Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 336, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF POSITIVE SYMMETRIC SOLUTIONS FOR AN INTEGRAL BOUNDARY-VALUE PROBLEM WITH ϕ-LAPLACIAN OPERATOR

YONGHONG DING

Abstract. In this article, we show the existence of three positive symmetric solutions for the integral boundary-value problem with ϕ-Laplacian

$$
\begin{gathered}
(\phi(u'(t)))' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\
u(0) = u(1) = \int_0^1 u(r)g(r)\,\mathrm{d}r,
\end{gathered}
$$

where ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Our main tool is a fixed point theorem due to Avery and Peterson. An example shows an applications of the obtained results.

1. Introduction

The aim of this article is to show the existence of positive symmetric solutions for the integral boundary-value problem

$$
\begin{gathered}
(\phi(u'(t)))' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\
u(0) = u(1) = \int_0^1 u(r)g(r)\,\mathrm{d}r,
\end{gathered} \tag{1.1}
$$

where ϕ, f, g satisfy the following assumptins:

(H1) ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and there exist two increasing homeomorphism ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$
\psi_1(u)\phi(v) \le \phi(uv) \le \psi_2(u)\phi(v) \quad \forall u, v > 0.
$$

Moreover, $\phi, \phi^{-1} \in C^1(\mathbb{R})$, where ϕ^{-1} denotes the inverse of ϕ.

(H2) $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to (0,+\infty)$ is continuous, and

$$
f(t,u,v) = f(1-t,u,-v), \quad (t,u,v) \in [0,1] \times [0,+\infty) \times (-\infty,+\infty).
$$

(H3) $g \in L^1[0,1]$ is nonnegative, and $0 < \int_0^1 g(t)\,\mathrm{d}t < 1$, $g(t) = g(1-t)$, $t \in [0,1]$.

Condition (H1) was first introduced by Wang [17, 18], and it includes two important cases when $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$, $p > 1$. Many authors have studied the positive solutions for two-point and multi-point boundary-value problems when $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$, $p > 1$, see [2, 3, 6, 7, 8, 10, 11, 12, 13, 15, 16, 19, 20]

2010 *Mathematics Subject Classification.* 34B15, 34B18.
Key words and phrases. ϕ-Laplacian; fixed point; cone; positive symmetric solutions.

Submitted May 18, 2016. Published December 28, 2016.

and references therein. In 1997, Wang [19] proved the existence of at least one positive for the equation

$$(\phi_p(u'(t)))' + a(t)f(u(t)) = 0, t \in (0,\ 1), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. The technique used there is the well-known Krasnoselskii's fixed point theorem. In [12], by using the upper and lower solutions method associated with Leray-Schauder degree theory, Kim showed the existence of three-solutions to the boundary-value problem

$$\begin{gathered} (w(t)\phi_p(u'(t)))' + \lambda f(t, u(t), u'(t)) = 0, \quad t \in (0,\ 1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$.

Boundary-value problem with integral boundary conditions constitute a very interesting and important class of problems. They include two, three and multipoint boundary value problems as special cases, hence they have been considerably developed, and the numerous properties of their solutions have been studied, see [4, 9, 14, 21, 22] and references therein. The main tools in these papers are the a priori estimate method and the fixed point theorems. However, there are few papers dealing with the existence of positive solutions for integral boundary-value problem with p-Laplacian operator, especially, when ϕ satisfies (H1) and f depends on both u and u'. The purpose of this paper is to investigate the existence of positive symmetric solutions for the integral boundary-value problem (1.1). By means of a fixed point theorem due to Avery and Peterson, sufficient conditions are obtained that guarantee the existence of at least three positive symmetric solutions for (1.1).

By a positive symmetric solution of (1.1), we mean a solution u of (1.1) satisfying $u(t) > 0, t \in (0, 1)$ and $u(t) = u(1-t)$ for $t \in [0, 1]$.

For the convenience of the reader, we provided some background material from the theory of cones in Banach spaces. We also state in this section the Avery-Peterson fixed point theorem.

Definition 1.1. A map α is said to be nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha : K \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone K of a real Banach space E provided that $\gamma : K \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Let γ and θ be a nonnegative continuous convex functionals on a cone K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real number a, b, c and d, we define the following convex sets:

$$\begin{gathered} P(\gamma, d) = \{u \in K : \gamma(u) < d\}, \\ P(\gamma, \alpha, b, d) = \{u \in K : \alpha(u) \geq b, \gamma(u) \leq d\}, \\ P(\gamma, \theta, \alpha, b, c, d) = \{u \in K : \alpha(u) \geq b, \theta(u) \leq c, \gamma(u) \leq d\}, \end{gathered}$$

$$R(\gamma, \psi, a, d) = \{u \in K | \psi(u) \geq a, \gamma(u) \leq d\}.$$

The following theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Theorem 1.2 ([3]). *Let K be a cone in a real Banach space E. Let γ and θ be a nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for positive number M and d,*

$$\alpha(u) \leq \psi(u), \quad \|u\| \leq M\gamma(u) \tag{1.4}$$

for all $u \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive number a, b and c with $a < b$ such that

1. *$\{u \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma, \theta, \alpha, b, c, d)$;*
2. *$\alpha(Tu) > b$ for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$;*
3. *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tu) < a$ for $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$, such that $\gamma(u_i) \leq d$ for $i = 1, 2, 3$, $\alpha(u_1) > b$, $\psi(u_2) > a$ with $\alpha(u_2) < b$, $\psi(u_3) < a$.

The organization of this paper is as follows. In Section 2, some lemmas will be established In Section 3, the main results of problem (1.1) will be stated and proved. An example is also given to show our main results.

2. Preliminaries

The basic space used in this paper is a real Banach space $C^1[0, 1]$ endowed the norm $\|\cdot\|_1$ defined by $\|u\|_1 = \max\{\|u\|_c, \|u'\|_c\}$, where $\|u\|_c = \max_{0 \leq t \leq 1} |u(t)|$. Let

$$K = \Big\{u \in C^1[0, 1] : u(t) \geq 0, u(0) = u(1) = \int_0^1 u(t)g(t)\,\mathrm{d}t,$$
$$u \text{ is concave and } u(t) = u(1 - t),\ t \in [0, 1]\Big\}.$$

It is obvious that K is a cone in $C^1[0, 1]$.

Lemma 2.1 ([15]). *Let $u \in K, \eta \in (0, 1/2)$, then $u(t) \geq \eta \max_{0 \leq t \leq 1} |u(t)|$, $t \in [\eta, 1 - \eta]$.*

Lemma 2.2. *Let $u \in K$, then there exists a constant $M > 0$ such that $\max_{0 \leq t \leq 1} |u(t)| \leq M \max_{0 \leq t \leq 1} |u'(t)|$.*

Proof. The Mean Value Theorem implies that there exists $\rho \in [0, 1]$, such that

$$u(1) = u(\rho) \int_0^1 g(t)\,\mathrm{d}t.$$

Furthermore, the Mean Value Theorem of differential implies that there exists $\sigma \in [\rho, 1]$, such that

$$\Big(\int_0^1 g(t)\,\mathrm{d}t - 1\Big) u(\rho) = u(1) - u(\rho) = (1 - \rho)u'(\sigma).$$

Therefore,

$$u(\rho) = \frac{(1 - \rho)u'(\sigma)}{1 - \int_0^1 g(t)\,\mathrm{d}t}.$$

So we obtain

$$
\begin{aligned}
|u(t)| &\le |u(\rho)| + |\int_\rho^t u'(s)\,\mathrm{d}s| \\
&\le \Big(\frac{1-\rho}{1-\int_0^1 g(t)\,\mathrm{d}t} + 1\Big) \max_{0\le t\le 1} |u'(t)| \\
&\le \frac{2-\int_0^1 g(t)\,\mathrm{d}t}{1-\int_0^1 g(t)\,\mathrm{d}t} \max_{0\le t\le 1} |u'(t)|.
\end{aligned}
$$

Denote $M = \frac{2-\int_0^1 g(t)\,\mathrm{d}t}{1-\int_0^1 g(t)\,\mathrm{d}t}$, then the proof is complete. $\square$

Now, by a similar argument as in [5], we define an operator $T : C^1[0,1] \to C^1[0,1]$ by

$$
Tx(t) = \begin{cases}
\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_0^1 g(r) \int_0^r \phi^{-1}\big(\int_s^{1/2} f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s\,\mathrm{d}r \\
+ \int_0^t \phi^{-1}\big(\int_s^{1/2} f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s, & 0 \le t \le \frac{1}{2}, \\
\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_0^1 g(r) \int_r^1 \phi^{-1}\big(\int_{\frac{1}{2}}^s f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s\,\mathrm{d}r \\
+ \int_t^1 \phi^{-1}\big(\int_{\frac{1}{2}}^s f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s, & \frac{1}{2} \le t \le 1.
\end{cases} \tag{2.1}
$$

Lemma 2.3. *$T : K \to K$ is completely continuous.*

Proof. Let $u \in K$. It follows from the definition of T that

$$
(Tu)'(t) = \begin{cases}
\phi^{-1}\big(\int_t^{1/2} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\big) \ge 0, & 0 \le t \le \frac{1}{2}, \\
-\phi^{-1}\big(\int_{\frac{1}{2}}^t f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\big) \le 0, & \frac{1}{2} \le t \le 1.
\end{cases} \tag{2.2}
$$

From (2.1), (2.2), we have $(Tu)'(0) \ge 0$ and $(Tu)'(t)$ is positive on $[0, 1/2]$. Moreover, $(Tu)'(t)$ is monotone decreasing continuous and $(Tu)'(\frac{1}{2}) = 0$. It implies that $(Tu)(t)$ is nonnegative and concave on $[0,1]$. By computation, we have $Tu(0) = Tu(1) = \int_0^1 Tu(t)g(t)\,\mathrm{d}t$. Now, we show that Tu is symmetric about t on $[0,1]$.

In fact, for all $t \in [0, 1/2]$, we note that $(1-t) \in [1/2, 1]$. So, by H(2), H(3), we have

$$
\begin{aligned}
(Tu)(1-t) &= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_0^1 g(r) \int_r^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_{1-t}^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= -\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_1^0 g(1-r) \int_{1-r}^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_{1-t}^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_0^1 g(r) \int_{1-r}^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_{1-t}^1 \phi^{-1}\Big(\int_{\frac{1}{2}}^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{\frac12}^{1-s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{1-t}^1 \phi^{-1}\Big(\int_{\frac12}^{s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(1-\tau,u(1-\tau),u'(1-\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{1-t}^1 \phi^{-1}\Big(\int_{\frac12}^{s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(1-\tau,u(\tau),-u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{1-t}^1 \phi^{-1}\Big(\int_{\frac12}^{s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(1-\tau,u(\tau),-u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{0}^t \phi^{-1}\Big(\int_{\frac12}^{1-s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{0}^t \phi^{-1}\Big(\int_{\frac12}^{1-s} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{0}^t \phi^{-1}\Big(\int_{s}^{1/2} f(1-\tau,u(1-\tau),u'(1-\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_{s}^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_{0}^t \phi^{-1}\Big(\int_{s}^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= (Tu)(t).
\end{aligned}
$$

Using the same method, we may prove that $Tu(t) = Tu(1-t)$ for $t \in [1/2, 1]$. Thus, $T(K) \subset K$. Next, we prove T is compact.

Let D be a bounded subset of K and $m > 0$ is a constant such that $\int_0^1 f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau < m$ for $u \in D$. We know from (2.1) and (2.2) that for any $u \in D$,

$$
|Tu(t)| < \begin{cases} \frac{\phi^{-1}(m)}{1-\int_0^1 g(r)\,\mathrm{d}r}, & 0 \le t \le \frac12,\\ \frac{\phi^{-1}(m)}{1-\int_0^1 g(r)\,\mathrm{d}r}, & \frac12 \le t \le 1, \end{cases}
$$

$$
|(Tu)'(t)| < \phi^{-1}(m), 0 \le t \le 1.
$$

Hence, TD is uniformly bounded and equicontinuous. By Arzela-Ascoli theorem, TD is compact on $C[0,1]$. From (2.2), we know that for all $\varepsilon>0$ there exists $\kappa>0$, such that when $|t_1-t_2|<\kappa$, we have $|\phi(Tu)'(t_1)-\phi(Tu)'(t_2)|<\varepsilon$. So $\phi(TD)'$ is compact on $C[0,1]$, it follows that $(TD)'$ is compact on $C[0,1]$. Therefore, TD is compact on $C^1[0,1]$.

Thus, $T:K\to K$ is completely continuous. □

It is easy to verify that each fixed point of T is a solution for problem (1.1).

Lemma 2.4 ([17]). *Assume that* (H1) *holds. Then for* $u,v\in(0,\infty)$,

$$\psi_2^{-1}(u)v\le\phi^{-1}(u\phi(v))\le\psi_1^{-1}(u)v.$$

3. Existence of three positive symmetric solutions

For convenience, we introduce the nonation

$$L=\frac{\int_0^1\psi_1^{-1}(1-s)\,\mathrm{d}s}{1-\int_0^1 g(s)\,\mathrm{d}s},\quad N=\int_\eta^{1/2}\psi_2^{-1}(\frac{1}{2}-s)\,\mathrm{d}s.$$

In this section, we impose growth conditions on f which allow us to apply Theorem 1.2 to establish the existence of three positive symmetric solutions of problem (1.1).

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ,θ, and nonnegative continuous functional ψ be defined on cone K by

$$\gamma(u)=\max_{0\le t\le 1}|u'(t)|,\quad \psi(u)=\theta(u)=\max_{0\le t\le 1}|u(t)|,\quad \alpha(u)=\min_{\eta\le t\le 1-\eta}|u(t)|. \tag{3.1}$$

Lemmas 2.1 and 2.2 imply that the functionals defined above satisfy

$$\eta\theta(u)\le\alpha(u)\le\psi(u)=\theta(u),\quad \|u\|_1=\max\{\gamma(u),\theta(u)\}\le M\gamma(u), \tag{3.2}$$

for all $u\in K$. Therefore, the condition (1.4) of Theorem 1.2 is satisfied.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. Let*

$$0<a<b\le\frac{d\eta}{1+\frac{1-\int_0^1 g(t)\,\mathrm{d}t}{\int_0^1 g(t)t(1-t)\,\mathrm{d}t}}.$$

If f satisfies the following conditions:

(H6) $f(t,u,v)\le\phi(d)$ *for* $(t,u,v)\in[0,1]\times[0,Md]\times[-d,d]$;

(H7) $f(t,u,v)>\phi(\frac{b}{\eta N})$ *for* $(t,u,v)\in[\eta,1-\eta]\times[b,\frac{b}{\eta}(1+\frac{1-\int_0^1 g(t)\,\mathrm{d}t}{\int_0^1 g(t)t(1-t)\,\mathrm{d}t})]\times[-d,d]$;

(H8) $f(t,u,v)<\phi(\frac{a}{L})$ *for* $(t,u,v)\in[0,1]\times[0,a]\times[-d,d]$.

Then problem (1.1) *has at least three positive symmetric solutions* u_1,u_2*, and* u_3 *satisfying*

$$\max_{0\le t\le 1}|u_i'(t)|\le d\quad \text{for } i=1,2,3,\ \min_{\eta\le t\le 1-\eta}|u_1(t)|>b,$$
$$\max_{0\le t\le 1}|u_2(t)|>a\quad \text{with}\ \min_{\eta\le t\le 1-\eta}|u_2(t)|<b,\ \max_{0\le t\le 1}|u_3(t)|<a.$$

Proof. We shall show that all the conditions of Theorem 1.2 are satisfied. If $u\in\overline{P(\gamma,d)}$, then $\gamma(u)=\max_{0\le t\le 1}|u'(t)|\le d$. Lemma 2.2 implies that $\max_{0\le t\le 1}|u(t)|\le Md$, so by (H6), we have $f(t,u(t),u'(t))\le\phi(d)$ when $0\le t\le 1$. Thus

$$\gamma(Tu)=\max_{0\le t\le 1}|(Tu)'(t)|$$

$$
\begin{aligned}
&= \max\Big\{\phi^{-1}\Big(\int_t^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big), \phi^{-1}\Big(\int_{\frac12}^{t} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\Big\}\\
&\le \phi^{-1}\Big(\int_0^1 f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\\
&\le \phi^{-1}(\phi(d)) = d
\end{aligned}
$$

This proves that $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$.

To check condition (1) of Theorem 1.2, we choose

$$
u_0(t)=\frac{b}{\eta}+\frac{b(1-\int_0^1 g(t)\,\mathrm{d}t)}{\eta\int_0^1 g(t)t(1-t)\,\mathrm{d}t}t(1-t), 0\le t\le 1.
$$

Let

$$
c=\frac{b}{\eta}\Big(1+\frac{1-\int_0^1 g(t)\,\mathrm{d}t}{\int_0^1 g(t)t(1-t)\,\mathrm{d}t}\Big).
$$

Then $u_0(t)\in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(u_0)>b$, so $\{u\in P(\gamma,\theta,\alpha,b,c,d)|\alpha(u)>b\}\neq\emptyset$. Therefore, for $u\in P(\gamma,\theta,\alpha,b,c,d)$, we have

$$
b\le u(t)\le c, |u'(t)|\le d,\quad \eta\le t\le 1-\eta.
$$

Thus, assumption (H7) implies that

$$
f(t,u(t),u'(t))>\phi(\frac{b}{\eta N})t\in[\eta,1-\eta]. \tag{3.3}
$$

This inequality and Lemmas 2.1, and 2.4 imply

$$
\begin{aligned}
\alpha(Tu)&=\min_{\eta\le t\le 1-\eta}|(Tu)(t)|\ge \eta\max_{0\le t\le 1}|(Tu)(t)|\\
&=\frac{\eta}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r\phi^{-1}\Big(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\eta\int_0^{1/2}\phi^{-1}\Big(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&>\eta\int_0^{1/2}\phi^{-1}\Big(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&>\eta\int_\eta^{1/2}\phi^{-1}\big(\phi(\frac{b}{\eta N})(\frac12-s)\big)\,\mathrm{d}s\\
&\ge\frac{b}{N}\int_\eta^{1/2}\psi_2^{-1}(\frac12-s)\,\mathrm{d}s=b.
\end{aligned}
$$

This shows that condition (1) of Theorem 1.2 is satisfied.

Secondly, for $u\in P(\gamma,\alpha,b,d)$ with $\theta(Tu)>c$, we have

$$
\alpha(Tu)\ge\eta\theta(Tu)\ge\eta c>b.
$$

Thus condition (2) of Theorem 1.2 holds.

Finally, as $\psi(0)=0<a$, there holds $0\notin R(\gamma,\psi,a,d)$. Suppose that $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$, then by the assumption (H8),

$$
f(t,u(t),u'(t))<\phi(\frac{a}{L}),\quad t\in[0,1]. \tag{3.4}
$$

This inequality and Lemma 2.4 imply

$$\begin{aligned}
\psi(Tu) &= \max_{0\le t\le 1} |(Tu)(t)| \\
&< \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^1 \phi^{-1}\Big(\int_s^1 f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^1 \phi^{-1}\Big(int_s^1 f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&< \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 \phi^{-1}\big((1-s)\phi(\frac{a}{L})\big)\,\mathrm{d}s \\
&\le \frac{a}{L}\frac{\int_0^1 \psi_1^{-1}(1-s)\,\mathrm{d}s}{1-\int_0^1 g(s)\,\mathrm{d}s} = a.
\end{aligned}$$

Hence condition (3) of Theorem 1.2 is also satisfied.

Therefore, (1.1) has at least three positive symmetric solutions u_1, u_2, and u_3 satisfying

$$\begin{gathered}
\max_{0\le t\le 1} |u_i'(t)| \le d \quad \text{for } i=1,2,3, \min_{\eta\le t\le 1-\eta} |u_1(t)| > b, \\
\max_{0\le t\le 1} |u_2(t)| > a \quad \text{with } \min_{\eta\le t\le 1-\eta} |u_2(t)| < b,]; \max_{0\le t\le 1} |u_3(t)| < a.
\end{gathered}$$

□

Example 3.2. Let $\phi(u) = u^3$ and $g(t) = 1/2$. Consider the boundary-value problem

$$\begin{gathered}
(\phi(u'))' + f(t,u(t),u'(t)) = 0, \quad t\in[0,1], \\
u(0) = u(1) = \frac{1}{2}\int_0^1 u(t)\,\mathrm{d}t,
\end{gathered} \tag{3.5}$$

where

$$f(t,u,v) = \begin{cases} \frac{t(1-t)}{10^7} + 85000u^6 + \frac{1}{10^7}(\frac{v}{10^6})^2, & u\le 21; \\ \frac{t(1-t)}{10^7} + 85000\cdot 21^6 + \frac{1}{10^7}(\frac{v}{10^6})^2, & u > 21. \end{cases}$$

Let $\psi_1^{-1}(u) = \psi_2^{-1}(u) = u^3, u>0$. Choosing $a=1/100$, $b=1$, $\eta=1/3$, $d=10^6$, we have

$$L = \frac{3}{2}, \quad N = \frac{3}{4}\big(\frac{1}{6}\big)^{4/3}, \quad \phi\big(\frac{b}{\eta N}\big) = 82944, \quad \phi\big(\frac{a}{L}\big) = \frac{1}{3375000}.$$

So, $f(t,u,v)$ satisfies

$$\begin{gathered}
f(t,u,v) < \phi(d) = 10^{18}, \quad 0\le t\le 1,\ 0\le u\le 3\cdot 10^6,\ -10^6\le v\le 10^6, \\
f(t,u,v) > 82944, \quad \frac{1}{3}\le t\le \frac{2}{3},\ 1\le u\le 21,\ -10^6\le v\le 10^6, \\
f(t,u,v) < \frac{1}{3375000}, \quad 0\le t\le 1,\ 0\le u\le \frac{1}{100},\ -10^6\le v\le 10^6.
\end{gathered}$$

Therefore, by Theorem 3.1, boundary-value problem (3.5) has at least three positive symmetric solutions u_1, u_2, u_3 such that

$$\begin{gathered}
\max_{0\le t\le 1} |u_i'(t)| \le 10^6,\ i=1,2,3, \quad \min_{\frac14\le t\le\frac34} |u_1(t)| > 1, \\
\max_{0\le t\le 1} |u_2(t)| > \frac{1}{10} \quad \min_{\frac14\le t\le\frac34} |u_2(t)| < 1, \quad \max_{0\le t\le 1} |u_3(t)| < \frac{1}{10}.
\end{gathered}$$

References

[1] R. Agarwal, H. Lü, D. O'Regan; *Eigenvalue of boundary value problem for the one-dimensional p-Laplacian*, J. Math. Anal. Appl., 266 (2002), 383–400.

[2] R. Avery, J. Henderson; *Existence of three positive pseudo-symmetric solutions for a one-dimensional p-Laplacian*, J. Math. Anal. Appl., 277 (2003), 395–404.

[3] R. Avery, A. Peterson; *Three positive fixed points of nonlinear operator on ordered Banach spaces*, Comput. Math. Appl., 42 (2001), 313-332.

[4] A. Boucherif; *Second-order boundary value problem with integral boundary conditions*, Nonlinear Anal., 70 (2009), 364-371.

[5] Y. Ding; *Monotone iterative method for obtaining positive solutions of integral boundary value problems with ϕ-Laplacian operator*, Electronce J. Diff. Equ., 2012 (2012), 1–9.

[6] H. Feng, W. Ge, M. Jiang; *Multiple positive solutions for m-point boundary value problems with a one-dimensional p-Laplacian*, Nonlinear Anal., 68 (2008), 2269-2279.

[7] C. Gupta; *Impulsive A non-resonant multi-point boundary value problem for a p-Laplacian type operator*, Electronce J. Diff. Equ., 10 (2003), 143–152.

[8] V. Il'in, E. Moiseev; *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, J. Differential Equations., 23 (1987), 979-987.

[9] G. Infante; *Eigenvalue and positive solutions of ODEs involving integral boundary conditions*, Discrete Contin Dyn. Syst. (Suppl.) (2005), 436–442.

[10] D. Ji, W. Ge; *Multiple positive solutions for some p-Laplacian boundary value problems*, J. Appl. Math. Comput., 187 (2007), 1315-1325.

[11] D. Jiang, W. Gao; *Upper and lower solution method and a singular boundary value problem for the one-dimensional p-Laplacian*, J. Math. Anal. Appl., 252 (2000), 631–648.

[12] C. Kim; *The three-solutions theorem for p-Laplacian boundary value problems*, Nonlinear Anal., 75 (2012), 924-931.

[13] L. Kong, J. Wang; *Multiple positive solutions for the p-Laplacian*, Nonlinear Anal., 42 (2000), 1327-1333.

[14] L. Kong; *Second order singular boundary value problem with integral boundary conditions*, Nonlinear Anal., 72 (2010), 2628-2638.

[15] B. Liu; *Positive solutions of three-point boundary value problems for one-dimensional p-Laplacian with infinitely many singularities*, Appl. Math. Lett., 17 (2004), 655-661.

[16] D. O'Regan; *Some general existence principles and results for* $(\phi_p(u'))' = f(t, u(t), u'(t)) = 0, 0 < t < 1$, SIAM. J. Math. Appl. 24 (1993), 648–668.

[17] H. Wang; *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl., 281 (2003), 287-306.

[18] H. Wang; *On the structure of positive radial solutions for quasilinear equations in annular domain*, Adv. Differential Equations, 8 (2003), 111-128.

[19] J. Wang; *The existence of positive solutions for the one-dimensional p-Laplacian*, Proc. Amer. Math. Soc., 125 (1997), 2275-2283.

[20] Z. Wang, J. Zhang; *Positive solutions for one-dimensional p-Laplacian boundary value problems with dependence on the first order derivative*, J. Math. Anal. Appl., 314 (2006), 618-630.

[21] Z. Yang; *Existence and uniqueness of positive solutions for an integral boundary value problem*, Nonlinear Anal., 69 (2008), 3910-3918.

[22] Z. Yang; *Positive solutions of second-order integral boundary value problem*, J. Math. Anal. Appl., 321 (2006), 751-765.

Yonghong Ding

Department of Mathematics, Tianshui Normal University, Tianshui 741000, China

E-mail address: dyh198510@126.com