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SOLVING THE STOKES PROBLEM IN A DOMAIN WITH CORNERS BY THE MORTAR SPECTRAL ELEMENT METHOD

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Abstract. In this article, we implement the mortar spectral element method for the Stokes problem on a domain within corners. We consider the Strang and Fix algorithm, which permits to enlarge the discrete space of the velocity by the first singular function. The usefulness of this method is confirmed by the numerical results presented here.

1. Introduction

The solution of the Stokes equation in a domain of $\mathbb{R}^2$ with corners is divided into a regular part and a linear combination of singular functions [15, 16]. To take into account these singularities, we propose to decompose the domain. The domain decomposition method consists in dividing the domain of resolution into sub-domains of smaller sizes and simpler geometries. The emergence of parallel computing and the development of effective codes (Message Passing Interface) have motivated the use of the decomposition domain methods. For these methods the matching conditions at the interfaces are decisive for a good approximation of the solution. One of the most used method is the *Mortar Element Method* [8]. The advantage of this method is that it admits a weak matching condition type, which gives flexibility on the geometry and on the choice of the functional spaces.

The approximation of singular functions which appear on the decomposition of the solution was done for the first time by Babuška et al [3, 4] for the p-version of the finite element method. This work was adapted by Bernardi et al [6] for the spectral method. They proved that the approximation order of these singular functions by polynomials is better than that given by the general theory of the spectral approximation [7]. We use these approximation results to implement the Strang and Fix [20] algorithm in the case of the Stokes problem. This algorithm consists in enlarging the discrete space of the velocity by the first singular function. We are interested in this work by the implementation of the mortar spectral element method for the Stokes problem in a domain with corners. We first deduce the matrix system which is solved using Uzawa algorithm [13]. Next, we present some numerical results showing the interest of using the Strand and Fix algorithm for this type of problems.

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An outline of this paper is as follows. In section 2, we present the geometry of the domain and the continuous problem, we give the singular functions and some regularity results. In section 3, we present the discrete problem and the error result obtained from the discretization of the Stokes problem by the mortar spectral method. Section 4 is devoted to the implementation of the mortar spectral element method. We describe the matrix system and its resolution algorithm. Finally, we present some numerical results which confirm the interest of the method.

2. Continuous problem and singular functions

Let Ω be a polygonal domain of $\mathbb{R}^2$ simply connected and Γ be its connected boundary. The generic point in Ω is denoted by $\mathbf{x} = (x, y)$. We suppose that there exists a finite number of vertex Γ_j for $j \in \{1, \dots, J\}$, J is a positive integer such that

$$\Gamma = \cup_{j=1}^{J} \Gamma_j .$$

We consider $\mathbf{c}_j$ the corner of Ω made by Γ_j and Γ_{j+1}, and α_j its measure. Let $H^s(\Omega)$ be the Sobolev space of order $s \in \mathbb{R}$, $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions which vanish on the boundary Γ and the space

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0\}.$$

We introduce the Stokes problem on the domain Ω in the velocity and pressure formulation:

For a forcing term $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, we try to find the velocity $\mathbf{u} \in [H_0^1(\Omega)]^2$ and the pressure $p \in L_0^2(\Omega)$ such that

$$\begin{gathered} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \\ \mathbf{u} = 0 \quad \text{on } \Gamma, \end{gathered} \tag{2.1}$$

where ν is the viscosity of the fluid that is supposed to be positive and constant. Problem (2.1) has the following variational formulation:

For $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, find $\mathbf{u}$ in $[H_0^1(\Omega)]^2$ and p in $L_0^2(\Omega)$ such that for all $\mathbf{v}$ in $[H_0^1(\Omega)]^2$ and for all q in $L^2(\Omega)$:

$$\begin{gathered} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle \\ b(\mathbf{u}, q) = 0, \end{gathered} \tag{2.2}$$

where

$$\begin{gathered} a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla \mathbf{u} \nabla \mathbf{v} d\mathbf{x}, \\ b(\mathbf{u}, q) = -\int_\Omega (\operatorname{div} \mathbf{u}) q d\mathbf{x}. \end{gathered}$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

We use the Cauchy-Schwarz and Poincaré inequalities to prove that the bilinear form $a(\cdot, \cdot)$ is continuous on the space $[H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2$, elliptic on $[H_0^1(\Omega)]^2$, and that the bilinear form $b(\cdot, \cdot)$ is continuous on the space $[H_0^1(\Omega)]^2 \times L^2(\Omega)$.

The bilinear form $b(\cdot,\cdot)$ satisfies the following Inf-Sup condition [9]: There exists a positive constant γ such that

$$\forall t \in L_0^2(\Omega), \qquad \sup_{\mathbf{v}\in[H_0^1(\Omega)]^2} \frac{b(\mathbf{v},t)}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}} \geq \gamma \|t\|_{L^2(\Omega)}.$$

Then we deduce that for all $\mathbf{f}$ in the space $[H^{-1}(\Omega)]^2$, the problem (2.2) has a unique solution $(\mathbf{u},p)$ in $[H_0^1(\Omega)]^2 \times L_0^2(\Omega)$, such that

$$\|\mathbf{u}\|_{[H^1(\Omega)]^2} + \gamma\|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{[H^{-1}(\Omega)]^2},$$

where C is a positive constant [14, 22].

The incompressibility condition $\operatorname{div}\mathbf{u} = 0$ on the connected domain Ω allows us to deduce that there exists a stream function ψ in the space $H_0^2(\Omega)$ such that [14, 18]

$$\mathbf{u} = \operatorname{curl}(\psi).$$

This allows us to deduce that the problem (2.2) is equivalent to the following problem: For $\mathbf{f} \in [L^2(\Omega)]^2$, find $\psi \in H_0^2(\Omega)$ such that

$$\begin{gathered} -\nu\Delta^2\psi = \operatorname{curl}(\mathbf{f}) \quad \text{in } \Omega \\ \psi = 0 \quad \text{on } \Gamma \\ \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma\,, \end{gathered}$$

where n is the outward normal unit vector to Ω on Γ.

Because of its fundamental importance in the study of singularities and the regularity of the solution ψ, we consider the characteristic equation of the bi-Laplacian operator [16, 19]:

$$\sin^2\alpha_j z = z^2 \sin^2\alpha_j. \tag{2.3}$$

Next, we consider a partition of the domain Ω in rectangles Ω_i, $1 \leq i \leq I$, satisfying

$$\bar{\Omega} = \cup_{i=1}^I \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j.$$

We suppose that the intersection of each $\bar{\Omega}_i$, $1 \leq i \leq I$, with the boundary Γ is either empty or a corner or one of several entire edges of Ω_i. The edge of the Ω_i are parallel to the coordinate axes. We choose a non-convex domain, and we assume that the non-convex angle α is equal to $\frac{3\pi}{2}$ or to 2π (case of the crack). The treatment of the singular function is processed locally, then we suppose that the non-convex corner is unique.

Assumption 1. Let $\mathbf{c}$ be the corresponding corner of α. We define $\sum$ the open domain in Ω such that $\bar{\sum}$ is the union of the $\bar{\Omega}_k$ which contains $\mathbf{c}$. We consider that the origin of the coordinate axes is at the point $\mathbf{c}$ and we introduce a system of polar coordinates (r,θ). For technical reasons, we lead to assume the following conforming property: if the intersection of $\bar{\Omega}_i$ and $\bar{\Omega}_j$, $i \neq j$, it contains either $\mathbf{c}$, or both an edge of Ω_i and Ω_j.

If the angle α is equal to $3\pi/2$, equation (2.3) has two roots in the band $0 < \operatorname{Re}(z) < 1$. When we approximate those roots by the Newton method we find $z_1 \simeq 0,544484$ and $z_2 \simeq 0,908529$.

Let V be a neighborhood of the corner $\mathbf{c}$, we introduce the functions

$$\eta_1(r,\theta) = r^{1+z_1}\beta_1(\theta), \quad \eta_2(r,\theta) = r^{1+z_2}\beta_2(\theta),$$

where

$$\begin{aligned}\beta_1(\theta) &= 2.093\big(\cos(0.459\theta) - \cos(1.544\theta)\big) + 1.093\big(2.193\sin(0.459\theta) \\ &\quad - 0.647\sin(1.544\theta)\big), \\ \beta_2(\theta) &= 4.302\big(\cos(0.092\theta(-\cos(1.908\theta)\big) - 1.815\big(10.869\sin(0.092\theta) \\ &\quad - 0.524\sin(1.908\theta)\big).\end{aligned}$$

If $\mathbf{f}$ belongs to $[L^2(\Omega)]^2$, the velocity $\mathbf{u}$ is decomposed as:

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s$$

such that $\mathbf{u}_r$ is in the space $H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\mathbf{u}_s = \lambda_1\kappa_1 + \lambda_2\kappa_2$$

where $\kappa_i(r,\theta) = \operatorname{curl}(\eta_i(r,\theta)) \in [H^{(1+z_i)-\epsilon}(V)]^2$, $\lambda_i \in \mathbb{R}$ are the two singularity coefficients, $i \in \{1,2\}$, for all $\epsilon > 0$.

In the case of the crack, for $\mathbf{f}$ in $[L^2(\Omega)]^2$, the velocity is written in the form

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s,$$

$\mathbf{u}_r$ is in $[H^2(\Omega) \cap H_0^1(\Omega)]^2$ and there exists two real constants λ_1 and λ_2 such that

$$\mathbf{u}_s = \lambda_1\kappa_1 + \lambda_2\kappa_2$$

with

$$\begin{gathered}\kappa_1(r,\theta) = r^{1/2}(3\sin\theta\sin\frac{\theta}{2}, 3(1-\cos\theta)\sin\frac{\theta}{2}), \\ \kappa_2(r,\theta) = r^{1/2}(2\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2}, (1-\cos\theta)\cos\frac{\theta}{2}),\end{gathered}$$

where κ_1 and κ_2 belong to the space $[H^{\frac{3}{2}-\epsilon}(V)]^2$, for all ϵ positive [11, 16].

3. Discrete problem and the error estimate

We start by recalling the basic results of the mortar spectral element method. Since the discretization is based on the Galerkin method, we have to define the discrete problem and give the quadrature formula which is used for the numerical integration.

We denote by $\delta = (N_1, N_2, \dots, N_I)$ the discretization parameter where $N_i \geq 2$, $1 \leq i \leq I$. Let $\mathbb{P}_N(\Omega_i)$ be the space of polynomials on Ω_i. The restriction of the discrete functions to Ω_i will belong to $\mathbb{P}_{N_i}(\Omega_i)$.

Let us recall the Gauss-Lobatto quadrature formula: there exists a unique set of nodes such that $\xi_0 = -1$, $\xi_N = 1$, $\xi_j^N \in]-1,1[$, $1 \leq j \leq (N-1)$, and a set of $(N+1)$ positive weights ρ_j^N, $0 \leq j \leq N$, such that for any polynomial φ_{2N-1} with degree less than or equal to $2N-1$,

$$\int_{-1}^{1} \varphi_{2N-1}(\varsigma)d\varsigma = \sum_{j=0}^{N} \varphi_{2N-1}(\xi_j^N)\rho_j^N. \tag{3.1}$$

Let F^i be the affine bijection from $]-1,1[^2$ into Ω_i. We consider the local discrete scalar product: For u, v continuous functions on $\bar{\Omega}_i$,

$$(u,v)_{N_i} = \frac{|\Omega_i|}{4} \sum_{j=0}^{N_i}\sum_{l=0}^{N_i} (uoF^i)(\xi_j^{N_i}, \xi_l^{N_i})(voF^i)(\xi_j^{N_i}, \xi_l^{N_i})\rho_j^{N_i}\rho_l^{N_i}, \tag{3.2}$$

where $|\Omega_i|$ is the measure of Ω_i.

We consider $\Gamma^{i,j}$, $1 \le i \le I$, $1 \le j \le 4$, the edges of Ω_i and we denote by $\gamma^{il} = \bar{\Omega}_i \cap \bar{\Omega}_l$, $i \neq l$, γ^{il} is not necessarily an entire edge $\Gamma^{i,j}$ since the decomposition is in general not conforming. We suppose that the boundary $\partial\Omega$ is composed of entire edges of the Ω_i.

We define the skeleton of the decomposition:

$$S = \cup_{k=1}^{K} \gamma_k \quad \text{and} \quad \gamma_k \cap \gamma_{k'} = \emptyset \text{ for } k \neq k',$$

where γ_k is called mortar. γ_k is assumed to be an entire edge of one rectangle Ω_i, denoted by $\Omega_{i(k)}$. For any nonnegative integer n and for any segment γ, $\mathbb{P}_n(\gamma)$ is the space of polynomials with degree less or equal to n on γ. We define W_δ the mortar functions space by:

$$W_\delta = \{\psi \in L^2(S) : \forall k, 1 \le k \le K, \psi/_{\gamma_k} \in \mathbb{P}_{N_{i(k)}}(\gamma_k)\}.$$

The discrete space X_δ represents the space of functions w_δ in $L^2(\Omega)$ such that we have the following properties: [8, Chap 3, §1]

- the restriction of w_δ to Ω_i, $1 \le i \le I$, belongs to $\mathbb{P}_{N_i}(\Omega_i)$;
- w_δ vanishes on $\partial\Omega$;
- the mortar function ϕ defined on S by

$$\phi/_{\gamma_k} = w_\delta/\Omega_{i(k)}/\gamma_k, \ 1 \le k \le K,$$

satisfies, for $1 \le i \le I$ and for any edge Γ of Ω_i contained in S: for all $\chi \in \mathbb{P}_{N_i-2}(\Gamma)$,

$$\int_\Gamma (w_\delta/_{\Omega_i} - \phi)(\tau)\chi(\tau)d\tau = 0. \tag{3.3}$$

Let $Y_\delta = X_\delta \times X_\delta$ be the discrete space of the discrete velocity. For the discrete pressure we consider the space:

$$M_\delta = \{p_\delta \in L^2(\Omega) : p_{\delta/\Omega_i} \in \mathbb{P}_{N_i-2}(\Omega_i) \text{ and } \int_\Omega p_\delta(\mathbf{x})d\mathbf{x} = 0\}.$$

The space M_δ corresponds to the case where the pressure has no spurious modes (see [5, 17]). We define the following norm on Y_δ:

$$\|\mathbf{w}_\delta\| = \Big(\sum_{i=1}^{I} \|\mathbf{w}_{\delta/\Omega_i}\|^2_{[H^1(\Omega_i)]^2}\Big)^{1/2}.$$

We are expanding the discrete space of the velocity Y_δ. Let $\kappa_1 = (\kappa_1^1, \kappa_1^2)$ be the first singular function of the velocity [12]. We denote by Y_δ^* the space

$$Y_\delta^* = Y_\delta + \mathbb{R}\kappa_1.$$

If $\mathbf{u}_\delta^*$ is in Y_δ^*, there exists $\mathbf{u}_\delta = (u_{\delta 1}, u_{\delta 2})$ in Y_δ and λ_1 in $\mathbb{R}$ such that

$$\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1\kappa_1.$$

Since κ_1 is in the space $[H^1(\Omega)]^2$, we consider the norm on the space Y_δ^* for all $\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1\kappa_1$ in Y_δ^*:

$$\|\mathbf{u}_\delta\|_* = \big(\|\mathbf{u}_\delta\|^2 + |\lambda_1|^2\|\kappa_1\|^2\big)^{1/2}.$$

Therefore, the discrete problem is defined as follows: for a continuous data function $\mathbf{f} = (f_1, f_2)$ on $\bar{\Omega}$, find $\mathbf{u}_\delta^* = (u_{\delta 1}^*, u_{\delta 2}^*) = (u_{\delta 1} + \lambda_1\kappa_1^1, u_{\delta 2} + \lambda_1\kappa_1^2)$ in Y_δ^* and p_δ in

M_δ such that for all $\mathbf{v}_\delta^* = (v_{\delta 1}^*, v_{\delta 2}^*) = (v_{\delta 1} + \mu_1 \kappa_1^1, v_{\delta 2} + \mu_1 \kappa_1^2)$ in Y_δ^* and for all q_δ in M_δ,

$$\begin{gathered} a_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*) + b_\delta^*(\mathbf{v}_\delta^*, p_\delta) = (\mathbf{f}, \mathbf{v}_\delta^*)_\delta \\ b_\delta^*(\mathbf{u}_\delta^*, q_\delta) = 0, \end{gathered} \tag{3.4}$$

such that

$$a_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*) = a_{1\delta}^*(u_{\delta 1}^*, v_{\delta 1}^*) + a_{2\delta}^*(u_{\delta 2}^*, v_{\delta 2}^*),$$

where the bilinear form $a_{k\delta}^*(\cdot,\cdot)$, $k \in \{1,2\}$ is defined by

$$\begin{aligned} a_{k\delta}^*(u_{\delta k}^*, v_{\delta k}^*) = \sum_{i=1}^{I} \Big(& (\nabla u_{\delta k}, \nabla v_{\delta k})_{N_i} + \lambda_1 \int_{\Omega_i} \nabla \kappa_1^k \nabla v_{\delta k} dx + \mu_1 \int_{\Omega_i} \nabla u_{\delta k} \nabla \kappa_1^k dx \\ & + \lambda_1 \mu_1 \int_{\Omega_i} (\nabla \kappa_1^k)^2 dx \Big). \end{aligned}$$

Since we know κ_1^k, $k \in \{1,2\}$, we can compute exactly the value of the integral $\int_{\Omega_i} (\nabla \kappa_1^k)^2 dx$. To approximate the integral $\int_{\Omega_i} \nabla \kappa_1^k \nabla v_{\delta k} d\mathbf{x}$, and

$$(\mathbf{f}, \mathbf{v}_\delta^*)_\delta = (f_1, v_{\delta 1}^*)_\delta + (f_2, v_{\delta 2}^*)_\delta$$

such that

$$(f_k, v_{\delta k}^*)_\delta = (f_k, v_{\delta k})_\delta + \mu_1 \int_\Omega f_k(x,y) \kappa_1^k(x,y) d\mathbf{x}$$

we use the algorithm presented in [1].

As $\operatorname{div} \kappa_1 = 0$, for $\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1 \kappa_1$ in Y_δ^* and q_δ in M_δ, we have

$$b_\delta^*(\mathbf{u}_\delta^*, q_\delta) = -\big(\sum_{i=1}^{I} (\operatorname{div} \mathbf{u}_\delta, q_\delta)_{N_i} + \lambda_1 \int_{\Omega_i} \operatorname{div} \kappa_1 q_\delta dx \big) = b_\delta(\mathbf{u}_\delta, q_\delta).$$

Let

$$V_\delta^* = \{ \mathbf{v}_\delta^* \in Y_\delta^*,\ b_\delta^*(\mathbf{v}_\delta^*, q_\delta) = 0,\ \forall q_\delta \in M_\delta \}$$

be the kernel of the bilinear form $b_\delta^*(\cdot,\cdot)$.

The bilinear form $a_\delta^*(\cdot,\cdot)$ is continuous on the space Y_δ^* by respecting the norm $\|.\|_*$. There exists a positive constant ν independent of δ such that for all $\mathbf{u}_\delta^*, \mathbf{v}_\delta^*$ in Y_δ^*

$$|a_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*)| \le \nu \|\mathbf{u}_\delta^*\|_* \|\mathbf{v}_\delta^*\|_*.$$

In order to prove that the problem (3.4) is well posed, we need to define the following norm on the space Y_δ^*:

$$\|\mathbf{v}_\delta^*\|_{**} = \Big(\sum_{i=1}^{I} \|\mathbf{v}_{\delta/\Omega_i}^*\|_{[H^1(\Omega_i)]^2}^2 \Big)^{1/2} \quad \text{for all } \mathbf{v}_\delta \in Y_\delta^*.$$

The bilinear form $a_\delta^*(\cdot,\cdot)$ is elliptic relative to the norm $\|\cdot\|_{**}$. There exists a positive constant β independent of δ such that for all $\mathbf{u}_\delta^*$ in Y_δ^*,

$$a_\delta^*(\mathbf{u}_\delta^*, \mathbf{u}_\delta^*) \ge \beta \|\mathbf{u}_\delta^*\|_{**}^2.$$

We have the following Inf-Sup condition on the bilinear form $b_\delta^*(\cdot,\cdot)$, which establishes the compatibility between the spaces Y_δ^* and M_δ: there exists a constant $\gamma_\delta = \inf(N_i^{-\frac{1}{2}}),\ 1 \le i \le I$ such that [5]

$$\forall t_\delta \in M_\delta, \quad \sup_{\mathbf{u}_\delta^* \in Y_\delta^*} \frac{b_\delta^*(\mathbf{u}_\delta^*, t_\delta)}{\|\mathbf{u}_\delta^*\|_*} \ge \gamma_\delta \|t_\delta\|_{L^2(\Omega)}.$$

Then we can conclude that problem (3.4) has a unique solution $(\mathbf{u}_\delta^*, p_\delta)$ in the space $Y_\delta^* \times M_\delta$ such that

$$\|\mathbf{u}_\delta^*\|_{**} + \gamma_\delta \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{[L^2(\Omega)]^2},$$

where C is a positive constant.

Since the equivalence constant for the two norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ depends on the discrete parameter δ, we prefer to keep the norm $\|\cdot\|_*$ and proving an Inf-Sup condition on the bilinear form $a_\delta^*(\cdot,\cdot)$. There exists a constant $\mu > 0$ independent of δ such that [10] for all $\mathbf{u}_\delta^* \in V_\delta^*$,

$$\sup_{\mathbf{v}_\delta^* \in V_\delta^*} \frac{a_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*)}{\|\mathbf{v}_\delta^*\|_*} \geq \mu \|\mathbf{u}_\delta^*\|_{L^2(\Omega)}. \tag{3.5}$$

From the Inf-Sup condition (3.5) on the bilinear form $a_\delta^*(\cdot,\cdot)$ and the Strang's lemma, we conclude the following error estimate

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_\delta^*\|_* \leq C\Big(&\inf_{v_\delta^* \in V_\delta^*} \Big\{ \|\mathbf{u} - \mathbf{v}_\delta^*\|_* + \sup_{\mathbf{w}_\delta^* \in \ Y_\delta^*} \frac{(a - a_\delta^*)(\mathbf{v}_\delta^*, \mathbf{w}_\delta^*)}{\|\mathbf{w}_\delta^*\|_*} \Big\} \\
&+ \sup_{\mathbf{w}_\delta^* \in \ Y_\delta^*} \frac{\int_{\Omega_i} \mathbf{f}\mathbf{w}_\delta^* \, dx\, dy - (f, \mathbf{w}_\delta^*)_\delta}{\|\mathbf{w}_\delta^*\|_*} \\
&+ \sup_{\mathbf{w}_\delta \in V_\delta^*} \frac{\sum_{i=1}^I \sum_{i=1}^I \int_{\gamma^{il}} (-\nu \frac{\partial \mathbf{u}}{\partial n} + pn)[\mathbf{w}_\delta^*]}{\|\mathbf{w}_\delta^*\|_*} \Big),
\end{aligned} \tag{3.6}$$

where $\mathbf{u}$ is the solution of (2.1), $\mathbf{u}_\delta^*$ is the solution of (3.4), $[\mathbf{w}_\delta^*]$ is the jump of $\mathbf{w}_\delta^*$ through the edge γ^{il} and C is a positive constant independent of δ.

If we decompose the solution as:

$$\mathbf{u} = \mathbf{u}_r + \lambda_1 \kappa_1 + \lambda_2 \kappa_2,$$

and we estimate each term of (3.6), the error between the continuous velocity $\mathbf{u}$ of problem (2.1) and the discrete velocity $\mathbf{u}_\delta^*$, of problem (3.4) is

$$\|\mathbf{u} - \mathbf{u}_\delta^*\|_* \leq C \sup \Big\{ \sum_{i=1}^I N_i^{-\sigma_i}, \sum_{i=1}^I N_i^{-\rho_i} \Big\} \|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}$$

where $\mathbf{f}$ belongs to the space $[H^{s-2}(\Omega)]^2$, $s > 0$, such that $\mathbf{f}/_{\Omega_i}$ in the space $[H^{\rho_i}(\Omega_i)]^2$, $\rho_i > 1$ and for all $\epsilon > 0$, σ_i is given by

$$\sigma_i = \begin{cases} s - 1 & \text{if } \bar{\Omega}_i \text{ contains no corner } \bar{\Omega}_i \\ \inf\{s-1, 2z_2(\frac{\pi}{2}) - \epsilon\} & \text{if } \bar{\Omega}_i \text{ contains corners different of } \mathbf{c} \\ \inf\{s-1, 2z_2(\alpha) - \epsilon\} & \text{if } \bar{\Omega}_i \text{ contains } \mathbf{c} \end{cases}$$

$z_2(\alpha)$ is the second real solution of the equation (2.3) in the band $0 < \mathrm{Re}(z) < 1$ for the angle α.

For a regular data $\mathbf{f}$ and $N = \inf_{1 \leq i \leq I} N_i$, then for all ϵ positive we remark that

- If $\alpha = \frac{3\pi}{2}$, then the order of convergence is $N^{\epsilon - 1,816}$;
- If $\alpha = 2\pi$, then the order of convergence is $N^{\epsilon - 1}$.

We conclude that we double the order of convergence when we compare with the case without the Strang and Fix algorithm [6, 12].

4. Implementation and numerical results

We present in this section the implementation of the Strang and Fix algorithm for the spectral element method of the Stokes problem. We consider just the first singular function on the velocity for $\alpha = \frac{3\pi}{2}$ and the case of the crack $\alpha = 2\pi$. To simplify the implementation we neglect the singularity of the pressure. We start by choosing the basis for the discrete spaces of the velocity and the pressure. Then, we write the matrix system corresponding to the discrete problem (3.4). Afterward, we describe the iterative method for solving the matrix system. Finally, we present some numerical results showing that the Strang and Fix algorithm is an efficient method. We notice that the computation are performed with MATLAB code.

4.1. **Choice of basis.** To define the matrix system of the discrete problem, we have to choose a basis of the discrete spaces $Y_\delta^* = X_\delta^* \times X_\delta^*$ and M_δ. These basis are naturally defined through a local basis (on each sub-domain) and determine the structure of the matrix system. The choice of the resolution method is inherent in this structure. Let φ_j^N the Lagrange interpolating polynomials defined on $[-1,1]$ of degree less or equal to N such that

$$\varphi_j^N \in \mathbb{P}_N([-1,1]), \quad \varphi_j^N(\xi_l) = \delta_{lj}, \quad 0 \le l, j \le N$$

where δ_{lj} is the Kronecker symbol. The polynomial φ_j^N is defined as

$$\varphi_j^N(\xi) = \frac{-1}{N(N+1)} \frac{(1-\xi^2)L_N^{'}(\xi)}{(\xi - \xi_j)} \quad \forall \xi \in [-1,1],$$

where L_N is the Legendre polynomials defined on $[-1,1]$ of degree less than or equal to N. We denote $\xi_l^i = F^i(\xi_l)$ and the polynomial $\varphi_l^{N_i}$ satisfies $\varphi_l^{N_i} \circ F^i = \varphi_l^N$, then for all v_δ in the space X_δ,

$$v_\delta(x,y)_{/\Omega_i} = \sum_{l=0}^{N_i} \sum_{j=0}^{N_i} v_{N_i}^{l,j} \varphi_l^{N_i}(x) \varphi_j^{N_i}(y)$$

where $v_{N_i}^{l,j} = v_\delta(\xi_l^i, \xi_j^i)_{/\Omega_i}$.

For each v_δ^* in X_δ^*, there exists (v_δ, λ_1) in $X_\delta \times \mathbb{R}$ such that $v_\delta^* = v_\delta + \lambda_1 \kappa_1^k$, $k \in \{1,2\}$ then

$$v_\delta^*(x,y)_{/\Omega_i} = \sum_{l=0}^{N_i} \sum_{j=0}^{N_i} v_{N_i}^{l,j} \varphi_l^{N_i}(x) \varphi_j^{N_i}(y) + \lambda_1 \kappa_{1/\Omega_i}^k. \tag{4.1}$$

Now, we have to choose a basis for the space of the discrete pressure M_δ. Let the polynomial $\bar{\varphi}_j^N$ in the space $\mathbb{P}_{N-2}(]-1,1[)$ defined by

$$\bar{\varphi}_j^N(x) = \frac{\varphi_j^N(x)}{(1-x^2)}.$$

We recall the following equalities that will be used later. For $0 \le l, j \le N$ we have

$$\varphi_l^{N'}(\xi_j) = \frac{L_N(\xi_j)}{L_N(\xi_l)(\xi_j - \xi_l)}, \quad \varphi_l^{N'}(\xi_l) = 0$$

$$\bar{\varphi}_l^N(\xi_0) = \frac{1}{2}\varphi_l'(\xi_0), \quad \bar{\varphi}_l^N(\xi_N) = \frac{1}{2}\varphi_l'(\xi_N).$$

The family of polynomials $\bar\varphi_l^N \bar\varphi_j^N$ for $1 \le l, j \le N-1$ is a basis of the space $\mathbb{P}_{N-2}(\Omega)$. Let $\bar\varphi_l^{N_i}$ the polynomials such that $\bar\varphi_l^{N_i} \circ F^i = \bar\varphi_l^N$, then for all q_δ belongs to M_δ

$$q_\delta(x,y)/_{\Omega_i} = \sum_{l=1}^{N_i-1} \sum_{j=1}^{N_i-1} q_{N_i}^{l\,j} \bar\varphi_l^{N_i}(x) \bar\varphi_j^{N_i}(y). \tag{4.2}$$

where $q_{N_i}^{l\,j} = q_\delta(\xi_l^i, \xi_j^i)(1-{\xi_l^i}^2)(1-{\xi_j^i}^2)$ (see [2]).

4.2. **Matrix system and algorithm of resolution.** In this section we formulate the discret problem (3.4) as a matrix system using the basis of the velocity and the pressure spaces.

For u_δ^* in the space Y_δ^* we have $u_\delta^* = u_\delta + \lambda_1 \kappa_1$, where λ_1 is the first singular coefficient related to the first singular function $\kappa_1 = (\kappa_1^1, \kappa_1^2)$. Although in the reality we have only one singular coefficient λ_1 in $\mathbb{R}$, we will suppose that we have two unknowns λ_1^1 and λ_1^2 to maintain the symmetry of the problem. The numerical results of λ_1^1 and λ_1^2 are approximatively equal to λ_1.

Having (4.1) and (4.2) we obtain the matrix system

$$\begin{gathered} AU + B^T P = F \\ BU = 0 \end{gathered} \tag{4.3}$$

where (U, P) are, respectively, the vector of admissible unknowns of the velocity and the vector of admissible unknowns of the pressure. The components of U are the values of the solution on the nodes of the sub-domains and the nodes on the corresponding boundaries. The components of P are the values of the pressure on the internal nodes of the sub-domains.

Construction of the matrix A. We note that the matrix A is written in the form

$$A = \begin{pmatrix} \tilde A_1 & 0 \\ 0 & \tilde A_2 \end{pmatrix}$$

where the matrices $\tilde A_k$, for $k \in \{1,2\}$ are symmetric matrices such that:

- the diagonal entries are $[\tilde A_k]_{i,i} = \Big(\nabla(\varphi_l^{N_i}\varphi_j^{N_i}); \nabla(\varphi_p^{N_i}\varphi_q^{N_i})\Big)_{N_i}$, $1 \le l, j \le N_i - 1$ and $1 \le p, q \le N_i - 1$, which represents the Laplace operator with the Neumann boundary condition on each sub-domain Ω_i, $1 \le i \le I$;
- the entries on the last row and last column are $[\tilde A_k]_{i,I+1} = [\tilde A_k]_{I+1,i} = \int_{\Omega_i} \nabla\kappa_1^k \nabla(\varphi_p^{N_i}\varphi_q^{N_i})\,dx\,dy$ where $1 \le i \le N_\Sigma$, and N_Σ is the number of rectangles contained in $\sum$;
- the bottom right entry is $[\tilde A_k]_{I+1,I+1} = \int_\Omega (\nabla\kappa_1^k)^2\,dx\,dy$;
- the other entries are zero.

Construction of the matrix B. The matrix B is written as $B = [B_1,\ B_2]$ where $B_k = [\mathcal{B}, 0]$, $k \in \{1,2\}$, $\mathcal{B}$ is composed by the matrices

$$\mathcal{B}^i = -\Big(\operatorname{div}(\varphi_l^{N_i}\varphi_j^{N_i}, \varphi_l^{N_i}\varphi_j^{N_i}), (\bar\varphi_p^{N_i}\bar\varphi_q^{N_i})\Big)_{N_i},$$

for $1 \le l, j \le (N_i - 1)$, $1 \le p, q \le N_i - 1$ and $1 \le i \le I$, which represents the gradient operator on each sub-domain Ω_i.

In the following i is omitted for simplicity,

$$-\Big(\operatorname{div}(\varphi_l^N\varphi_j^N, \varphi_l^N\varphi_j^N), (\bar\varphi_p^N\bar\varphi_q^N)\Big)_N$$

$$= -\sum_{m=0}^{N}\sum_{n=0}^{N}\Big(\varphi_l^{N\prime}(\xi_m)\varphi_j^N(\xi_n)+\varphi_l^N(\xi_m)\varphi_j^{N\prime}(\xi_n)\Big)\bar{\varphi}_p^N(\xi_m)\bar{\varphi}_q^N(\xi_n)\rho_m\rho_n,$$

thus for $p \neq l$ we have

$$\begin{aligned}\Big(\varphi_l^{N\prime}\varphi_j^N,\bar{\varphi}_p^N\bar{\varphi}_q^N\Big)_N &= -\frac{1}{2}\frac{\rho_j}{(1-\xi_j^2)L_N(\xi_l)}\Big(\frac{\rho_0}{L_N(\xi_p)(\xi_0-\xi_l)(\xi_0-\xi_p)} \\ &\quad + \frac{2L_N(\xi_p)\rho_p}{(\xi_p-\xi_l)(1-{\xi_p}^2)} - \frac{\rho_N}{(\xi_N-\xi_p)(\xi_N-\xi_l)L_N(\xi_p)}\Big)\delta_{jq}\end{aligned}$$

and for $p = l$,

$$\begin{aligned}&\Big(\varphi_l^{N\prime}\varphi_j^N,\bar{\varphi}_p^N\bar{\varphi}_q^N\Big)_N \\ &= -\frac{1}{2}\frac{\rho_j}{(1-\xi_j^2)L_N(\xi_l)L_N(\xi_p)}\Big(\frac{\rho_0}{(\xi_0-\xi_l)(\xi_0-\xi_p)} - \frac{\rho_N}{(\xi_N-\xi_p)(\xi_N-\xi_l)}\Big)\delta_{jq}.\end{aligned}$$

We note that if $l = j = 0$ and $l = j = N$ are made in the same way, with some modifications. We denote by $F = \big(F_1\ F_2\ 0\big)^{\mathrm{T}}$ the second member where $F_k, k \in \{1,2\}$ are given by

$$F_k = \begin{pmatrix}(\varphi_p^{N_1}\varphi_q^{N_1}, f_k)_{N_1} \\ \dots \\ (\varphi_p^{N_I}\varphi_q^{N_I}, f_k)_{N_I} \\ \int_\Omega f_k\kappa_1^k dx\end{pmatrix}.$$

Solution of the matrix system. System (4.3) has false degrees of freedom, which are the values of the velocity on the boundary nodes of the sub-domains. These values are found by the action of the matching matrix $\widetilde{Q}$ on the mortar functions. They are linked by the integral matching condition (3.3). The calculation of this matrix is local for each pair edge-mortar.

We consider ϕ a mortar function such that:

$$\phi_{/\gamma_k} = \sum_{j=0}^{N_{i(k)}}\phi_j^k\varphi_j^{N_{i(k)}}(s), \quad 1 \le k \le K, \quad s \in [-1,1].$$

Now, we determine a basis of the test function space $\mathbb{P}_{N_i-2}(\Gamma)$ that identifies with $\mathbb{P}_{N_i-2}(]-1,1[)$

$$\chi_{/\Gamma} = \sum_{p=1}^{N_i-1}\chi_p\eta_p^{N_i-2}(s)$$

where

$$\eta_p^{N_i-2}(s) = \frac{(-1)^{N_k-p}L_{N_i}(s)}{(\xi_p-s)}, \quad p \in \{1,\dots,N_i-1\},$$

(more details are given in [21]). Hence, the integral matching condition (3.3) is written as $w_\delta = \widetilde{Q}\phi$. Then, we present the matching global matrix Q.

$$\underbrace{\begin{pmatrix} w_{lj}^i/\text{internal} \\ w_{lj}^i/\text{boundary} \\ \lambda_1\end{pmatrix}}_{w_\delta^*} = \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & \widetilde{Q} & 0 \\ 0 & 0 & 1\end{pmatrix}}_{Q}\underbrace{\begin{pmatrix} w_{lj}^i/\text{internal} \\ \phi_j^k \\ \lambda_1\end{pmatrix}}_{\tilde{w_\delta^*}},$$

for $1 \le i \le I$, and $1 \le k \le K$.

The matrix Q is used to uncouple the system (4.3) to be solved locally. The transpose of the matrix Q purges the vector of unknowns from the false degrees of freedom. So the resulting system is

$$\begin{gathered}\mathbf{Q}^T A\mathbf{Q}\tilde{U} + \mathbf{Q}^T B^T P = \mathbf{Q}^T F \\ B\mathbf{Q}\tilde{U} = 0,\end{gathered} \tag{4.4}$$

where $\mathbf{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$, $\tilde{U}$ is composed with the values of the velocity on the internal nodes and the mortar functions on the skeleton. To solve the system (4.4) we apply the Uzawa algorithm which is appropriate for the Stokes problem (see [14, 22]). We consider

$$\tilde{A} = \mathbf{Q}^T A\mathbf{Q}, \tilde{B} = B\mathbf{Q} \text{ and } \tilde{F} = \mathbf{Q}^T F.$$

We uncouple the two unknowns $\tilde{U}$ and P. The matrix $\tilde{A}$ is invertible since the bilinear form $a_\delta^*(\cdot,\cdot)$ is elliptic. The first equation of the system (4.4) is written as

$$\tilde{U} = \tilde{A}^{-1}\big(\tilde{F} - \tilde{B}P\big). \tag{4.5}$$

By inserting (4.5) in the second equation of the system (4.4) we obtain

$$\big(\tilde{B}\tilde{A}^{-1}\tilde{B}^T\big)P = \tilde{A}^{-1}\tilde{F}. \tag{4.6}$$

To find the discrete pressure P, the linear system (4.6) is solved by the gradient conjugate method since the matrix $\tilde{B}\tilde{A}^{-1}\tilde{B}^T$ is symmetric and positive define. We apply the same method on the system (4.5) to find the discrete velocity $\tilde{U}$.

Uzawa algorithm

- Let P_0 arbitrary.
- Initialization step: $\tilde{A}\tilde{U}_0 = \tilde{F} - \tilde{B}^T P_0$
- Iterations: $m \geq 0$ From $\tilde{U}_m$ and P_m:

$$\begin{gathered}G_m = -\tilde{B}\tilde{U}_m \\ \tilde{A}V_m = \tilde{B}^T G_m \\ \rho_m = \frac{\|G_m\|^2}{(\tilde{B}^T G_m, V_m)} \\ P_{m+1} = P_m - \rho_m G_m \\ \tilde{U}_{m+1} = \tilde{U}_m + \rho_m V_m.\end{gathered}$$

4.3. **Numerical results.** In this section we present some numerical tests which are in accordance with the theoretical results given in [12]. We provide the behavior of the error between the continuous solution and the discrete one. The convergence tests are done with analytical and singular solutions. The tests are established on two non convex domains which correspond respectively to the case where $\alpha = 2\pi$ and $\alpha = \frac{3\pi}{2}$ see Figure 1.

If we denote by N the degree of the polynomial in $\bar{\Omega}_i$, $1 \leq i \leq I$ which contains the singular point $\mathbf{c}$. Then, the degree of the polynomial in the other rectangular sub-domains is chosen less than N and is fixed for each numerical tests.

Next, we present some numerical cases to show the efficiency of the method which consists in the use of the domain decomposition method associated to Strang and Fix algorithm.

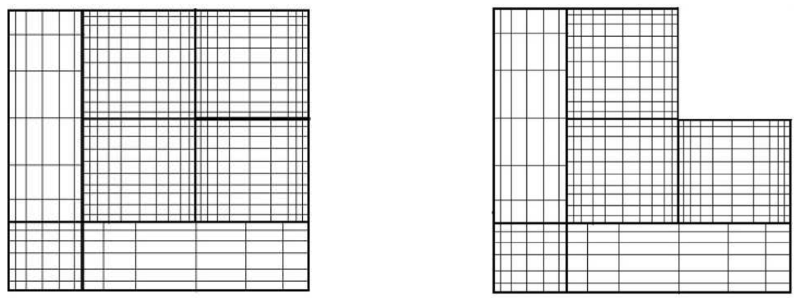

FIGURE 1. Spectral mesh of the domain when $\alpha = 2\pi$ (left) and $\alpha = \frac{3\pi}{2}$ (right).

We consider the viscosity $\nu = 1$, and the pressure equal to $p(x, y) = xy$ for all the error curves test. The given solution for the velocity is the first singular function in the two situations:

(i) for $\alpha = 3\pi/2$ the first singularity is constructed using the formula

$$\kappa_1(r,\theta) = \operatorname{curl}(\eta_1(r,\theta)) = \Big(\frac{\cos(\theta)}{r}\frac{\partial \eta_1}{\partial \theta} + \sin(\theta)\frac{\partial \eta_1}{\partial r}, \frac{\sin(\theta)}{r}\frac{\partial \eta_1}{\partial \theta} - \cos(\theta)\frac{\partial \eta_1}{\partial r}\Big) \tag{4.7}$$

where

$$\begin{aligned}\eta_1(r,\theta) = r^{1.544}\Big(&2.093\big(\cos(0.459\theta) - \cos(1.544\theta)\big)\\ &+ 1.093\big(2.193\sin(0.459\theta) - 0.647\sin(1.544\theta)\big)\Big).\end{aligned}$$

(ii) for $\alpha = 2\pi$ the first singularity is

$$\kappa_1(r,\theta) = r^{1/2}(3\sin\theta\sin\frac{\theta}{2}, 3(1-\cos\theta)\sin\frac{\theta}{2}). \tag{4.8}$$

Figure 2 (resp. 3) presents the convergence curves of the error on the velocity for the solution issued from (4.7) (resp. (4.8)). We present the semi logarithmic scale (for N varying from 5 to 60) and the Logarithmic scale for proving the slope. We remark that the obtained results for the error and slopes using Strang and Fix algorithm are better than those without.

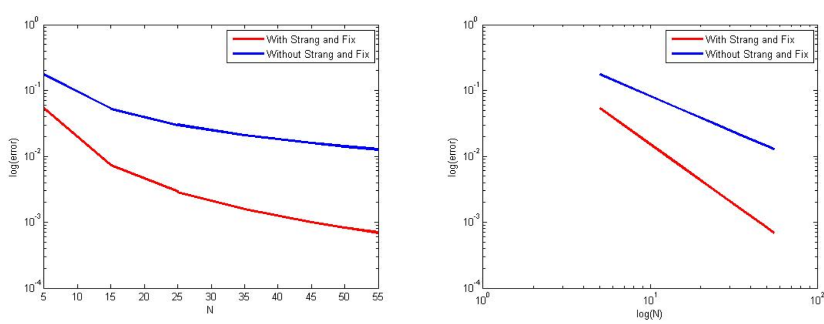

FIGURE 2. Error curves for the solution defined from (4.7)

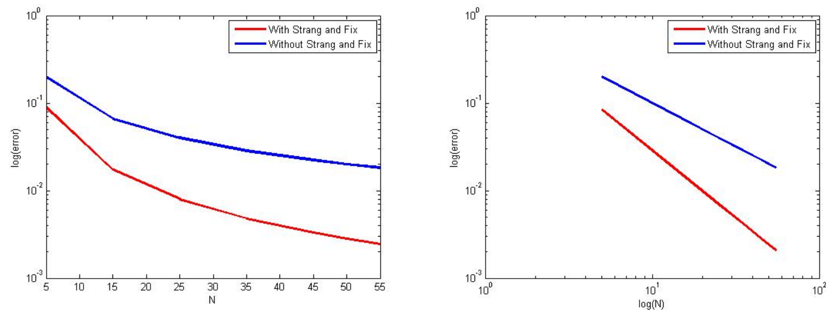

FIGURE 3. Error curves for the solution defined from (4.8).

In the second test we choose an analytic velocity $u = \mathrm{curl}(\phi)$, ϕ is the stream function

$$\phi(x, y) = \sin(\pi x)^2 \sin(\pi y)^2. \tag{4.9}$$

Figure 4 corresponds from left to right to the semi logarithmic error curves with respect to N for $\alpha = 3\pi/2$ and $\alpha = 2\pi$ in the case of the regular solution issued from (4.9). We obtain a good convergence with or without Strang and Fix algorithm. This is so because the Strang and Fix algorithm does not improve the convergence for a regular function.

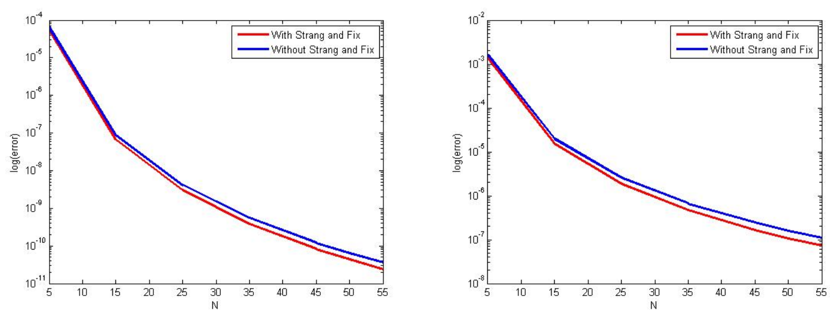

FIGURE 4. Error curves for the solution from (4.9), for $\alpha = 3\pi/2$ and $\alpha = 2\pi$.

The given solution for the velocity is the second singular function in the two situations:
(i) for $\alpha = \frac{3\pi}{2}$ the second singularity constructed using the formula

$$\kappa_2(r,\theta) = \mathrm{curl}(\eta_2(r,\theta)) = \big(\frac{\cos(\theta)}{r}\frac{\partial \eta_2}{\partial \theta} + \sin(\theta)\frac{\partial \eta_2}{\partial r}, \frac{\sin(\theta)}{r}\frac{\partial \eta_2}{\partial \theta} - \cos(\theta)\frac{\partial \eta_2}{\partial r}\big) \tag{4.10}$$

where

$$\begin{aligned}\eta_2(r,\theta) = r^{1.908}\Big(&4.302\big(\cos(0.092\theta) - \cos(1.908\theta)\big)\\ &- 1.815\big(10.869\sin(0.092\theta) - 0.524\sin(1.908\theta)\big)\Big).\end{aligned}$$

(ii) for $\alpha = 2\pi$ the second singularity is

$$\kappa_2(r,\theta) = r^{1/2}(2\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2}, (1-\cos\theta)\cos\frac{\theta}{2}). \tag{4.11}$$

We present in Figure 5 from left to right the convergence curves of the error corresponding to the second singular function defined in (4.10) for $\alpha = 3\pi/2$ (respectively in (4.11) for $\alpha = 2\pi$). We remark that we do not obtain a good convergence with or without Strang and Fix algorithm. If we want to improve this convergence we have to add the second singular function to the discrete space X_δ^* which is difficult to implement numerically.

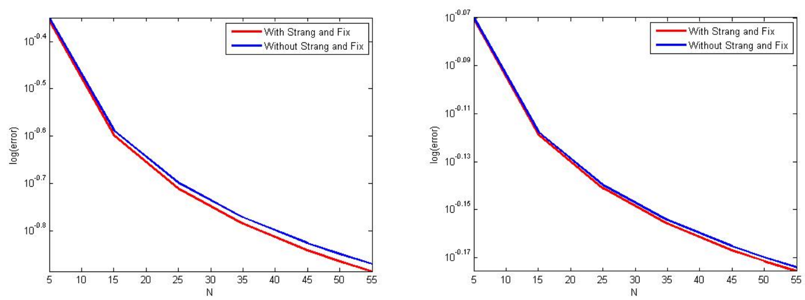

FIGURE 5. Error curves for the solution defined from (4.10), respectively from (4.11).

Figure 6 presents from left to right and top to bottom the values of the two components of the velocity and of the pressure corresponding to the less regular data function $\mathbf{f} = (f_1, f_2)$, with

$$f_1 = |x|^{0.5} \quad f_2 = |y|^{0.5} \tag{4.12}$$

obtained with $N = 60$.

Conclusion. This paper dealt with the implementation of the mortar decomposition domain. This technique is applied to the spectral element method for approximating the standard velocity pressure formulation of the Stokes problem. We implemented Strang and Fix algorithm, which consists to enlarge the discrete space of the velocity by the first singular function. The obtained error curves confirm that Strang and Fix algorithm applied with the mortar spectral method is a good tool to improve the convergence order of the error. The extension of this method to the non linear Navier-Stokes equations and to the three axisymmetric domain is presently under consideration.

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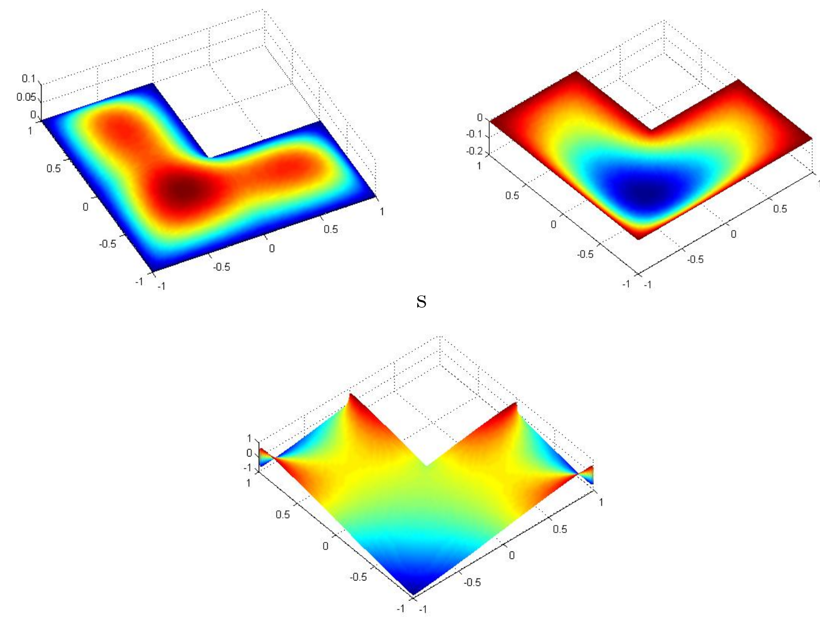

FIGURE 6. The solution (u_1, u_2, p) for the data function (4.12) and the mesh $\alpha = 3\pi/2$.

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