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INFINITELY MANY SOLUTIONS FOR SCHRÖDINGER-KIRCHHOFF TYPE EQUATIONS INVOLVING THE FRACTIONAL p-LAPLACIAN AND CRITICAL EXPONENT

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Abstract. In this article, we show the existence of infinitely many solutions for the fractional p-Laplacian equations of Schrödinger-Kirchhoff type equation

$$M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \alpha|u|^{p_s^*-2}u + \beta k(x)|u|^{q-2}u \quad x \in \mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, $[u]_{s,p}$ is the Gagliardo p-seminorm, $0 < s < 1 < p < \infty$, $N > sp$, $1 < q < p$, M is a continuous and positive function, V is a continuous and positive potential function and $k(x)$ is a non-negative function in an appropriate Lebesgue space. By means of the concentration-compactness principle in fractional Sobolev space and Kajikiya's new version of the symmetric mountain pass lemma, we obtain the existence of infinitely many solutions which tend to zero for suitable positive parameters α and β.

1. Introduction and statement of main result

In this article, we consider the following fractional p-Laplacian equations of Schrödinger-Kirchhoff type:

$$\begin{gathered} M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \alpha|u|^{p_s^*-2}u + \beta k(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ [u]_{s,p}^p := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy, \end{gathered} \tag{1.1}$$

where $0 < s < 1 < p < \infty$, $1 < q < p$, $N > sp$, $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent, M, V and k are functions satisfying some suitable conditions which will be given later, $(-\Delta)_p^s$ is the fractional p-Laplace operator which, up to normalization factors, by the Riesz potential as

$$(-\Delta)_p^s u(x) := 2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $B_\epsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \epsilon\}$. Consistent, up to some normalization constant depending upon n and s, with the linear fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. As for some recent results on the fractional p-Laplacian, we refer to for example [15, 37, 38] and the references therein.

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Recently, a great deal of attention has been focused on studying of problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for concrete applications, since they naturally arise in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. For more details, we can see [9, 10, 25] and the references therein.

Problem (1.1) is related to the stationary analogue of the Kirchhoff model

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{p_0}{\lambda}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0$$

which was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation for free vibrations of elastic strings, where ρ, p_0, λ, E, L are constants which represent some physical meanings respectively. Indeed, Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In particular, Kirchhoff's equation models several physical and biological systems, we refer to [2] for more details. Recently, Fiscella and Valdinoci [13] proposed a stationary Kirchhoff model involving the fractional Laplacian by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see [13, Appendix A] for further details.

When $p=2$ and $M\equiv 1$, problem (1.1) becomes the fractional Schrödinger equation with a critical nonlinearity

$$(-\Delta)^s u+V(x)u=\alpha|u|^{p_s^*-2}u+\beta k(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which was first proposed by Laskin in [17, 18] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In recent years, a lot of interesting results about problem (1.2) have been obtained, here we just quote a few, see for example [23, 41, 40].

For our problem, we first assume that the Kirchhoff function $M:\mathbb{R}_0^+\to\mathbb{R}^+$, the potential function $V(x)$ and the weight function $k(x)$ satisfy the following assumptions:

- (A1) $M\in C(\mathbb{R}_0^+,\mathbb{R}^+)$ satisfies $\inf_{t\in\mathbb{R}_0^+}M(t)\geq m_0>0$, where m_0 is a constant.
- (A2) There exists $\theta\in[1,\frac{N}{N-ps})$ such that $\theta\mathcal{M}(t):=\theta\int_0^t M(\tau)\mathrm{d}\tau\geq M(t)t$ for any $t\in\mathbb{R}_0^+$.
- (A3) $V\in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N}V(x)\geq V_0>0$, where $V_0>0$ is a constant.
- (A4) $0\leq k(x)\in L^r(\mathbb{R}^N)$, where $r=\frac{p_s^*}{p_s^*-q}$.

A typical example for M is $M(t)=m_0+b_1t^{\theta-1}$ with $\theta\geq 1$, $m_0\in\mathbb{R}^+$ and $b_1\in\mathbb{R}_0^+$. When M is of this type, the Kirchhoff problem is said to be *non-degenerate* if $m_0>0$, while it is called *degenerate* if $m_0=0$.

Next we state some recent advance related with our problem. First of all, we consider the case that M satisfies (A1) and (A2). Xiang, Zhang and Ferrara [35] studied the existence of solutions for the following Kirchhoff type problem driven by the fractional p-Laplacian operator with homogeneous Dirichlet boundary conditions:

$$\begin{gathered} M([u]_{s,p}^p)(-\Delta)_p^s u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.3}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. By using variational methods, they gave some existence results with respect to $f(x,u) = a(x)|u|^{q-2}u$ with $1 < q < p$ and $p < q < p_s^*$. In [36], Xiang, Zhang and Guo obtained the existence of infinitely many solutions for problem (1.3) with $p = 2$ by applying the fountain theorem and the dual fountain theorem. More precisely, they considered mainly two cases: for any $\lambda \in \mathbb{R}$, the above result holds as $f(x,u) = |u|^{q-2}u + \lambda u$ with $q \in (2, 2_s^*)$; there exists $\Lambda^* > 0$ such that for any for $\lambda \in (0, \Lambda^*)$, the above result holds when $f(x,u) = \alpha|u|^{\xi-2}u + \beta|u|^{\eta-2}u + \lambda u$ for any $\alpha \in \mathbb{R}, \beta > 0$ or for any $\alpha > 0, \beta \in \mathbb{R}$, where $1 < \xi < 2 \le 2\theta < \eta < 2_s^*$, see also [6] for similar applications of the fountain theorem. By appealing to Krasnoselskii's genus theory, Fiscella in [12] obtained the existence of infinitely many solutions for problem (1.3) with $p = 2$ and $f(x,u) = \lambda g(x,u)[\int_\Omega G(x,u)dx]^r + |u|^{2_s^*-2}u$, where $G(x,u) = \int_0^u g(x,\mu)d\mu$, r and λ are positive parameters, see also [11, 24, 21, 30] for similar results involving variational methods. By using a truncation argument and the mountain pass theorem, Autuori, Fiscella and Pucci [3] considered the existence of solutions for problem (1.3) with $f(x,u) = \lambda g(x,u) + |u|^{2_s^*-2}u$ in the degenerate and non-degenerate cases.

On the other hand, Pucci, Xiang and Zhang [29] were concerned with the nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian

$$M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + g(x) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where M satisfies (A1) and (A2), $f(x,u)$ satisfied the subcritical growth. with the help of the Ekeland variational principle and the mountain pass theorem, the authors obtained the existence of at least two solutions for problem (1.4), see also [4] for related results. Subsequently, in [30] they considered the existence and multiplicity of solutions for the equation

$$M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda\omega(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $h(x)$ is a non-negative function satisfying some ratio of integration with $\omega(x)$, $1 < q < r < \infty$, see also [28, 37] for related results. In this case, the existence of infinitely many solutions for problem (1.5) was obtained by genus theory in the degenerate case.

In [32], with the help of classical variational techniques, Servadei consider the existence of infinite solutions for problem (1.3), in which $M \equiv 1$ and $f(x,t) = |u|^{q-2}u$ with $2 < q < (2N-2s)/(N-2s)$, but in presence of a perturbation $h \in L^2(\Omega)$. In [22], by means of the symmetric mountain pass theorem, Molica Bisci obtained the existence of infinite solutions for(1.3) with $M \equiv 1$. Concerning the study of elliptic equations with critical Sobolev exponent, we refer to the seminal works of Brézis and Nirenberg in [8]. In order to overcome the lack of compactness, Lions in [19, 20] developed the concentrate-compactness principle. Based on the principle of concentration compactness in the fractional Sobolev space in [27], Zhang, Zhang and Xiang [41] obtained the existence of ground state solution for problem (1.2) with $\alpha = 1$, see also [40] for extensive discussions for this kind of problem. Xiang, Zhang and Zhang [39] studied the multiplicity of solutions for problem (1.1) in some special cases. For this, they extended the concentrate-compactness principle in [27] to the setting of fractional p-Laplacian. In the context of fractional Laplacian, the discussions about the existence of infinitely many solutions, we also refer to [14].

Motivated by the above works, in the present paper we are interested in the existence of infinitely many solutions for problem (1.1) by means of Kajikiya's new version of the symmetric mountain pass lemma. To our best knowledge, there is no result in the literature on problem (1.1). There is no doubt that we encounter serious difficulties because of the lack of compactness and of the nonlocal nature of the fractional p-Laplacian. To this end, we will use the concentrate-compactness principle in [39] to conquer the difficulty due to the lack of compactness.

Now we first give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W$ is a weak solution of problem (1.1), if $u \in W$ and

$$\begin{aligned}
&M([u]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&+\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi\mathrm{d}x\\
&=\alpha\int_{\mathbb{R}^N}|u|^{p_s^*-2}u\varphi\mathrm{d}x+\beta\int_{\mathbb{R}^N}k(x)|u|^{q-2}u\varphi\mathrm{d}x
\end{aligned}$$

for all $\varphi \in W$.

In the sequel we will omit the term weak when referring to solutions that satisfy the conditions of Definition 1.1. Our main result of this paper is stated as follows.

Theorem 1.2. *Let* (A1)–(A4) *and* $1 < q < p$ *hold. Then*

- (i) *for all* $\alpha > 0$ *there exists* $\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$, *then* (1.1) *has a sequence of solutions* $\{u_n\}_n$ *with* $I(u_n) < 0$, $I(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n \to 0$.
- (ii) *for all* $\beta > 0$ *there exists* $\alpha_0 > 0$ *such that if* $0 < \alpha < \alpha_0$, *then* (1.1) *has a sequence of solutions* $\{u_n\}_n$ *with* $I(u_n) < 0$, $I(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n \to 0$.

Remark 1.3. From Theorem 1.2 it is natural to raise the following open problems: (i) What if $\theta p < q < p_s^*$? (ii) Are our result still valid in the degenerate case? These problems would be investigated by the authors in future works.

The rest of this paper is organized as follows. The functional framework and some preliminaries are given in Section 2. In Section 3, behavior of (PS) sequences are established. The proof of the main result Theorem 1.2 is given in Section 4.

$L^t(\mathbb{R}^N)$ is the usual Lebesgue space with the norm $\|u\|_p^p = \int_{\mathbb{R}^N} |u|^p\mathrm{d}x, 1 \le p < +\infty$. Various positive constants are denoted by C and C_i.

2. Preliminaries

In this section, we first give some basic results of fractional Sobolev space and then provide some useful technical lemmas, which will be used in the sequel.

Let $0 < s < 1 < p < \infty$ be real numbers. The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \Big)^{1/p}. \tag{2.1}$$

The fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) = \big\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y < \infty \big\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \Big(\|u\|_p^p + [u]_{s,p}^p \Big)^{1/p}. \tag{2.2}$$

As it is well-known that this space is a uniformly convex Banach space. For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$, we refer to [10].

Let W denote the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$\|u\|_W := \Big([u]_{s,p}^p + \|u\|_{p,V}^p \Big)^{1/p}, \quad \|u\|_{p,V}^p = \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x. \tag{2.3}$$

Clearly the definition makes sense since every $\varphi \in C_0^\infty(\mathbb{R}^N)$ has finite Gagliardo norm as well finite norm $\|\varphi\|_{p,V}$. Indeed, $L^p(\mathbb{R}^N, V) = (L^p(\mathbb{R}^N, V), \|\cdot\|_{p,V})$ is a uniformly convex Banach space thanks to $(V1)$. By standard arguments, it is clear that W is a uniformly convex Banach space, see [29, Lemma 10] for details. The embedding $W \hookrightarrow L^t(\mathbb{R}^N)$ is continuous for any $t \in [p, p_s^*]$ by [10, Theorem 6.7]; that is, there exists a positive constant C_* such that

$$\|u\|_{L^t(\mathbb{R}^N)} \le C_* \|u\|_W \quad \text{for all } u \in W. \tag{2.4}$$

In our context, the Sobolev constant is given by

$$S := \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y}{(\int_{\mathbb{R}^N} |u|^{p_s^*} \mathrm{d}x)^{p/p_s^*}} \tag{2.5}$$

is the associated Rayleigh quotient. The constant S is well defined, as can be seen in [1, Theorem 7.58].

Next we recall the the concentration-compactness principle in the setting of the fractional p-Laplacian, see [39, Definition 2.1,Theorem 2.1 and Theorem 2.2].

Definition 2.1. Let $\widetilde{\mathcal{M}}(\mathbb{R})$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$. For any $\mu \in \widetilde{\mathcal{M}}(\mathbb{R}^N), \mu(\mathbb{R}^N) = \|\mu\|$ holds. We say that $\mu \rightharpoonup \mu$ weakly $*$ in $\widetilde{\mathcal{M}}(\mathbb{R}^N)$, if $(\mu_n, \eta) \to (\mu, \eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Proposition 2.2. *Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ with upper bound $C > 0$ for all $n \ge 1$ and*

$$\begin{gathered} u_n \rightharpoonup u \quad \textit{weakly in } D^{s,p}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y \rightharpoonup \mu \quad \textit{weak} * \textit{in } \widetilde{\mathcal{M}}(\mathbb{R}^N), \\ |u_n(x)|^{p_s^*} \rightharpoonup \nu \quad \textit{weak} * \textit{in } \widetilde{\mathcal{M}}(\mathbb{R}^N). \end{gathered}$$

Then

$$\mu = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y + \sum_{j \in \mathcal{J}} \mu_j \delta_{x_j} + \tilde{\mu}, \quad \mu(\mathbb{R}^N) \le C^p,$$

$$\nu = |u|^{p_s^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \quad \nu(\mathbb{R}^N) \le S^{p_s^*} C^p,$$

where J is at most countable, sequences $\{\mu_j\}_j, \{\nu_j\}_j \subset \mathbb{R}_0^+, \{x_j\}_j \subset \mathbb{R}^N, \delta_{x_j}$ is the Dirac mass centered at x_j, $\tilde{\mu}$ is a non-atomic measure,

$$\nu(\mathbb{R}^N) \le S^{-\frac{p_s^*}{p}} \mu(\mathbb{R}^N)^{\frac{p_s^*}{p}},$$
$$\nu_j \le S^{-\frac{p_s^*}{p}} \mu_j^{\frac{p_s^*}{p}}, \quad \forall j \in J,$$

and $S > 0$ is the best constant of $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$

Proposition 2.3. *Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y \rightharpoonup \mu \quad \text{weak} * \text{ in } \widetilde{\mathcal{M}}(\mathbb{R}^N),$$
$$|u_n(x)|^{p_s^*} \rightharpoonup \nu \quad \text{weak} * \text{ in } \widetilde{\mathcal{M}}(\mathbb{R}^N),$$

and define

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y\mathrm{d}x,$$
$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} |u_n|^{p_s^*} \mathrm{d}x.$$

Then the quantities μ_∞ and ν_∞ are well defined and satisfy

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N} \mathrm{d}\mu + \mu_\infty,$$
$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} \mathrm{d}x = \int_{\mathbb{R}^N} \mathrm{d}\nu + \nu_\infty.$$

Moreover,

$$S\nu_\infty^{\frac{p}{p_s^*}} \le \mu_\infty.$$

Lemma 2.4 ([39, Lemma 2.3])**.** *Assume $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ is the sequence given by Lemma 3.1 and for $\varepsilon > 0$, let $\phi_j(x)$ be a smooth cut-off function centered at x_j such that $0 \le \phi_j(x) \le 1$, $\phi_j(x) \equiv 0$ on $|x - x_j| \ge 2$, $\phi_j(x) \equiv 1$ on $|x - x_j| \le 1$, and $|\nabla\phi_j(x)| \le 2$ for all $x \in \mathbb{R}^N$. Set $\phi_j^\varepsilon(x) = \phi_j(x/\varepsilon)$ for all $x \in \mathbb{R}^N$. Then*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\phi_j^\varepsilon(x) - \phi_j^\varepsilon(y)|^p |u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = 0.$$

Lemma 2.5 ([29, Lemma 2])**.** *Let* (**M1**) *and* (**V1**) *hold. Then $J : W \to \mathbb{R}$ is of class $C^1(W)$ and*

$$\begin{aligned}\langle J'(u), v\rangle = M([u]_{s,p}^p) &\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2} u(x)v(x)\mathrm{d}x,\end{aligned}$$

for all $u, v \in W$. Moreover, J is weakly lower semi-continuous in W.

Lemma 2.6 ([33, Lemma 2.1]). *For each k in $L^r(\mathbb{R}^N)$, the functional $\mathcal{F}(u) = \int_{\mathbb{R}^N} k(x)|u|^q \mathrm{d}x$ is well defined and weakly continuous on W. Moreover, $\mathcal{F}(u)$ is continuously differentiable, its derivative $\mathcal{F}' : W \to W^*$ is given by*

$$\langle \mathcal{F}'(u), \varphi\rangle = q \int_{\mathbb{R}^N} k(x)|u|^{q-2}u\varphi \mathrm{d}x, \quad \forall \varphi \in W.$$

Lemma 2.7 ([29, Theorem2.1]). *Let (V1) hold. Let $\vartheta \in [p, p_s^*)$ be a fixed exponent and let $\{v_j\}_j$ be a bounded sequence in W. Then there exists $v \in W \cap L^\vartheta(\mathbb{R}^N)$ such that up to a subsequence, $v_j \to v$ strongly in $L^\vartheta(\mathbb{R}^N)$ as $j \to \infty$.*

3. Behavior of (PS) sequences

In this section, we perform a careful analysis of the behavior of minimizing sequences with the aid of the concentration–compactness principle in fractional Sobolev space stated above, which allows to recover compactness below some critical threshold.

Let E be a real Banach space and $I : E \to \mathbb{R}$ be a function of class C^1. We say that $\{u_n\}_n \subset E$ is a $(PS)_c$ sequence if $I(u_n) \to c$ and $I'(u_n) \to 0$. I is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if any $(PS)_c$ sequence contains a convergent subsequence.

Lemma 3.1. *Let (A1)–(A4), $1 < q \le p$ and $c < 0$ hold. Then*

- (i) *there exists $C > 0$ such that, for all $n \in \mathbb{N}$, $\|u_n\|_W \le C$;*
- (ii) *for each $\alpha > 0$ there exists $\beta_* > 0$ such that if $0 < \beta < \beta_*$, then I satisfies $(PS)_c$;*
- (iii) *for each $\beta > 0$ there exists $\alpha_* > 0$ such that if $0 < \alpha < \alpha_*$, then I satisfies $(PS)_c$.*

Proof. We first prove that $\{u_n\}_n$ is bounded in W. Let $\{u_n\}_n$ be a $(PS)_c$ sequence in W such that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} c + o_n(\|u_n\|_W) &= I(u_n) \\ &= \frac{1}{p}\Big[\mathcal{M}([u_n]_{s,p}^p) + \|u_n\|_{p,V}^p\Big] - \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x - \frac{\beta}{q}\int_{\mathbb{R}^N} k(x)|u_n|^q \mathrm{d}x. \end{aligned} \tag{3.1}$$

$$\begin{aligned} o_n(\|u_n\|_W) &= \langle I'(u_n), u_n\rangle \\ &= M([u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x \\ &\quad - \alpha\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x - \beta\int_{\mathbb{R}^N} k(x)|u_n|^q \mathrm{d}x. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} 0 > c + o_n(\|u_n\|_W) &= I(u_n) - \frac{1}{p_s^*}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{p}\mathcal{M}([u_n]_{s,p}^p) - \frac{1}{p_s^*}M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|u_n\|_{p,V}^p \\ &\quad - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\int_{\mathbb{R}^N} k(x)|u_n|^q \mathrm{d}x \\ &\ge \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|u_n\|_{p,V}^p \end{aligned}$$

$$
\begin{aligned}
&\quad - \beta\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)\|k(x)\|_r\Big(\int_{\mathbb{R}^N}|u_n|^{p_s^*}\mathrm{d}x\Big)^{\frac{q}{p_s^*}}\\
&\ge \Big(\frac{1}{p\theta}-\frac{1}{p_s^*}\Big)m_0[u_n]_{s,p}^p+\Big(\frac{1}{p}-\frac{1}{p_s^*}\Big)\|u_n\|_{p,V}^p\\
&\quad - \beta\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)\|k(x)\|_r S^{-\frac{q}{p}}[u_n]_{s,p}^q\\
&\ge \min\Big\{\Big(\frac{1}{p\theta}-\frac{1}{p_s^*}\Big)m_0,\Big(\frac{1}{p}-\frac{1}{p_s^*}\Big)\Big\}\|u_n\|_W^p\\
&\quad - \beta\Big(\frac{1}{q}-\frac{1}{p_s^*}\Big)\|k(x)\|_r S^{-\frac{q}{p}}\|u_n\|_W^q.
\end{aligned}
$$

Since $\theta\in[1,\frac{N}{N-ps})$ and $q<p$, it follows that $\{u_n\}_n$ is bounded in W. Then, there exist u_0 and a subsequence, still denoted by $\{u_n\}_n\subset W$, such that

$$
\begin{gathered}
u_n \rightharpoonup u_0 \quad \text{weakly in } W,\\
u_n \to u_0 \quad \text{strongly in } L^t_{\rm loc}(\mathbb{R}^N) \quad \text{for all } t\in[1,p_s^*),\\
u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}
$$

From Proposition 2.2, we have

$$
\begin{gathered}
u_n \rightharpoonup u_0 \quad \text{weakly in } D^{s,p}(\mathbb{R}^N),\\
\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}y \rightharpoonup \mu \quad \text{weakly} * \text{ in } \widetilde{\mathcal{M}}(\mathbb{R}^N),\\
|u_n(x)|^{p_s^*} \rightharpoonup \nu \quad \text{weak} * \text{ in } \widetilde{\mathcal{M}}(\mathbb{R}^N).
\end{gathered}
$$

Then

$$
\begin{gathered}
\mu=\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}y+\sum_{j\in\mathcal{J}}\mu_j\delta_{x_j}+\tilde{\mu},\quad \mu(\mathbb{R}^N)\le C^p,\\
\nu=|u|^{p_s^*}+\sum_{j\in\mathcal{J}}\nu_j\delta_{x_j},\quad \nu(\mathbb{R}^N)\le S^{p_s^*}C^p,
\end{gathered}
$$

where J is at most countable, sequences $\{\mu_j\}_j,\{\nu_j\}_j\subset\mathbb{R}_0^+$, $\{x_j\}_j\subset\mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\tilde{\mu}$ is a non-atomic measure,

$$
\nu(\mathbb{R}^N)\le S^{-\frac{p_s^*}{p}}\mu(\mathbb{R}^N)^{\frac{p_s^*}{p}}, \tag{3.3}
$$

$$
\nu_j\le S^{-\frac{p_s^*}{p}}\mu_j^{\frac{p_s^*}{p}},\quad \forall j\in J, \tag{3.4}
$$

Concentration at infinity of the sequence $\{u_n\}_n$ is described by the following quantities:

$$
\begin{gathered}
\mu_\infty:=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}y\mathrm{d}x,\\
\nu_\infty:=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}|u_n|^{p_s^*}\mathrm{d}x,
\end{gathered}
$$

We claim that $\mathcal{J}$ is finite and, for $j\in\mathcal{J}$, either $\nu_j=0$ or $\nu_j\ge(m_0\alpha^{-1}S)^{N/ps}$.

In fact, for $\varepsilon>0$, let $\phi_j^\varepsilon(x)$ be a smooth cut-off function centered at x_j, such that $0\le\phi_j^\varepsilon(x)\le1$, $\phi_j^\varepsilon(x)\equiv0$ on $|x-x_j|\ge2\varepsilon$, $\phi_j^\varepsilon(x)\equiv1$ on $|x-x_j|\le\varepsilon$, and

$|\nabla\phi_j^\varepsilon(x)| \le \frac{2}{\varepsilon}$ for all $x \in \mathbb{R}^N$. Then, it is seen that $\{u_n\phi_j^\varepsilon\}$ is bounded in W. Testing $I'(u_n)$ with $u_n\phi_j^\varepsilon$, we obtain $\lim_{n\to\infty}\langle I'(u_n),\ u_n\phi_j^\varepsilon\rangle = 0$; that is,

$$
\begin{aligned}
&M([u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}}\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\
&\times(u_n(x)\phi_j^\varepsilon(x)-u_n(y)\phi_j^\varepsilon(y))/|x-y|^{N+ps}\Big)\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}V(x)|u_n|^p\phi_j^\varepsilon\mathrm{d}x\\
&-\alpha\int_{\mathbb{R}^N}|u_n|^{p_s^*}\phi_j^\varepsilon\mathrm{d}x-\beta\int_{\mathbb{R}^N}k(x)|u_n|^q\phi_j^\varepsilon(x)\mathrm{d}x=0.
\end{aligned}
\tag{3.5}
$$

Next we estimate each term in (3.5).

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n(x)\phi_j^\varepsilon(x)-u_n(y)\phi_j^\varepsilon(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&=\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\phi_j^\varepsilon(x)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&\quad+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_j^\varepsilon(x)-\phi_j^\varepsilon(y))u_n(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.
\end{aligned}
\tag{3.6}
$$

In fact, in the first double integral of the right-hand side of (3.5), we can use a compactness result (see Proposition 2.2),

$$
\begin{aligned}
&\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\phi_j^\varepsilon(x)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N}\phi_j^\varepsilon\mathrm{d}\mu=\iint_{\mathbb{R}^{2N}}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}\phi_j^\varepsilon(x)\mathrm{d}x\mathrm{d}y+\mu_j.
\end{aligned}
\tag{3.7}
$$

For the second double integral of the right-hand side of (3.6), we obtain

$$
\begin{aligned}
&\Big|\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_j^\varepsilon(x)-\phi_j^\varepsilon(y))u_n(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big|\\
&\le\Big(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{p-1}{p}}\\
&\quad\times\Big(\iint_{\mathbb{R}^{2N}}\frac{|\phi_j^\varepsilon(x)-\phi_j^\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big)^{1/p}\\
&\le C\Big(\iint_{\mathbb{R}^{2N}}\frac{|\phi_j^\varepsilon(x)-\phi_j^\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{p-1}{p}}.
\end{aligned}
\tag{3.8}
$$

By Lemma 2.4, we obtain

$$
\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|\phi_j^\varepsilon(x)-\phi_j^\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y=0.
\tag{3.9}
$$

So from (3.6), (3.7) and (3.9), we deduce

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\lim_{n\to\infty}M[u_n]_{s,p}^p\iint_{\mathbb{R}^{2N}}\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\
&\times\big(u_n(x)\phi_j^\varepsilon(x)-u_n(y)\phi_j^\varepsilon(y)\big)/|x-y|^{N+ps}\Big)\mathrm{d}x\mathrm{d}y\\
&\ge m_0\mu_j.
\end{aligned}
\tag{3.10}
$$

Also we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^p\phi_j^\varepsilon \mathrm{d}x=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_{2\varepsilon}(x_j)} V(x)|u_n|^p\phi_j^\varepsilon \mathrm{d}x=0, \tag{3.11}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\phi_j^\varepsilon \mathrm{d}x=\int_{\mathbb{R}^N}\phi_j^\varepsilon \mathrm{d}\nu=\int_{\mathbb{R}^N}|u_0|^{p_s^*}\phi_j^\varepsilon \mathrm{d}x+\nu_j. \tag{3.12}$$

By assumption (A4), we arrive at

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} k(x)|u_n|^q\phi_j^\varepsilon \mathrm{d}x&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_{2\varepsilon}(x_j)} k(x)|u_n|^q\phi_j^\varepsilon \mathrm{d}x\\&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\|k(x)\|_{L^r(B_{2\varepsilon}(x_j))}\|u_n\|^q_{L^{p_s^*}(B_{2\varepsilon}(x_j))}\\&=0.\end{aligned}\tag{3.13}$$

Therefore, from (3.5) and the aforementioned arguments we obtain

$$0\ge m_0\mu_j-\alpha\nu_j. \tag{3.14}$$

Combining this with (3.4), we obtain either (i) $\nu_j=0$ or (ii) $\nu_j\ge (m_0\alpha^{-1}S)^{\frac{N}{ps}}$, which implies that $\mathcal{J}$ is finite. The claim is thereby proved.

To analyze the concentration at ∞, by choosing a suitable cut-off function $\varphi\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\varphi(x)\equiv 0$ on $|x|\le 1$ and $\varphi(x)\equiv 1$ on $|x|\ge 2$. We set $\varphi_R(x)=\varphi(\frac{x}{R})$, then $\{u_n\varphi_R\}_n$ is bounded in W, and $\lim_{n\to\infty}\langle I'(u_n),u_n\varphi_R\rangle=0$; that is,

$$\begin{aligned}&M([u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}}\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\&\times(u_n(x)\varphi_R(x)-u_n(y)\varphi_R(y))/|x-y|^{N+ps}\Big)\mathrm{d}x\mathrm{d}y\\&+\int_{\mathbb{R}^N}V(x)|u_n|^p\varphi_R\mathrm{d}x-\alpha\int_{\mathbb{R}^N}|u_n|^{p_s^*}\varphi_R\mathrm{d}x-\beta\int_{\mathbb{R}^N}k(x)|u_n|^q\varphi_R\mathrm{d}x=0.\end{aligned}\tag{3.15}$$

Next we estimate each term in (3.15).

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n(x)\varphi_R(x)-u_n(y)\varphi_R(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\&=\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\varphi_R(x)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\&\quad+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_R(x)-\varphi_R(y))u_n(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{3.16}$$

Indeed, in the first double integral of the right-hand side of (3.16), we can use a compactness result (see Proposition 2.3),

$$\begin{aligned}&\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\varphi_R(x)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}^N}\varphi_R\mathrm{d}\mu\\&=\iint_{\mathbb{R}^{2N}}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}\varphi_R(x)\mathrm{d}x\mathrm{d}y+\mu_\infty.\end{aligned}\tag{3.17}$$

For the second double integral of the right-hand side of (3.16), it follows from the Hölder inequality that

$$\begin{aligned}&\Big|\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_R(x)-\varphi_R(y))u_n(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big|\\&\le C\Big(\iint_{\mathbb{R}^{2N}}\frac{|\varphi_R(x)-\varphi_R(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{p-1}{p}}.\end{aligned}\tag{3.18}$$

then, as in the proof of (3.9) we obtain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|\varphi_R(x)-\varphi_R(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y=0.$$

So we have

$$\begin{aligned}&\lim_{R\to\infty}\lim_{n\to\infty}M[u_n]_{s,p}^p\iint_{\mathbb{R}^{2N}}\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\&\times(u_n(x)\varphi_R(x)-u_n(y)\varphi_R(y))/|x-y|^{N+ps}\Big)\mathrm{d}x\mathrm{d}y\\&\ge m_0\mu_\infty.\end{aligned}\tag{3.19}$$

We also obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}V(x)|u_n|^p\varphi_R\mathrm{d}x=\lim_{R\to\infty}\lim_{n\to\infty}\int_{\{|x|>2R\}}V(x)|u_n|^p\varphi_R\mathrm{d}x=0,\tag{3.20}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\varphi_R\mathrm{d}x=\int_{\mathbb{R}^N}\varphi_R\mathrm{d}\nu=\int_{\mathbb{R}^N}|u_0|^{p_s^*}\varphi_R\mathrm{d}x+\nu_\infty.\tag{3.21}$$

By the weak continuity of $k(x)$, Hölder inequality and the definition of S,

$$\begin{aligned}\int_{\mathbb{R}^N}k(x)|u_n|^q\varphi_R\mathrm{d}x&\le\Big(\int_{\{|x|>2R\}}|u_n|^{p_s^*}\mathrm{d}x\Big)^{\frac{q}{p_s^*}}\Big(\int_{\{|x|>2R\}}|k(x)|^{\frac{p_s^*}{p_s^*-q}}\mathrm{d}x\Big)^{\frac{p_s^*-q}{p_s^*}}\\&\le S^{-\frac{q}{p_s^*}}[u_n]_{s,p}^q\Big(\int_{\{|x|>2R\}}|k(x)|^{\frac{p_s^*}{p_s^*-q}}\mathrm{d}x\Big)^{\frac{p_s^*-q}{p_s^*}}\\&\le S^{-\frac{q}{p_s^*}}\|u_n\|_W^q\Big(\int_{\{|x|>2R\}}|k(x)|^{\frac{p_s^*}{p_s^*-q}}\mathrm{d}x\Big)^{\frac{p_s^*-q}{p_s^*}},\end{aligned}$$

which implies

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}k(x)|u_n|^q\varphi_R\mathrm{d}x\le C\lim_{R\to\infty}\Big(\int_{\{|x|>2R\}}|k(x)|^{\frac{p_s^*}{p_s^*-q}}\mathrm{d}x\Big)^{\frac{p_s^*-q}{p_s^*}}=0.$$

Therefore, by letting $R\to\infty$ and $n\to\infty$ in (3.15), we have

$$m_0\mu_\infty\le\alpha\nu_\infty.\tag{3.22}$$

By Proposition 2.3 and (3.22), we conclude that either

(iii) $\nu_\infty=0$, or
(iv) $\nu_\infty\ge(m_0\alpha^{-1}S)^{\frac{N}{ps}}$.

Next, we claim that (ii) and (iv) cannot occur if α and β are chosen properly. To this, from the Hölder inequality and the weak continuity of $\mathcal{F}$, we have

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\Big[I(u_n) - \frac{1}{p_s^*}\langle I'(u_n), u_n\rangle\Big] - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\int_{\mathbb{R}^N} k(x)|u_n|^q \mathrm{d}x\Big] \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)M([u_0]_{s,p}^p)[u_0]_{s,p}^p + \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|u_0\|_{p,V}^p \\
&\quad - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\|k(x)\|_r\Big(\int_{\mathbb{R}^N}|u_0|^{p_s^*}\mathrm{d}x\Big)^{\frac{q}{p_s^*}} \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)m_0[u_0]_{s,p}^p + \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|u_0\|_{p,V}^p \\
&\quad - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\|k(x)\|_r S^{-\frac{q}{p}}[u_0]_{s,p}^{\frac{q}{p}} \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)m_0 S\|u_0\|_{p_s^*}^p - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\|k(x)\|_r\|u_0\|_{p_s^*}^q.
\end{aligned} \tag{3.23}$$

Thus, it follows that

$$\|u_0\|_{p_s^*} \leq C\beta^{\frac{1}{p-q}}. \tag{3.24}$$

If (iv) occurs, we obtain by (3.24) that

$$\begin{aligned}
0 > c &= \lim_{R\to\infty}\lim_{n\to\infty}\Big[I(u_n) - \frac{1}{p_s^*}\langle I'(u_n), \varphi_R\rangle\Big] \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)m_0\mu_\infty + \Big(\frac{1}{p} - \frac{1}{p_s^*}\Big)\|u_0\|_{p,V}^p - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\|k(x)\|_r\|u_0\|_{p_s^*}^q \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)m_0\mu_\infty - \beta\Big(\frac{1}{q} - \frac{1}{p_s^*}\Big)\|k(x)\|_r C\beta^{\frac{q}{p-q}} \\
&\geq \Big(\frac{1}{p\theta} - \frac{1}{p_s^*}\Big)m_0\alpha^{-\frac{N}{ps}}S^{\frac{N}{ps}} - C\beta^{\frac{p}{p-q}}\,.
\end{aligned}$$

However, since $\theta \in [1, \frac{N}{N-ps}), q < p$, if $\alpha > 0$ is given, we can take small β_* such that for every $0 < \beta < \beta_*$, the term on the right-hand side above is greater than zero, which is a contradiction. Similarly, if $\beta > 0$ is given, we can choose small α_* such that for every $0 < \alpha < \alpha_*$, the term on the right-hand side above is greater than zero. Similarly, we can prove that (ii) cannot occur. Hence

$$\int_{\mathbb{R}^N}|u_n|^{p_s^*}\mathrm{d}x \to \int_{\mathbb{R}^N}|u_0|^{p_s^*}\mathrm{d}x \quad \text{as } n\to\infty.$$

In view of $u_n \rightharpoonup u_0$ in $W^{s,p}(\mathbb{R}^N)$ and the Brézis-Lieb lemma, we have

$$\int_{\mathbb{R}^N}|u_n - u_0|^{p_s^*}\mathrm{d}x \to 0 \quad \text{as } n\to\infty. \tag{3.25}$$

We are now in a position to show that $\{u_n\}_n$ converges strongly to u_0 in W. Firstly, we have

$$\langle I'(u_n) - I'(u_0), u_n - u_0\rangle \to 0 \quad \text{as } n\to\infty.$$

By the boundedness of $\{u_n\}_n$ in W and (3.25), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|u_n|^{p_s^*-2}|u_n| - |u_0|^{p_s^*-2}u_0)(u_n-u_0)\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} |u_n|^{p_s^*-1}(u_n-u_0)\mathrm{d}x + \int_{\mathbb{R}^N} |u_0|^{p_s^*-1}(u_n-u_0)\mathrm{d}x \\
&\le \Big(\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x\Big)^{\frac{p_s^*-1}{p_s^*}} \Big(\int_{\mathbb{R}^N} |u_n-u_0|^{p_s^*}\mathrm{d}x\Big)^{1/p_s^*} \\
&\quad + \Big(\int_{\mathbb{R}^N} |u_0|^{p_s^*}\mathrm{d}x\Big)^{\frac{p_s^*-1}{p_s^*}} \Big(\int_{\mathbb{R}^N} |u_n-u_0|^{p_s^*}\mathrm{d}x\Big)^{1/p_s^*} \to 0,
\end{aligned} \tag{3.26}$$

as $n \to \infty$. Since $k \in L^r(\mathbb{R}^N)$, by the weak lower continuity of $\mathcal{F}$, we have

$$\int_{\mathbb{R}^N} k(x)|u_n|^{q-2}u_n(u_n-u_0)\mathrm{d}x \le \|k(x)\|_{L^r(\mathbb{R}^N)}\|u_n\|_{p_s^*}^{q-1}\|u_n-u_0\|_{p_s^*} \to 0,$$

as $n \to \infty$. Therefore, as $n \to \infty$, we have

$$[u_n - u_0]_{s,p} \to 0,$$

thanks to $I'(u_0) = 0$. By Lemma 2.7, as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n(u_n-u_0)\mathrm{d}x \to 0.$$

Thus we prove that $\{u_n\}_n$ strongly converges to u_0 in W. □

4. Proof of Theorem 1.2

In this section, we use minimax procedure (see [31]) to prove the existence of infinitely many solutions. Let X be a Banach space and Σ be the class of subsets of $X \setminus \{0\}$ which are closed and symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \inf\{n \in \mathbb{N} : \exists \phi \in C(A, \mathbb{R}^n \setminus \{0\}), \phi(z) = -\phi(-z)\},$$
$$N_\delta(A) = \{x \in X : \mathrm{dist}(x-A) \le \delta\}, \quad \mathrm{dist}(x-A) = \inf\{\|x-A\| : y \in A\}.$$

If there is no mapping as above for any $n \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Σ_n denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \ge n$. We summarize the property of genus, which will be used in the proof of Theorem 1.2. We refer the readers to [31] for the proof of the next result.

Proposition 4.1. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

1. *If there exists an odd continuous mapping from A to B, then $\gamma(A) \le \gamma(B)$;*
2. *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$;*
3. *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \ge \gamma(A) - \gamma(B)$;*
4. *n-dimensional sphere S_n has a genus of $n+1$ by the Borsuk-Ulam Theorem;*
5. *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [16].

Proposition 4.2. *Let E be an infinite-dimensional space and $I \in C^1(E, \mathbb{R})$ and suppose the following conditions hold.*

(A5) *$I(u)$ is even, bounded from below, $I(0)=0$ and $I(u)$ satisfies the local Palais-Smale condition (PS for short).*

(A6) *For each $n\in\mathbb{N}$, there exists an $A_n\in\Sigma_n$ such that $\sup_{u\in A_n} I(u)<0$.*

Then either

(i) *There exists a sequence $\{u_n\}$ such that $I'(u_n)=0, I(u_n)<0$ and $\{u_n\}$ converges to zero, or*

(ii) *There exist two sequences $\{u_n\}$ and $\{v_n\}$ such that $I'(u_n)=0, I(u_n)=0$, $u_n\neq 0$, $\lim_{n\to\infty} u_n=0$; $I'(v_n)=0$, $I(v_n)<0$, $\lim_{n\to\infty} I(v_n)=0$, and $\{v_n\}$ converges to a non-zero limit.*

Remark 4.3. From Proposition 4.2 we have a sequence $\{u_n\}_n$ of critical points such that $I(u_n)\le 0, u_n\neq 0$ and $\lim_{n\to\infty} u_n=0$.

To obtain infinitely many solutions, we need some technical lemmas. Let $I(u)$ be the functional defined as above, $1<q<p$, $\alpha>0$ and $\beta>0$. Then

$$\begin{aligned}
I(u)&=\frac{1}{p}\big[\mathcal{M}([u]_{s,p}^p)+\|u\|_{p,V}^p\big]-\frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\mathrm{d}x-\frac{\beta}{q}\int_{\mathbb{R}^N}k(x)|u|^q\mathrm{d}x\\
&\ge\frac{1}{p\theta}M([u]_{s,p}^p)[u]_{s,p}^p+\frac{1}{p}\|u\|_{p,V}^p-\frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\mathrm{d}x-\frac{\beta}{q}\int_{\mathbb{R}^N}k(x)|u|^q\mathrm{d}x\\
&\ge\frac{1}{p\theta}m_0[u]_{s,p}^p-\frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\mathrm{d}x-\frac{\beta}{q}\|k(x)\|_r\|u\|_{p_s^*}^q\\
&\ge\frac{1}{p\theta}m_0[u]_{s,p}^p-\frac{\alpha}{p_s^*}\Big(S^{-1}[u]_{s,p}^p\Big)^{\frac{p_s^*}{p}}-\frac{\beta}{q}\|k(x)\|_r\Big(S^{-1}[u]_{s,p}^p\Big)^{\frac{q}{p}}\\
&\ge C_1[u]_{s,p}^p-\alpha C_2[u]_{s,p}^{p_s^*}-\beta C_3[u]_{s,p}^q.
\end{aligned}$$

Define

$$g(t)=C_1t^p-\alpha C_2t^{p_s^*}-\beta C_3t^q.$$

Then, it is easy to see that, for the given $\alpha>0$, we can choose $\beta^*>0$ so small that if $0<\beta<\beta^*$, there exists $0<t_0<t_1$ such that $g(t)<0$ for $0<t<t_0$; $g(t)>0$ for $t_0<t<t_1$; $g(t)<0$ for $t>t_1$.

Similarly, for the given $\beta>0$, we can choose $\alpha^*>0$ so small that if $0<\alpha<\alpha^*$, there exists $0<t_0^*<t_1^*$ such that $g(t)<0$ for $0<t<t_0^*$; $g(t)>0$ for $t_0^*<t<t_1^*$; $g(t)<0$ for $t>t_1^*$.

Clearly, $g(t_0)=0=g(t_1)$. Following the same idea as in [5], we consider the truncated functional

$$\tilde I(u)=\frac{1}{p}\Big[\mathcal{M}([u]_{s,p}^p)+\|u\|_{p,V}^p\Big]-\frac{\alpha}{p_s^*}\psi(u)\int_{\mathbb{R}^N}|u|^{p_s^*}\mathrm{d}x-\frac{\beta}{q}\int_{\mathbb{R}^N}k(x)|u|^q\mathrm{d}x.$$

where $\psi(u)=\tau(\|u\|_W)$ and $\tau:\mathbb{R}^+\to[0,1]$ is a non-increasing C^∞ function such that $\tau(t)=1$ if $t\le t_0$ and $\tau(t)=0$ if $t\ge t_1$. Obviously, $\tilde I(u)$ is even. Thus, from Lemma 3.1, we obtain the following lemma.

Lemma 4.4. *Let $c<0$ and $1<q<p$. Then*

(1) *$\tilde I\in C^1$ and $\tilde I$ is bounded from below.*

(2) *If $\tilde I(u)<0$, then $\|u\|_W<t_0$ and $\tilde I(u)=I(u)$.*

(3) *for each $\alpha>0$ there exists $\widetilde{\beta^*}=\min\{\beta_*,\beta^*\}>0$ such that if $0<\beta<\widetilde{\beta^*}$, then $\tilde I$ satisfies $(PS)_c$;*

(4) *for each* $\beta > 0$ *there exists* $\widetilde{\alpha^*} = \min\{\alpha_*, \alpha^*\} > 0$ *such that if* $0 < \alpha < \widetilde{\alpha^*}$, *then* $\widetilde{I}$ *satisfies* $(PS)_c$.

Proof. Obviously, (1) and (2) are immediate. To prove (3) and (4), observe that all $(PS)_c$ sequences for $\tilde{I}$ with $c < 0$ must be bounded, similar to the proof of Lemma 3.1, there exists a strong convergent subsequence in $W^{s,p}(\mathbb{R}^N)$. □

Remark 4.5. Denote $K_c = \{u \in W : \tilde{I}'(u) = 0, \tilde{I}(u) = c\}$ If α, β are as in (3) or (4) above, then, it follows from $(PS)_c$ that $K_c(c < 0)$ is compact.

Lemma 4.6. *Denote* $\tilde{I}_c := \{u \in W : \tilde{I}'(u) = 0, \tilde{I}(u) \le c\}$. *Given* $n \in \mathbb{N}$, *there exists* $\epsilon_n < 0$, *such that*

$$\gamma(\tilde{I}^{\epsilon_n}) := \gamma(\{u \in W : \tilde{I}(u) \le \epsilon_n\}) \ge n.$$

Proof. Let X_n be a n-dimensional subspace of W. For any $u \in X_n, u \neq 0$, write $u = r_n w$ with $w \in X_n, \|w\|_W = 1$ and then $r_n = \|u\|_W$. From the assumptions $k(x)$, it is easy to see that, for every $w \in X_n$ with $\|w\|_W = 1$, there exists $d_n > 0$ such that $\int_{\mathbb{R}^N} k(x)|w|^q \mathrm{d}x \ge d_n$. Thus for $0 < r_n < t_0$, by the continuity of M, we have

$$\begin{aligned}
\tilde{I}(u) &= \frac{1}{p}\Big[\mathcal{M}([u]^p_{s,p}) + \|u\|^p_{p,V}\Big] - \frac{\alpha}{p^*_s}\psi(u)\int_{\mathbb{R}^N} |u|^{p^*_s}\mathrm{d}x - \frac{\beta}{q}\int_{\mathbb{R}^N} k(x)|u|^q \mathrm{d}x \\
&\le \frac{1}{p} r_n^p \Big[\mathcal{M}([w]^p_{s,p}) + \|w\|^p_{p,V}\Big] - \frac{\alpha}{p^*_s} r_n^{p^*_s}\int_{\mathbb{R}^N} |w|^{p^*_s}\mathrm{d}x - \frac{\beta}{p} r_n^q \int_{\mathbb{R}^N} k(x)|w|^q \mathrm{d}x \\
&\le \frac{C_1}{p} r_n^p - \frac{\alpha}{p^*_s} r_n^{p^*_s}\int_{\mathbb{R}^N} |w|^{p^*_s}\mathrm{d}x - \frac{\beta}{p} d_n r_n^q \\
&= \epsilon_n.
\end{aligned}$$

Therefore, we can choose $r_n \in (0, t_0)$ so small that $\tilde{I}(u) \le \epsilon_n < 0$. Let

$$S_{r_n} = \{u \in X_n : \|u\|_W = r_n\}. \tag{4.1}$$

Then $S_{r_n} \cap X_n \subset \tilde{I}^{\epsilon_n}$. Hence by Proposition 4.1,

$$\gamma(\tilde{I}^{\epsilon_n}) \ge \gamma(S_{r_n} \cap X_n) = n.$$

As desired. □

According to Lemma 4.4 we denote $\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}$ and let

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \tilde{I}(u). \tag{4.2}$$

Then

$$-\infty < c_n \le \epsilon_n < 0 \tag{4.3}$$

because $\tilde{I}^{\epsilon_n} \in \Sigma_n$ and $\tilde{I}$ is bounded from below.

Lemma 4.7. *Let* α, β *be as in* (3) *or* (4) *of Lemma 4.4. Then all* c_n *(given by* (4.2)*) are critical values of* $\tilde{I}$ *and* $c_n \to 0$.

Proof. Since $\Sigma_{n+1} \subset \Sigma_n$, it is clear that $c_n \le c_{n+1}$. By (4.3), we have $c_n < 0$. Hence there is a $\bar{c} \le 0$ such that $c_n \to \bar{c} \le 0$. Moreover, since that all c_n are critical values of $\tilde{I}$ (see [31]), we claim that $\bar{c} = 0$. If $\bar{c} < 0$, then by Remark 4.5, $K_{\bar{c}} = \{u \in W : \tilde{I}'(u) = 0, \tilde{I}(u) = \bar{c}\}$ is compact and $K_{\bar{c}} \in \Sigma$, then $\gamma(K_{\bar{c}}) = n_0 < +\infty$ and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = n_0$, here $N_\delta(K_{\bar{c}}) = \{x \in X :$

$\|x - K_{\bar c}\| \le \delta\}$. By the deformation lemma (see [34]), there exist $\epsilon > 0$ ($\bar c + \epsilon < 0$) and an odd homeomorphism $\eta : W \to W$ such that

$$\eta(\tilde I^{\bar c+\epsilon} \setminus N_\delta(K_{\bar c})) \subset \tilde I^{\bar c-\epsilon}.$$

Since c_n is increasing and converges to $\bar c$, there exists $n \in \mathbb{N}$ such that $c_n > \bar c - \epsilon$ and $c_{n+n_0} \le \bar c$. Choose $A \in \Sigma_{n+n_0}$ such that $\sup_{u\in A} \tilde I(u) < \bar c + \epsilon$, that is $A \subset \tilde I^{\bar c+\epsilon}$. By the properties of γ, we have

$$\gamma(\overline{A \setminus N_\delta(K_{\bar c})}) \ge \gamma(A) - \gamma(N_\delta(K_{\bar c}))) \ge n, \quad \gamma(\overline{\eta(A \setminus N_\delta(K_{\bar c}))}) \ge n.$$

Hence, we have $\overline{\eta(A \setminus N_\delta(K_{\bar c}))} \in \Sigma_n$. Consequently,

$$\sup_{u\in\overline{\eta(A\setminus N_\delta(K_{\bar c}))}} \tilde I(u) \ge c_n > \bar c - \epsilon,$$

a contradiction, hence $c_n \to 0$. □

Proof of Theorem 1.2. By Lemma 4.4(2), $\tilde I(u) = I(u)$ if $\tilde I(u) < 0$. Then, by Lemmas 4.4–4.7, one can see that all the assumptions of Proposition 4.2 are satisfied. This completes the proof. □

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