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CRITICAL EXPONENT FOR THE HEAT EQUATION IN α-MODULATION SPACES

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Abstract. In this article, we propose a method for finding the critical exponent for heat equations in α-modulation space $M^{s,\alpha}_{p,q}$. We define an index $\sigma(s,p,q)$, and use it to determine the critical exponent of the heat equation. Then we use this exponent to describe well and ill-posedness of the heat equation in $L^\infty([0,T];M^{s,\alpha}_{p,q})$. In some special case our conclusions are sharp. Furthermore, our method may be applied to other evolution equations.

1. Introduction and statement of main results

It is well known that many dispersive equations have their critical exponents in either Sobolev spaces or Besov spaces, or both. For instance, the critical exponent of nonlinear Schrödinger (NLS) equation is $\frac{n}{2}-\frac{2}{k-1}$ when the nonlinear term is $|u|^{k-1}u$ in Sobolev spaces. Cazenave and Weissler [3] showed that NLS is local well-posedness in $C([-T,T];\dot H^s)$ when $s\geq 0$ and $s>\frac{n}{2}-\frac{2}{k-1}$. Christ, Colliander and Tao [5] proved that when $s<\max\{0,\frac{n}{2}-\frac{2}{k-1}\}$, NLS is ill-posed in $C([-T,T];\dot H^s)$ for any fixed $T>0$. We can see that the domain of well and ill-posedness is completely described by their critical exponents. Furthermore, the methods in [3] and [5] relay heavily on the scaling invariance of the work spaces. In recent years, modulation space emerges and plays a significant role in the study of certain nonlinear dispersive equations. (We will describe more details of the modulation space and α-modulation space in the following contents.) Although modulation space lacks the scaling property, in our previous work[16], we found the critical exponents for some dispersive equations in modulation space by different methods. Particularly, we found critical exponents for fractional heat equation in the modulation space without the scaling property. This exponent also could describe well and ill posedness in modulation space completely. That description is quite similar to above conclusions in [3] and [5].

Modulation space was introduced by Feichtinger in [6] to measure smoothness of a function or distribution in a way different from L^p space, and they are now recognized as a powerful tool for studying wavelet and pseudo-differential operators (see [2, 4, 10, 11, 17, 18, 19, 22]). The original definition of the modulation space is based on the short-time Fourier transform and window function. Wang and Hudizk [20] gave an equivalent definition of the discrete version on modulation space by

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the frequency-uniform-decomposition. With this discrete version, they were able to find global solutions for nonlinear Schrödinger equation and nonlinear Klein-Gordon equation in lower regularity space. After then, there have been many studies on nonlinear PDEs in modulation space. So far, people have found that modulation has many advantages in study of PDE's problems.

The α-modulation space $M^{s,\alpha}_{p,q}$ was first introduced by Göbner in his unpublished thesis [9]. Later, the definition was refined by Han and Wang in [13]. They used the α-covering and a corresponding bounded admissible partition of unity of order p (BAPU) to define α-modulation space. The parameter $\alpha \in [0,1]$ determines a segmentation of the frequency spaces. When $\alpha = 0$, $M^{s,0}_{p,q}$ is equivalent to the classical modulation space; When $\alpha = 1$, $M^{s,1}_{p,q}$ is equivalent to the classical Besov space. Obviously, it is proposed to be an intermediate function space between Besov space and modulation space. Hence, it is very important to study some analysis and PDE's problems in α-modulation space. So far, there are many good results on this topic. Below we list some of them, among many others. Guo and Chen [12] proved the Stricharz estimates on $M^{s,\alpha}_{p,q}$. For Cauchy problem in α-modulation space, Han and Wang studied the derivative nonlinear Schrödinger equation in [14]; Chen and Huang studied dispersive equations with noninteger term in [15]. For the boundness of operators, Wu and Chen [21] obtained the sharp conditions for the boundness of fractional integral operators and bilinear fractional integral operators in $M^{s,\alpha}_{p,q}$; Feichtinger, Huang and Wang [7] studied trace operators in $M^{s,\alpha}_{p,q}$.

In this article, we find the critical exponents for heat equation in $M^{s,\alpha}_{p,q}$. Moreover, we use this exponent to describe well and ill-posedness for heat equation, and get sharp results in some special cases. First, we recall some important properties of Besov space [8] and modulation space [20]. The first one is Sobolev-type embedding that says $B^{s_1}_{p_1,q} \subset B^{s_2}_{p_2,q}$ if and only if

$$s_2 \le s_1 \quad \text{and} \quad s_1 - \frac{n}{p_1} \ge s_2 - \frac{n}{p_2}.$$

$M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2}$ if and only if

$$s_2 \le s_1 \quad \text{and} \quad s_1 - \frac{n}{q_1'} \ge s_2 - \frac{n}{q_2'}.$$

The second one is algebra property that says $B^s_{p,q}$ forms a multiplication algebra if $s - \frac{n}{p} > 0$, and $M^s_{p,q}$ forms a multiplication algebra if $s - \frac{n}{q'} > 0$. By comparing these properties to embedding in $M^{s,\alpha}_{p,q}$ (see proposition 2.3) and algebra property of $M^{s,\alpha}_{p,q}$ ([13, Theorem 5.1]), we observe that the index $s - \alpha\frac{n}{p} - (1-\alpha)\frac{n}{q'}$ in the α-modulation space is an analog of the index $s - \frac{n}{p}$ in the Besov space or $s - \frac{n}{q'}$ in the modulation space. Motivated by such an observation, heuristically, we may use the index $s - \alpha\frac{n}{p} - (1-\alpha)\frac{n}{q'}$ to describe the critical exponent for heat equation in $M^{s,\alpha}_{p,q}$. Of course, this heuristic idea will be technically supported in our following discussion. For convenience in the discussion, we denote $\sigma(s,p,q) = s - \alpha\frac{n}{p} - (1-\alpha)\frac{n}{q'}$, and $\sigma_i := \sigma(s_i,p_i,q_i) = s_i - \alpha\frac{n}{p_i} - (1-\alpha)\frac{n}{q_i'}$, we use the inequality

$$A(u,v,w\dots) \preceq B(u,v,w\dots)$$

to mean that there is a positive number C independent of all main variables $u,v,w\dots$, for which $A(u,v,w\dots) \le CB(u,v,w\dots)$.

Now we state main results in our paper. We only consider the case: $\mathfrak{D} = \{(p,q) \in \mathbb{R}^2 : 1 \le p \le \infty, 1 \le q \le \infty, q \ge p\}$ for the technical problem. Now we use the index $\sigma(s,p,q)$ to describe well and ill-posedness for following heat equation

$$u_t + \Delta u = u^2, \quad u(0) = u_0 \tag{1.1}$$

in $M^{s,\alpha}_{p,q}$. Following theorems are our main results in this paper:

Theorem 1.1. *Let $(p,q) \in \mathfrak{D}$ and $\sigma(s,p,q) > -\frac{2\alpha}{2-\alpha}$. There exists a $T > 0$ such that equation (1.1) is local well-posedness in $L^\infty([0,T]; M^{s,\alpha}_{p,q})$. Precisely, for every inial data $u_0 \in M^{s,\alpha}_{p,q}$, there exists $T > 0$ such that heat equation (1.1) has a unique solution in $L^\infty([0,T]; M^{s,\alpha}_{p,q})$.*

Theorem 1.2. *Let $s \in \mathbb{R}$,$1 \le p,q \le \infty$, when $\sigma(s,2,q) < -\frac{2}{k-1}$ or $s - \frac{n}{2} < -\frac{2}{k-1}$ for any $q \in [1,\infty)$, then equation (1.1) is ill-posed in $L^\infty([0,T]; M^{s,\alpha}_{p,q})$ for any fixed $T > 0$.*

Remark 1.3. Equation (1.1) is a special case of heat equations. For general case, if we replace the nonlinear term u^2 by u^k for $k \in \mathbb{Z}^+$ or replace Laplacian Δ by fractional Laplacian $\Delta^{\frac{\beta}{2}}$, we can also obtain similar results by the same method.

Remark 1.4. When $\alpha = 1$, we can see the results are sharp and same as that in Besov space. But for the case $\alpha \in (0,1)$, our results are not sharp for technical problem. Essentially, this difficulty is due to the shape of α-covering when we prove the algebra property of $M^{s,\alpha}_{p,q}$ (see Lemma 3.2). In the proof of Lemma 3.2, when $(p,q) = (1,1)$ we encounter to this difficulty. But for the case $1 \le p \le \infty$, $q = \infty$, we can obtain perfect conclusions. So, when $(p,q) = (2,\infty)$, our results are sharp for any $\alpha \in [0,1]$. Specifically, we have following corollary.

Corollary 1.5. *When $\sigma(s,2,\infty) > -2$, heat equation (1.1) is locally well-posedness in $L^\infty([0,T]; M^{s,\alpha}_{p,q})$; when $\sigma(s,2,\infty) < -2$, heat equation (1.1) is ill-posed in $L^\infty([0,T]; M^{s,\alpha}_{p,q})$ for any fix $T > 0$.*

This article is organized as follows. In Section 2, we will introduce some basic knowledge on α-modulation space, as well as some useful propositions that will be used in our proofs. All proofs of main theorems will be presented in Section 3.

2. Preliminaries

In this section, we give the definition and discuss some basic properties of α-modulation space. Before giving the definition of $M^{s,\alpha}_{p,q}$, we introduce some notation frequently used in this paper. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the Schwartz function. Its dual is $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$, the set of all tempered distribution on $\mathbb{R}^n$. For any $p \in [1,\infty)$, p' will stand for the dual index of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ We write L^p for $L^p(\mathbb{R}^n)$ and l^p the sequence Lebesgue space. For a vector $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote $|k| = (k_1^2 + k_2^2 + \dots + k_n^2)^{\frac{1}{2}}$, $|k|_\infty = \max_{i=1,\dots,n} |k_i|$, $\langle k\rangle = (1 + |k|^2)^{\frac{1}{2}}$. Now, we briefly introduce the definition of α-modulation. More details can be found in [13].

Definition 2.1. Let ρ be a nonnegative smooth radial bump function supported in $B(0,2)$, satisfying $\rho(\xi) = 1$ for $|\xi| < 1$ and $\rho(\xi) = 0$ for $|\xi| \ge 2$. For any $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we set

$$\rho^\alpha_k(\xi) = \rho\Big(\frac{\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k}{r\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\Big),$$

$$\varphi^\alpha_k = \rho^\alpha_k(\xi)\Big(\sum_{l\in\mathbb{Z}^n} \rho^\alpha_l(\xi)\Big)$$

We define the ball

$$B_k^r := \{\xi \in \mathbb{R}^n : |\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k| < r\langle k\rangle^{\frac{\alpha}{1-\alpha}}\}$$

It is easy to check that $\{\varphi_k^\alpha\}_{k\in\mathbb{Z}^n}$ satisfy

$$\begin{gathered}
\operatorname{supp}\varphi_k^\alpha \subset B_k^{2r},\\
\varphi_k^\alpha(\xi) = c, \quad \forall \xi \in B_k^r,\\
\sum_{k\in\mathbb{Z}^n} \varphi_k^\alpha(\xi) \equiv 1, \quad \xi \in \mathbb{R}^n,\\
\|\mathcal{F}^{-1}\varphi_k^\alpha\|_{L^1} \prec 1
\end{gathered}$$

Corresponding to the above sequence $\{\varphi_k^\alpha\}_{k\in\mathbb{Z}^n}$, we can construct an operator sequence $\{\Box_k^\alpha\}_{k\in\mathbb{Z}^n}$ by

$$\Box_k^\alpha = \mathcal{F}^{-1}\varphi_k^\alpha\mathcal{F}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ donate the standard Fourier transform and inverse Fourier transform respectively. For $\alpha \in [0,1), 0 \le p, q \le \infty, s \in \mathbb{R}$, using this decomposition, we define α-modulation space as

$$M_{p,q}^{s,\alpha} = \{f \in \mathcal{S}' : \|f\|_{M_{p,q}^{s,\alpha}} < \infty\}$$

where

$$\|f\|_{M_{p,q}^{s,\alpha}} = \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \|\Box_k^\alpha f\|_{L^p}^q\Big)^{1/q}$$

Proposition 2.2 (Isomorphism [13]). *Let $0 < p, q \le \infty, s, \sigma \in R$. $J_\sigma = (I-\triangle)^{\sigma/2} : M_{p,q}^{s,\alpha} \to M_{p,q}^{s-\sigma,\alpha}$ is an isomorphic mapping, where I is the identity mapping and Δ is the Laplacian.*

Proposition 2.3 (Embedding [13]). *Suppose $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, we have*

(i) *if $q_1 \le q_2$ and $s_1 \ge s_2 + n\alpha(\frac{1}{p_1} - \frac{1}{p_2})$, then*

$$M_{p_1,q_1}^{s_1,\alpha} \subset M_{p_2,q_2}^{s_2,\alpha} \tag{2.1}$$

(ii) *if $q_1 > q_2$ and $s_1 - \alpha\frac{n}{p_1} - (1-\alpha)\frac{n}{q_1'} > s_2 - \alpha\frac{n}{p_2} - (1-\alpha)\frac{n}{q_2'}$, then*

$$M_{p_1,q_1}^{s_1,\alpha} \subset M_{p_2,q_2}^{s_2,\alpha} \tag{2.2}$$

3. Proof of main results

Before proving Theorem 1.1, we state some key lemmas.

Lemma 3.1. *Let $1 \le p_1 \le p_2 \le \infty$, $1 \le q_1 \le q_2 \le \infty$, $s_2 \le s_1$. When $\sigma_1 - \sigma_2 > R$, heat semigroup $e^{t\Delta} := \mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}$ satisfy estimate*

$$\|e^{t\Delta}f\|_{M_{p_2,q_2}^{s_2,\alpha}} \preceq (1 + t^{-\frac{R}{2}})\|f\|_{M_{p_1,q_1}^{s_1,\alpha}}$$

Proof. We first consider the case $p = p_1 = p_2$, $q = q_1 = q_2$. For the low frequency part $|k| \le 100\sqrt{n}$, by the multiplier estimate of $e^{t\Delta}$, we have

$$\sum_{|k|\le 100\sqrt{n}} \langle k\rangle^{\frac{s_1 q}{1-\alpha}} \|\Box_k^\alpha e^{t\Delta} f\|_{L^p}^q \preceq \sum_{|k|\le 100\sqrt{n}} \langle k\rangle^{\frac{s_2 q}{1-\alpha}} \|\Box_k^\alpha f\|_{L^p}^q \preceq \|f\|_{M_{p,q}^{s_2}}^q.$$

For the high frequency part, note that the operator $\square_k^\alpha e^{t\Delta}$ can be written as

$$\square_k^\alpha e^{t\Delta} = \sum_{|\ell|\le 1} \square_{k+\ell}^\alpha e^{t\Delta}\square_k^\alpha$$

and $\square_{k+\ell}^\alpha e^{t\Delta}$ are convolution operators with the kernels

$$\Omega_{k+\ell}(y) = e^{i\langle k+\ell, y\rangle}\int_{\mathbb{R}^n} e^{-t|\xi+k+\ell|^{\frac{2}{1-\alpha}}} e^{i<y,\xi>}\varphi(\xi)d\xi.$$

Hence, when $|k| \ge 100\sqrt{n}$ it is easy to prove that

$$\|\square_k^\alpha e^{t\Delta} f\|_{L^p} \preceq e^{-\frac{t}{2}|k|^{\frac{2}{1-\alpha}}}\|\square_k^\alpha f\|_{L^p}.$$

Now, we have

$$\begin{aligned}
\langle k\rangle^{\frac{s_1}{1-\alpha}}\|\square_k e^{t\Delta} f\|_{L^p} &\preceq \langle k\rangle^{\frac{s_1-s_2}{1-\alpha}} e^{-\frac{t}{2}|k|^{\frac{2}{1-\alpha}}}\langle k\rangle^{\frac{s_2}{1-\alpha}}\|\square_k^\alpha f\|_{L^p}\\
&\preceq t^{-\frac12(s_1-s_2)}\langle k\rangle^{\frac{s_2}{1-\alpha}}\|\square_k^\alpha f\|_{L^p}.
\end{aligned}$$

Taking the l^q norm in both sides, we obtain

$$\|e^{t\Delta} f\|_{M_{p,q}^{s_2,\alpha}} \preceq (1+t^{-\frac12(s_1-s_2)})\|f\|_{M_{p,q}^{s_1,\alpha}} \tag{3.1}$$

Next, we estimate the case $1 \le p_1 < p_2, 1 \le q_1 < q_2$ and $s_1 \ge s_2$. By (2.2) and (3.1), we have

$$\|e^{t\Delta} f\|_{M_{p_1,q_1}^{s_1,\alpha}} \preceq \|e^{t\Delta} f\|_{M_{p_2,q_2}^{s_2-R,\alpha}} \preceq (1+t^{-\frac{R}{2}})\|f\|_{M_{p_2,q_2}^{s_2,\alpha}}.$$

□

Lemma 3.2. *Let $(p,q) \in \mathfrak{D}$, $s_0 > 0$. When $\sigma(s,p,q) > -\frac{s_0\alpha}{2-\alpha}$, we have following estimate:*

$$\|u^2\|_{M_{p,q}^{s-s_0,\alpha}} \preceq \|u\|^2_{M_{p,q}^{s,\alpha}}.$$

Proof. We start with some notation and basic conclusions which were obtained in [13]. For every $(k_1,k_2) \in \mathbb{Z}^{2n}$, we introduce

$$\Lambda(k_1,k_2) = \{k \in \mathbb{Z}^n : \square_k^\alpha(\square_{k_1}^\alpha u \square_{k_2}^\alpha u) \ne 0\}$$

We write

$$\begin{gathered}
K_j(k_1,k_2) = \langle k_1\rangle^{\frac{\alpha}{1-\alpha}} k_{1j} + \langle k_2\rangle^{\frac{\alpha}{1-\alpha}} k_{2j},\\
K(k_1,k_2) = \max_{1\le j\le n}|K_j(k_1,k_2)|
\end{gathered}$$

To obtain a more precise estimate, we divide $\mathbb{Z}^{2n}$ of all (k_1,k_2) in to the sets

$$\begin{gathered}
\Omega_0 = \{(k_1,k_2 \in \mathbb{Z}^{2n}) : \langle k_1\rangle \sim \langle k_2\rangle\},\\
\Omega_1 = \{(k_1,k_2 \in \mathbb{Z}^{2n}) : \langle k_1\rangle \gg \langle k_2\rangle\},\\
\Omega_2 = \{(k_1,k_2 \in \mathbb{Z}^{2n}) : \langle k_1\rangle \ll \langle k_2\rangle\}
\end{gathered}$$

and separate Ω_0 into the sets

$$\begin{gathered}
\Omega_{0,1} = \{(k_1,k_2 \in \Omega_0 : K(k_1,k_2) \preceq \langle k_1\rangle^{\frac{\alpha}{1-\alpha}}\},\\
\Omega_{0,2} = \{(k_1,k_2 \in \Omega_0 : K(k_1,k_2) \gg \langle k_1\rangle^{\frac{\alpha}{1-\alpha}}\}.
\end{gathered}$$

In [13], it had been proved that when $(k_1,k_2) \in \Omega_{0,1}$, we have $\langle k\rangle \preceq \langle k_1\rangle^\alpha$; when $(k_1,k_2) \in \Omega_{0,2}$, we have $\langle k\rangle \preceq \langle k_1\rangle^y$ for some $y := y(k_1,k_2) \in (\alpha,1]$.

First, we consider the case $(p,q)=(1,1)$, by the triangle inequality, we have

$$\begin{aligned}
\|u^2\|_{M_{p,q}^{s-s_0,\alpha}} &= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \|\Box_k^\alpha u^2\|_{L^1} \\
&\le \sum_k \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \Big(\sum_{k_1,k_2} \|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1}\Big) \\
&= \sum_{l=0}^{2} \sum_{(k_1,k_2)\in\Omega_l} \sum_{k\in\Lambda(k_1,k_2)} \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1}
\end{aligned}$$

By the multiplier estimate and Hölder's inequality, we have

$$\|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1} \preceq \|\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u\|_{L^1} \preceq \|\Box_{k_1}^\alpha u\|_{L^1} \|\Box_{k_2}^\alpha u\|_{L^\infty}$$

For $(k_1,k_2)\in\Omega_{0,1}$, choose $b=\sigma(s,1,1)-\varepsilon$, we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Omega_{01}} \sum_{k\in\Lambda(k_1,k_2)} \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1} \\
&\preceq \sum_{(k_1,k_2)} \langle k_1\rangle^{\frac{(s-s_0)\alpha}{1-\alpha}+n\alpha-\frac{b}{1-\alpha}} \|\Box_{k_1}^\alpha u\|_{L^1} \langle k_2\rangle^{\frac{b}{1-\alpha}} \|\Box_{k_2}^\alpha u\|_{L^\infty} \\
&\le \|u\|_{M_{1,1}^{(s-s_0)\alpha+n\alpha(1-\alpha)-b,\alpha}} \|u\|_{M_{\infty,1}^{b,\alpha}}
\end{aligned}$$

Choosing $\varepsilon\to 0^+$, the domain of $\sigma(s,1,1)$ guarantees that $(s-s_0)\alpha+n\alpha(1-\alpha)-b<s$. Hence, by (2.2) we have

$$\|u^2\|_{M_{1,1}^{s-s_0,\alpha}} \preceq \|u\|^2_{M_{1,1}^{s,\alpha}}$$

For $(k_1,k_2)\in\Omega_{0,2}$, we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Omega_{02}} \sum_{k\in\Lambda(k_1,k_2)} \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1} \\
&\preceq \sum_{(k_1,k_2)} \langle k_1\rangle^{\frac{(s-s_0)y}{1-\alpha}+\frac{n\alpha}{1-\alpha}(1-y)-\frac{b}{1-\alpha}} \|\Box_{k_1}^\alpha u\|_{L^1} \langle k_2\rangle^{\frac{b}{1-\alpha}} \|\Box_{k_2}^\alpha u\|_{L^\infty} \\
&\le \|u\|_{M_{1,1}^{(s-s_0)y+n\alpha(1-y)-b,\alpha}} \|u\|_{M_{\infty,1}^{b,\alpha}}
\end{aligned}$$

Similarly, Choosing $\varepsilon\to 0^+$, the domain of $\sigma(s,1,1)$ and $y\in[\alpha,1)$ also guarantees that $(s-s_0)y+n\alpha(1-y)-b<s$. So we also have

$$\|u^2\|_{M_{1,1}^{s-s_0,\alpha}} \preceq \|u\|^2_{M_{1,1}^{s,\alpha}}$$

For $(k_1,k_2)\in\Omega_1$, we recall the refined Hödler inequality:

$$\|fg\|_{L^p} \preceq \|J_a f\|_{L^{p_1}} \|J_b g\|_{L^{p_2}},$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, J_a and J_b are Bessel potentials which satisfy $a+b>0$. Also, it had been proved that $\sharp\Lambda(k_1,k_2)\sim 1$ in [13, 5.17]. By this conclusion and the above Hödler inequality, choosing $b=\sigma(s,1,1)-\varepsilon$, $a=2\varepsilon-\sigma(s,1,1)$, and $\varepsilon\to 0^+$ we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Omega_1} \sum_{k\in\Lambda(k_1,k_2)} \langle k\rangle^{\frac{s-s_0}{1-\alpha}} \|\Box_k^\alpha(\Box_{k_1}^\alpha u \Box_{k_2}^\alpha u)\|_{L^1} \\
&\preceq \sum_{k_1\in\mathbb{Z}^n} \langle k_1\rangle^{\frac{(s-s_0)}{1-\alpha}} \|\Box_{k_1}^\alpha J_a u\|_{L^1} \sum_{k_2\in\mathbb{Z}^n} \|\Box_{k_2}^\alpha J_b u\|_{L^\infty}
\end{aligned}$$

$$\begin{aligned}
&\le \|u\|_{M^{s-s_0+a,\alpha}_{1,1}}\|u\|_{M^{b,\alpha}_{\infty,1}}\\
&\le \|u\|^2_{M^{s,\alpha}_{1,1}}
\end{aligned}$$

For $(k_1,k_2)\in\Omega_2$, we can get the same estimate by using the method above.

Next, we consider the case $1\le p\le\infty$, $q=\infty$. We also choose $b=\sigma(s,p,\infty)-\varepsilon$, $a=2\varepsilon-\sigma(s,p,\infty)$, and let $\varepsilon\to 0^+$. By the triangle inequality,

$$\begin{aligned}
\|u^2\|_{M^{s-s_0,\alpha}_{p,q}} &= \sup_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_k u^2\|_{L^p}\\
&\le \sup_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\sum_{k_1,k_2\in\Lambda(k)}\|\Box^\alpha_k(\Box^\alpha_{k_1}u\Box^\alpha_{k_2}u)\|_{L^p}\\
&= \sup_{k\in\mathbb{Z}^n}\sum_{l=0}^{2}\sum_{(k_1,k_2)\in\Lambda(k)\cap\Omega_l}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_k(\Box^\alpha_{k_1}u\Box^\alpha_{k_2}u)\|_{L^p}
\end{aligned}$$

For a $\Phi\subset\mathbb{Z}^{2n}$, we denote

$$\begin{aligned}
\Phi_1^* &= \{k_1\in\mathbb{Z}^n : \exists k_2\in\mathbb{Z}^n \text{ s.t. } (k_1,k_2)\in\Phi\},\\
\Phi_2^* &= \{k_2\in\mathbb{Z}^n : \exists k_1\in\mathbb{Z}^n \text{ s.t. } (k_1,k_2)\in\Phi\}.
\end{aligned}$$

It had been proved that $\sharp\Lambda(-k_2,k)\preceq 1$ in [13]. Then for any $k_2\in\{\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)\}_2^*$ with every fixed k, we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_k(\Box^\alpha_{k_1}u\Box^\alpha_{k_2}u)\|_{L^p}\\
&\preceq \sup_{k_1\in\{\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)\}_1^*}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_{k_1}J_a u\|_{L^p}\sum_{k_1\in\{\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)\}_1^*}\sum_{k_2\in\Lambda(-k_1,k)}\|\Box^\alpha_{k_2}J_b u\|_{L^\infty}\\
&\preceq \sup_{k_1\in\mathbb{Z}^n}\langle k_1\rangle\frac{s-s_0}{1-\alpha}\|\Box^\alpha_{k_1}J_a f\|_{L^p}\sum_{k_2\in\{\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)\}_2^*}\sum_{k_1\in\Lambda(-k_2,k)}\|\Box^\alpha_{k_2}J_b u\|_{L^\infty}\\
&\preceq \|u\|_{M^{s-s_0+a,\alpha}_{p,\infty}}\|u\|_{M^{b,\alpha}_{\infty,1}}\preceq\|u\|^2_{M^{s,\alpha}_{p,q}}
\end{aligned}$$

For $k_2\in\{\{\Omega_0\cup\Omega_1\}\cap\Lambda(k)\}_2^*$ with every fixed k, symmetrically, we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Omega_2\cap\Lambda(k)}\langle k\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_k(\Box^\alpha_{k_1}u\Box^\alpha_{k_2}u)\|_{L^p}\\
&\preceq \sup_{k_2\in\mathbb{Z}^n}\langle k_2\rangle^{\frac{s-s_0}{1-\alpha}}\|\Box^\alpha_{k_2}J_a u\|_{L^p}\sum_{k_1\in\{\Omega_2\cap\Lambda(k)\}_1^*}\sum_{k_2\in\Lambda(-k_1,k)}\|\Box^\alpha_{k_1}J_b u\|_{L^\infty}\\
&\preceq \|u\|_{M^{s-s_0+a,\alpha}_{p,\infty}}\|u\|_{M^{b,\alpha}_{\infty,1}}\preceq\|u\|^2_{M^{s,\alpha}_{p,q}}
\end{aligned}$$

Finally, using the complex interpolation (see [13, Theorem 2.2]) and combining above estimates, we obtain the desire conclusion. $\square$

Based on the above lemmas, now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that heat equation (1.1) is equivalent to the integral equation

$$u=\Phi(u):=e^{t\Delta}u_0+\int_0^t e^{(t-\tau)\Delta}u^2d\tau\,.$$

To prove that the above equation is local well-posed in $M^{s,\alpha}_{p,q}$, we use the standard contraction method. To this end, we define the space

$$X = \{u : \|u\|_{L^\infty([0,T];M^{s,\alpha}_{p,q})} \le C_0\}$$

with the metric

$$d(u,v) = \|u-v\|_{L^\infty([0,T];M^{s,\alpha}_{p,q})}\,,$$

where the positive numbers C_0 and T will be chosen later when we invoke the contraction. Choosing $\varepsilon > 0$ small enough to ensure that $\sigma(s,p,q) > -\frac{(2-\varepsilon)\alpha}{1-\alpha}$, by (3.1) and Lemma 3.2, we have

$$\begin{aligned}
\|\Phi(u)\|_X &\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + \|\int_0^t e^{(t-\tau)\Delta}u^2 d\tau\|_X \\
&\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + \sup_{t\in(0,T]}\int_0^t (t-\tau)^{-\frac{2-\varepsilon}{2}}\|u^2\|_{M^{s-2+\varepsilon,\alpha}_{p,q}} d\tau \\
&\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + \sup_{t\in(0,T]}\int_0^t (t-\tau)^{-\frac{2-\varepsilon}{2}} d\tau \|u\|^2_{M^{s-2+\varepsilon,\alpha}_{p,q}} d\tau \\
&\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + T^{\frac{\varepsilon}{2}}\|u\|_X^2\,.
\end{aligned}$$

By the contraction mapping argument, we obtain the conclusion of Theorem 1.1 after choosing T such that $T^{\varepsilon/2} < 1/2$, and $C_0 = 2\|u_0\|_{M^{s,\alpha}_{p,q}}$. $\square$

Proof of Theorem 1.2. Before the proof, we recall a crucial conclusion which was obtained by Bejenaru and Tao [1]. They consider equation

$$u = L(u_0) + N_k(u,\dots,u)$$

where L is a linear operator, u_0 is the initial data, $N_k(u,\dots,u)$ is a k-linear operator. Also we define $A_1(u_0) := L(u_0)$,

$$A_n(u_0) := \sum_{n_1,\dots,n_k\ge 1; n_1+\dots+n_k=n} N_k(A_{n_1(u_0)},\dots,A_{n_k(u_0)}) forn \in \mathbb{Z}^+$$

They proved that if above equation is well posed from space X to Y, then for each $i \in \mathbb{Z}^+$, A_i is continuous from X to Y (see [1, Proposition 1]). So, if we want to prove ill-posedness of equation (1.1), we only need to choose a special $i \in \mathbb{Z}^+$ and prove A_i is discontinuous. Here, we choose $i = 2$. So, it suffices to show that the map from $M^{s,\alpha}_{p,q}$ to $L^\infty([0,T]; M^{s,\alpha}_{p,q})$ defined by

$$u_0 \to \int_0^t e^{(t-\tau)\Delta}(e^{\tau\Delta}u_0)^2 d\tau. \tag{3.2}$$

is discontinuous in our domain of s,p,q. Actually, if the map is continuous, we will have

$$\sup_{t\in[0,T]}\Big\|\int_0^t e^{(t-\tau)\Delta}(e^{\tau\Delta}u_0)^2 d\tau\Big\|_{M^{s,\alpha}_{p,q}} \preceq \|u_0\|^2_{M^{s,\alpha}_{p,q}}. \tag{3.3}$$

So, we only need to find a u_0 such that (3.3) fails.

First, we consider the case $\sigma(s,p,q) < -\frac{2}{k-1}$. We choose

$$\widehat{u_0}(\xi) = \chi_{[N^{1/(1-\alpha)},3N^{1/(1-\alpha)}]^n}(\xi),$$

where $N \gg 1$. Obviously, the number of $j \in \mathbb{Z}^n$ that satisfy

$$\operatorname{supp}\varphi_j^\alpha \cap [N^{1/(1-\alpha)}, 3N^{1/(1-\alpha)}]^n \ne \emptyset$$

is CN^n. By the definition of $M^{s,\alpha}_{p,q}$, we have

$$\begin{aligned}
\|u_0\|_{M^{s,\alpha}_{p,q}} &= \Big(\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\Box^\alpha_j u_0\|^q_{L^2}\Big)^{1/q}\\
&\preceq \Big(\sum_{j\in\mathbb{Z}^n}\langle N\rangle^{\frac{sq}{1-\alpha}}\|\varphi^\alpha_j\widehat{u_0}\|^q_{L^2}\Big)^{1/q}\\
&\preceq N^{\frac{s}{1-\alpha}+\frac{\alpha}{1-\alpha}\frac{n}{2}+\frac{n}{q}}
\end{aligned}$$

Now, we estimate $\big\|\int_0^t e^{(t-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\big\|_{M^{s,\alpha}_{p,q}}$. It is easy to obtain

$$(\widehat{u_0}*\widehat{u_0})(\eta)=\begin{cases}\prod_{i=1}^n 6N^{\frac{1}{1-\alpha}}-\eta_i, & \eta\in[4N^{\frac{1}{1-\alpha}},6N^{\frac{1}{1-\alpha}}]^n\\ \prod_{i=1}^n \eta_i-2N^{\frac{1}{1-\alpha}}, & \eta\in[2N^{\frac{1}{1-\alpha}},4N^{\frac{1}{1-\alpha}}]^n\\ 0, & \text{otherwise}.\end{cases}$$

Then by taking $t=N^{-\frac{2}{1-\alpha}}$, we obtain

$$\begin{aligned}
&\|\int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{M^{s,\alpha}_{2,q}}\\
&=\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\Box^\alpha_j\int_0^{N^{-\frac{2}{1-\alpha}}} e^{(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{L^2}
\end{aligned}$$

It is easy to see that

$$e^{-\tau|\xi|^2}\geq C>0$$

for $\tau\in[0,N^{-\frac{2}{1-\alpha}}]$ and $\xi\in\operatorname{supp}\widehat{u_0}(\xi)$, and that

$$e^{-(\frac{1}{N^\alpha}-\tau)|\eta|^2}\geq C>0$$

for $\tau\in[0,N^{-\frac{2}{1-\alpha}}]$ and $\eta\in\operatorname{supp}(\widehat{u_0}*\widehat{u_0})(\eta)$.

We denote

$$E_N=[(\frac{5}{2})N^{\frac{1}{1-\alpha}},(\frac{7}{2})N^{\frac{1}{1-\alpha}}]^n\cup[(\frac{9}{2})N^{\frac{1}{1-\alpha}},(\frac{11}{2})N^{\frac{1}{1-\alpha}}]^n.$$

Also, the number of $j\in\mathbb{Z}^n$ which satisfy $\operatorname{supp}\varphi^\alpha_j\cap E_N\neq\emptyset$ is CN^n. By the Plancharel theorem, for such set of j, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{M^{s,\alpha}_{2,q}}\\
&\geq\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\Box^\alpha_j\int_0^{N^{-\frac{2}{1-\alpha}}} e^{(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{L^2}\\
&=\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\varphi^\alpha_j(\xi)\int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)|\xi|^2}\Big\{(e^{\tau|\cdot|^2}\widehat{u_0})*(e^{\tau|\cdot|^2}\widehat{u_0})\Big\}d\tau\|^q_{L^2}\\
&\succeq\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\Big(\int_0^{N^{-\frac{2}{1-\alpha}}}\|\widehat{u_0}*\widehat{u_0}\|_{L^2(E_N\cap\operatorname{supp}\varphi^\alpha_j)}d\tau\Big)^q\\
&\succeq N^{n+\frac{sq}{1-\alpha}-\frac{2q}{1-\alpha}+\frac{\alpha}{1-\alpha}\frac{qn}{2}+\frac{qn}{1-\alpha}}
\end{aligned}$$

So, one has

$$\| \int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2 d\tau\|_{M^{s,\alpha}_{2,q}} \succeq N^{\frac{s}{1-\alpha}+\frac{n}{2}\frac{\alpha}{1-\alpha}+\frac{n}{1-\alpha}+\frac{n}{q}-\frac{2}{1-\alpha}}.$$

Hence, when $\sigma(s,2,q)<-2$, map (3.3) fails to be continuous; this leads to the heat equation (1.1) being ill-posed. $\square$

Next, we consider the case $s<-2$. Here we choose

$$\widehat{u_0}(\xi)=\chi_{[N^{\frac{1}{1-\alpha}}-N^{\frac{\alpha}{1-\alpha}},N^{\frac{1}{1-\alpha}}+N^{\frac{\alpha}{1-\alpha}}]^n}(\xi)\,.$$

Similarly, by the almost orthogonal property of $\{\varphi^\alpha_j\}$(see [13]), the number of $j\in\mathbb{Z}^n$ satisfying $\operatorname{supp}\varphi^\alpha_j\cap\operatorname{supp}\widehat{u_0}\neq\emptyset$ is a constant. Hence, we have

$$\begin{aligned}
\|u_0\|_{M^{s,\alpha}_{p,q}} &= \Big(\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\Box^\alpha_j u_0\|^q_{L^2}\Big)^{1/q}\\
&\preceq \Big(\sum_{j\in\mathbb{Z}^n}\langle N\rangle^{\frac{sq}{1-\alpha}}\|\varphi^\alpha_j\widehat{u_0}\|^q_{L^2}\Big)^{1/q}\\
&\preceq N^{\frac{s}{1-\alpha}+\frac{\alpha}{1-\alpha}\frac{n}{2}}
\end{aligned}$$

Also, by simple calculations, we have

$$(\widehat{u_0}*\widehat{u_0})(\eta)=\begin{cases}\prod_{i=1}^n(2N^{\frac{1}{1-\alpha}}+2N^{\frac{\alpha}{1-\alpha}}-\eta_i), & \eta\in[2N^{\frac{1}{1-\alpha}}-2N^{\frac{\alpha}{1-\alpha}},2N^{\frac{1}{1-\alpha}}]^n,\\ \prod_{i=1}^n(\eta_i-2N^{\frac{1}{1-\alpha}}+2N^{\frac{\alpha}{1-\alpha}}), & \eta\in[2N^{\frac{1}{1-\alpha}},2N^{\frac{1}{1-\alpha}}+2N^{\frac{\alpha}{1-\alpha}}]^n,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\alpha\in[0,1)$, when choose $t=N^{-\frac{2}{1-\alpha}}$, we also have

$$e^{-\tau|\xi|^2}\geq C>0$$

for $\tau\in[0,N^{-\frac{2}{1-\alpha}}]$ and $\xi\in\operatorname{supp}\widehat{u_0}(\xi)$, and that

$$e^{-(\frac{1}{N^\alpha}-\tau)|\eta|^2}\geq C>0$$

for $\tau\in[0,N^{-\frac{2}{1-\alpha}}]$ and $\eta\in supp(\widehat{u_0}*\widehat{u_0})(\eta)$. Fixed $j_0\in\mathbb{Z}^n$ such that $\operatorname{supp}\varphi^\alpha_{j_0}\cap\operatorname{supp}(\widehat{u_0}*\widehat{u_0})(\eta)\neq\emptyset$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^n}\langle j\rangle^{\frac{sq}{1-\alpha}}\|\int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{M^{s,\alpha}_{2,q}}\\
&\geq \langle j_0\rangle^{\frac{sq}{1-\alpha}}\|\Box^\alpha_{j_0}\int_0^{N^{-\frac{2}{1-\alpha}}} e^{(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2d\tau\|^q_{L^2}\\
&= \langle j_0\rangle^{\frac{sq}{1-\alpha}}\|\varphi^\alpha_{j_0}(\xi)\int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)|\xi|^2}\Big\{(e^{\tau|\cdot|^2}\widehat{u_0})*(e^{\tau|\cdot|^2}\widehat{u_0})\Big\}d\tau\|^q_{L^2}\\
&\succeq \langle j_0\rangle^{\frac{sq}{1-\alpha}}\Big(\int_0^{N^{-\frac{2}{1-\alpha}}}\|\widehat{u_0}*\widehat{u_0}\|_{L^2(\operatorname{supp}\varphi^\alpha_{j_0})}d\tau\Big)^q\\
&\succeq N^{\frac{sq}{1-\alpha}-\frac{2q}{1-\alpha}+\frac{\alpha}{1-\alpha}\frac{qn}{2}+\frac{q\alpha n}{1-\alpha}}
\end{aligned}$$

So, we have

$$\Big\| \int_0^{N^{-\frac{2}{1-\alpha}}} e^{-(N^{-\frac{2}{1-\alpha}}-\tau)\Delta}(e^{\tau\Delta}u_0)^2 d\tau \Big\|_{M^{s,\alpha}_{2,q}} \succeq N^{\frac{s}{1-\alpha}+\frac{n}{2}\frac{\alpha}{1-\alpha}+\frac{\alpha n}{1-\alpha}-\frac{2}{1-\alpha}}.$$

Hence, when $s - \frac{n}{2}\alpha < -2$, map (3.3) fail to be continuous; this lead the heat equation (1.1) being ill-posed in $L^\infty([0,T]; M^{s;\alpha}_{2,q})$ for any fixed $T > 0$.

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