we refer to [15, Theorem 8.14].

Lemma 2.1. *Let f be any function in $H^1(\mathbb{R}^N)\setminus\{0\}$ and α be any positive number. Then,*

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(x)|^2 \log|f(x)|^2 dx\\ &\leq \frac{\alpha^2}{\pi}\|\nabla f\|_{L^2}^2 + (\log\|f\|_{L^2}^2 - N(1+\log\alpha))\|f\|_{L^2}^2.\end{aligned} \tag{2.1}$$

Moreover, there is equality if and only if f is, up to translation, a multiple of $e^{\{-\pi|x|^2/2\alpha^2\}}$.

Lemma 2.2. *Let $\omega \in \mathbb{R}$. Then, the quantity $d(\omega)$ is positive and satisfies*

$$d(\omega) \geq \frac{1}{2}\pi^{N/2}e^{\omega+N}. \tag{2.2}$$

Proof. Let $u \in W(\mathbb{R}^N)\setminus\{0\}$ be such that $I_\omega(u) = 0$. Using the logarithmic Sobolev inequality with $\alpha = \sqrt{\pi}$, we see that

$$(\omega + N(1+\log(\sqrt{\pi})))\|u\|_{L^2}^2 \leq (\log\|u\|_{L^2}^2)\|u\|_{L^2}^2,$$

which implies that $\|u\|_{L^2}^2 \geq \pi^{N/2}e^{\omega+N}$. Thus, by the definition of $d(\omega)$ given in (1.6), we obtain (2.2). $\square$

The following lemma is a variant of the Brézis-Lieb lemma from [5].

Lemma 2.3. *Let $\{u_n\}$ be a bounded sequence in $W(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then $u \in W(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\{|u_n|^2\log|u_n|^2 - |u_n-u|^2\log|u_n-u|^2\}dx = \int_{\mathbb{R}^N}|u|^2\log|u|^2 dx.$$

Proof. We first recall that, by (4.1) in the Appendix, $|z|^2\log|z|^2 = A(|z|) - B(|z|)$ for every $z \in \mathbb{C}$. We need only apply the Brézis-Lieb lemma (see Lemma 4.2) to the functions A and B. By the weak-lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm and Fatou lemma we have $u \in W(\mathbb{R}^N)$. It is clear that the sequence $\{u_n\}$ is bounded in $L^A(\mathbb{R}^N)$. Since A is convex and increasing function with $A(0) = 0$, it is follows by property (4.2) in the Appendix that the function A satisfies the assumptions of Lemma 4.2 in Appendix. Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|A(|u_n|) - A(|u_n-u|) - A(|u|)|dx = 0. \tag{2.3}$$

On the other hand, by the continuous embedding $W(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$, we have that $\{u_n\}$ is also bounded in $H^1(\mathbb{R}^N)$. By Hölder an Sobolev inequalities, for any u, $v \in H^1(\mathbb{R}^N)$ we have that (see [6, Lemma 1.1])

$$\int_{\mathbb{R}^N}|B(|u(x)|) - B(|v(x)|)|dx \leq C(1+\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2)\|u-v\|_{L^2}. \tag{2.4}$$

Thus, the function B satisfies the hypotheses (ii) and (iii) of Lemma 4.2. Furthermore, an easy calculation shows that the function B is convex, increasing and nonnegative with $B(0) = 0$. Then from Lemma 4.2 we see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |B(|u_n|) - B(|u_n - u|) - B(|u|)| dx = 0. \tag{2.5}$$

Thus the result follows from (2.3) and (2.5). □

Lemma 2.4. *Let $2 < p < 2^*$ and $\omega \in \mathbb{R}$. Assume that $\{u_n\} \subset W(\mathbb{R}^N)$, $I_\omega(u_n) = 0$ for any $n \in \mathbb{N}$ and $S_\omega(u_n) \to d(\omega)$ as n approaches $+\infty$. Then, there exist a constant $C > 0$ depending only on p such that $\|u_n\|^p_{L^p(\mathbb{R}^N)} \geq C$ for every $n \in \mathbb{N}$.*

The proof of the above lemma follows along the same lines as [6, Lemma 3.3]. We omit it.

Proof of Proposition 1.4. Let $\{u_n\} \subseteq W(\mathbb{R}^N)$ be a minimizing sequence for $d(\omega)$, then the sequence $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$. Indeed, it is clear that the sequence $\|u_n\|^2_{L^2}$ is bounded. Moreover, using the logarithmic Sobolev inequality and recalling that $I_\omega(u_n) = 0$, we obtain

$$\Big(1 - \frac{\alpha^2}{\pi}\Big)\|\nabla u_n\|^2_{L^2} \leq \Big(\log\big(\frac{e^{-(\omega+N)}}{\alpha^N}\big)\Big)\|u_n\|^2_{L^2} + \big(\log\|u_n\|^2_{L^2}\big)\|u_n\|^2_{L^2}.$$

Taking $\alpha > 0$ sufficiently small, we see that $\|\nabla u_n\|^2_{L^2}$ is bounded, so the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, using $I_\omega(u_n) = 0$ again, and (2.4) we obtain that there exist a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} A(|u_n|)dx \leq \int_{\mathbb{R}^N} B(|u_n|)dx + |\omega|\|u_n\|^2_{L^2} \leq C,$$

which implies, by (4.2) in the Appendix, that the sequence $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$.

Next, notice that for any sequence $x_n \in \mathbb{R}^N$ we have that $\{u_n(\cdot + x_n)\}$ is still a bounded minimizing sequence for $d(\omega)$. Moreover, if

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx = 0,$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*)$, where $B_1(y) = \{z \in \mathbb{R}^N : |y - z| < 1\}$. Therefore, from Lemma 2.4 and the compactness of the embedding $H^1(B_1(0)) \hookrightarrow L^2(B_1(0))$, we deduce that there exist a sequence $y_n \in \mathbb{R}^N$ such that the weak limit in $H^1(\mathbb{R}^N)$ of the sequence $\{u_n(\cdot + y_n)\}$ is not the trivial function. Let $v_n := u_n(\cdot+y_n)$. Then there exist $\varphi \in W(\mathbb{R}^N)\setminus\{0\}$ such that, up to a subsequence, $v_n \rightharpoonup \varphi$ weakly in $W(\mathbb{R}^N)$ and $v_n \to \varphi$ a.e. in $\mathbb{R}^N$.

Now we prove that $I_\omega(\varphi) = 0$ and $S(\varphi) = d(\omega)$. First, assume by contradiction that $I_\omega(\varphi) < 0$. By elementary computations, we can see that there is $0 < \lambda < 1$ such that $I_\omega(\lambda\varphi) = 0$. Then, from the definition of $d(\omega)$ and the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm, we have

$$d(\omega) \leq \frac{1}{2}\|\lambda\varphi\|^2_{L^2} < \frac{1}{2}\|\varphi\|^2_{L^2} \leq \frac{1}{2}\liminf_{n\to\infty} \|v_n\|^2_{L^2} = d(\omega),$$

it which is impossible. On the other hand, assume that $I_\omega(\varphi) > 0$. Since the embedding $W(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$ is continuous, we see that $v_n \rightharpoonup \varphi$ weakly in $H^1(\mathbb{R}^N)$. Thus, we have

$$\|v_n\|^2_{L^2} - \|v_n - \varphi\|^2_{L^2} - \|\varphi\|^2_{L^2} \to 0, \tag{2.6}$$

$$\|\nabla v_n\|_{L^2}^2 - \|\nabla v_n - \nabla\varphi\|_{L^2}^2 - \|\nabla\varphi\|_{L^2}^2 \to 0 \tag{2.7}$$

as $n \to \infty$. Combining (2.6), (2.7) and Lemma 2.3 leads to

$$\lim_{n\to\infty} I_\omega(v_n - \varphi) = \lim_{n\to\infty} I_\omega(v_n) - I_{\omega,\gamma}(\varphi) = -I_\omega(\varphi),$$

which combined with $I_\omega(\varphi) > 0$ give us that $I_\omega(v_n - \varphi) < 0$ for sufficiently large n. Thus, by (2.6) and applying the same argument as above, we see that

$$d(\omega) \leq \frac{1}{2}\lim_{n\to\infty} \|v_n - \varphi\|_{L^2}^2 = d(\omega) - \frac{1}{2}\|\varphi\|_{L^2}^2,$$

which is a contradiction because $\|\varphi\|_{L^2}^2 > 0$. Then, we deduce that $I_\omega(\varphi) = 0$. In addition, by the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm, we have

$$d(\omega) \leq \frac{1}{2}\|\varphi\|_{L^2}^2 \leq \frac{1}{2}\liminf_{n\to\infty} \|v_n\|_{L^2}^2 = d(\omega), \tag{2.8}$$

which implies, by the definition of $d(\omega)$, that $\varphi \in \mathcal{N}_\omega$. This proves the first part of the statement of Proposition 1.4.

By direct computations, we see that the Gausson (1.5) satisfies $I_\omega(\phi_\omega) = 0$ and $S_\omega(\phi_\omega) = \pi^{N/2}e^{\omega+N}/2$. Thus, by Lemma 2.2 we have that $d(\omega) = \pi^{N/2}e^{\omega+N}/2$ and

$$\{e^{i\theta}\phi_\omega(\cdot - y); \theta \in \mathbb{R},\, y \in \mathbb{R}^N\} \subseteq \mathcal{N}_\omega.$$

Next, let $\varphi \in \mathcal{N}_\omega$. Then, by definition of $d(\omega)$, $\|\varphi\|_{L^2}^2 = 2d(\omega) = \pi^{N/2}e^{\omega+N}$ and $I_\omega(\varphi) = 0$. This implies that φ satisfies the equality in (2.1) with $\alpha = \sqrt{\pi}$. Indeed, suppose that we have the strict inequality in (2.1) with $\alpha = \sqrt{\pi}$. Since φ satisfies $I_\omega(\varphi) = 0$, it is easy to show that in this case $\|\varphi\|_{L^2}^2 > \pi^{N/2}e^{\omega+N}$, it which is impossible. Therefore, from Lemma 2.1 we infer that there exist $r > 0$, $\theta_0 \in \mathbb{R}$ and $y \in \mathbb{R}^N$ such that

$$\varphi(x) = r\, e^{i\theta_0} e^{-\frac{1}{2}|x-y|^2}.$$

Elementary calculations show that $r^2 = e^{\omega+N}$. Thus, we have $\varphi(x) = e^{i\theta_0}\,\phi_\omega(x-y)$ and Proposition 1.4 is proved. □

3. Stability of ground state

The proof of Theorem 1.5 relies on the following compactness result.

Lemma 3.1. *Let $\{u_n\} \subseteq W(\mathbb{R}^N)$ be a minimizing sequence for $d(\omega)$. Then there exist a family $(y_n) \subset \mathbb{R}^N$ and a function $\varphi \in \mathcal{N}_\omega$ such that, possibly for a subsequence only,*

$$u_n(\cdot - y_n) \to \varphi \quad \text{strongly in } W(\mathbb{R}^N).$$

Proof. By Proposition 1.4, we see that exist $(y_n) \subset \mathbb{R}^N$ and a function $\varphi \in \mathcal{N}_\omega$ such that, up to a subsequence, $u_n(\cdot - y_n) \rightharpoonup \varphi$ weakly in $W(\mathbb{R}^N)$. Let $v_n := u_n(\cdot - y_n)$. From (2.6), we infer that $v_n \to \varphi$ in $L^2(\mathbb{R}^N)$. Then, since the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, from (2.4) we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} B(|v_n(x)|)dx = \int_{\mathbb{R}^N} B(|\varphi(x)|)dx,$$

which combined with $I_\omega(v_n) = I_\omega(\varphi) = 0$ for any $n \in \mathbb{N}$, gives

$$\lim_{n\to\infty}\Big[\int_{\mathbb{R}^N} |\nabla v_n|^2dx + \int_{\mathbb{R}^N} A(|v_n(x)|)dx\Big] = \int_{\mathbb{R}^N} |\nabla\varphi|^2dx + \int_{\mathbb{R}^N} A(|\varphi(x)|)dx. \tag{3.1}$$

Moreover, by (3.1), the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm and Fatou lemma, we deduce (see e.g. [12, Lemma 12 in chapter V])

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2dx=\int_{\mathbb{R}^N}|\nabla\varphi|^2dx, \tag{3.2}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}A(|v_n(x)|)dx=\int_{\mathbb{R}^N}A(|\varphi(x)|)dx. \tag{3.3}$$

Since $v_n \rightharpoonup \varphi$ weakly in $H^1(\mathbb{R}^N)$, it follows from (3.2) that $v_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Finally, by Proposition 4.1-ii) in Appendix and (3.3) we have $v_n \to \varphi$ in $L^A(\mathbb{R}^N)$. Thus, by definition of the $W(\mathbb{R}^N)$-norm, we infer that $v_n \to \varphi$ in $W(\mathbb{R}^N)$. Which completes the proof. □

Proof of Theorem 1.5. The result is proved by contradiction. Assume that there exist $\epsilon > 0$ and two sequences $\{u_{n,0}\} \subset W(\mathbb{R}^N)$, $\{t_n\} \subset \mathbb{R}$ such that

$$\|u_{n,0}-\phi_\omega\|_{W(\mathbb{R}^N)}\to 0, \quad \text{as } n\to\infty, \tag{3.4}$$

$$\inf_{\theta\in\mathbb{R}}\inf_{y\in\mathbb{R}^N}\|u(t_n)-e^{i\theta}\phi_\omega(\cdot-y)\|_{W(\mathbb{R}^N)}\geq\epsilon, \quad \text{for any } n\in\mathbb{N}, \tag{3.5}$$

where u_n is the solution of (1.1) with initial data $u_{n,0}$. Set $v_n(x) = u_n(x, t_n)$. By (3.4) and conservation laws, we obtain

$$\|v_n\|_{L^2}^2=\|u_n(t_n)\|_{L^2}^2=\|u_{n,0}\|_{L^2}^2\to\|\phi_\omega\|_{L^2}^2 \tag{3.6}$$

$$E(v_n)=E(u_n(t_n))=E(u_{n,0})\to E(\phi_\omega), \tag{3.7}$$

as $n \to \infty$. In particular, it follows from (3.6) and (3.7) that, as $n \to \infty$,

$$S_\omega(v_n)\to S_\omega(\phi_\omega)=d(\omega). \tag{3.8}$$

Moreover, by combining (3.6) and (3.8) lead us to $I_\omega(v_n) \to 0$ as $n \to \infty$. Next, define the sequence $f_n(x) = \rho_n v_n(x)$ with

$$\rho_n=\exp\Big(\frac{I_\omega(v_n)}{2\|v_n\|_{L^2}^2}\Big),$$

where $\exp(x)$ represent the exponential function. It is clear that $\lim_{n\to\infty}\rho_n = 1$ and $I_\omega(f_n) = 0$ for any $n \in \mathbb{N}$. Furthermore, since the sequence $\{v_n\}$ is bounded in $W(\mathbb{R}^N)$, we obtain $\|v_n - f_n\|_{W(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Then, by (3.8), we have that $\{f_n\}$ is a minimizing sequence for $d(\omega)$. Thus, by Lemma 3.1, up to a subsequence, there exist $(y_n) \subset \mathbb{R}^N$ and a function $\varphi \in \mathcal{N}_\omega$ such that

$$\|f_n(\cdot-y_n)-\varphi\|_{W(\mathbb{R}^N)}\to 0 \quad \text{as } n\to+\infty. \tag{3.9}$$

Now, by Proposition 1.4, there exist $\theta_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^N$ such that $\varphi(x) = e^{i\theta_0}\phi_\omega(x-y_0)$. Remembering that $v_n = u_n(t_n)$ and using (3.9), we obtain

$$\|u_n(t_n)-e^{i\theta_0}\phi_\omega(\cdot-(y_0-y_n))\|_{W(\mathbb{R}^N)}\to 0 \quad \text{as } n\to+\infty$$

which contradicts (3.5). This completes the proof. □

4. Appendix

The purpose of this Appendix is to describe the structure of space $W(\mathbb{R}^N)$. We need to introduce some notation. Define

$$F(z) = |z|^2 \log |z|^2 \quad \text{for every } z \in \mathbb{C},$$

and as in [6], we define the functions A, B on $[0,\infty)$ by

$$A(s) = \begin{cases} -s^2 \log(s^2), & \text{if } 0 \le s \le e^{-3}; \\ 3s^2 + 4e^{-3}s - e^{-6}, & \text{if } s \ge e^{-3}; \end{cases} \qquad B(s) = F(s) + A(s). \tag{4.1}$$

Note that A is a nonnegative convex and increasing function, and $A \in C^1([0,+\infty)) \cap C^2((0,+\infty))$. The Orlicz space $L^A(\mathbb{R}^N)$ corresponding to A is defined by

$$L^A(\mathbb{R}^N) = \{u \in L^1_{\rm loc}(\mathbb{R}^N) : A(|u|) \in L^1(\mathbb{R}^N)\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^A} = \inf\{k > 0 : \int_{\mathbb{R}^N} A(k^{-1}|u(x)|)dx \le 1\}.$$

Here as usual $L^1_{\rm loc}(\mathbb{R}^N)$ is the space of all locally Lebesgue integrable functions. It is proved in [6, Lemma 2.1] that A is a Young-function which is Δ_2-regular and $(L^A(\mathbb{R}^N), \|\cdot\|_{L^A})$ is a separable reflexive Banach space.

Next, we consider the reflexive Banach space $W(\mathbb{R}^N) = H^1(\mathbb{R}^N) \cap L^A(\mathbb{R}^N)$ equipped with the usual norm $\|u\|_{W(\mathbb{R}^N)} = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{L^A}$ (see (1.3)). It is easy to see that (see [6, Proposition 2.2] for more details)

$$W(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^N)\}.$$

Furthermore, it is known that the dual space (see [7, Proposition 1.1.3])

$$W'(\mathbb{R}^N) = H^{-1}(\mathbb{R}^N) + L^{A'}(\mathbb{R}^N),$$

where the Banach space $W'(\mathbb{R}^N)$ is equipped with its usual norm. Here, $L^{A'}(\mathbb{R}^N)$ is the dual space of $L^A(\mathbb{R}^N)$ (see [6]).

Now we list some properties of the Orlicz space $L^A(\mathbb{R}^N)$ that we have used above. For a proof of such statements we refer to [6, Lemma 2.1].

Proposition 4.1. *Let $\{u_m\}$ be a sequence in $L^A(\mathbb{R}^N)$, the following facts hold:*

(i) *If $u_m \to u$ in $L^A(\mathbb{R}^N)$, then $A(|u_m|) \to A(|u|)$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$.*

(ii) *Let $u \in L^A(\mathbb{R}^N)$. If $u_m \to u$ a.e. in $\mathbb{R}^N$ and if*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} A(|u_m(x)|)dx = \int_{\mathbb{R}^N} A(|u(x)|)dx,$$

then $u_m \to u$ in $L^A(\mathbb{R}^N)$ as $n \to \infty$.

(iii) *For any $u \in L^A(\mathbb{R}^N)$, we have*

$$\min\{\|u\|_{L^A}, \|u\|^2_{L^A}\} \le \int_{\mathbb{R}^N} A(|u(x)|)dx \; leq \max\{\|u\|_{L^A}, \|u\|^2_{L^A}\}. \tag{4.2}$$

We conclude this Appendix with Brézis-Lieb's lemma: see [5, Theorem 2 and Example (b)]

Lemma 4.2. *Suppose that j is a continuous, convex function from $\mathbb{C}$ to $\mathbb{R}$ with $j(0) = 0$ and let $f_n = f + g_n$ be a sequence of measurable functions from $\mathbb{R}^N$ to $\mathbb{C}$ such that:*

(i) $g_n \to 0$ *a.e. in* $\mathbb{R}^N$.
(ii) $j(Mf)$ *is in* $L^1(\mathbb{R}^N)$ *for every real* M.
(iii) *There exists some fixed* $k > 1$ *such that* $\{j(kg_n) - kj(g_n)\}$ *is uniformly bounded in* $L^1(\mathbb{R}^N)$.

Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |j(f + g_n) - j(g_n) - j(f)|dx = 0.$$

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