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BOUNDARY VALUE PROBLEMS FOR FOURTH-ORDER MIXED TYPE EQUATION WITH FRACTIONAL DERIVATIVE

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Abstract. In this work we study direct and inverse problems for fourth-order mixed type equations with the Caputo fractional derivative. Applying method of separation of variables we prove unique solvability of these problems.

1. Introduction and formulation of problems

It is known that the theory of boundary value problems for fractional order differential equations is one of the rapidly developing branches of the general theory of differential equations. Detailed information related with fractional calculus can be found in [6].

Many problems in diffusion and dynamical processes, electrochemistry, biosciences, signal processing, system control theory lead to differential equations of fractional order. More information can be found in [4],[8].

Mathematical modeling of many real life processes lead to problems of identifying coefficients or right-hand sides of differential equations, based on some known data of their solutions. These kinds of problems are called inverse problems.

Inverse problems appear also in various fields such physics (inverse problems of quantum scattering theory), geophysics (inverse problems of magneto metrics, seismology, theory of potentials), biology, medicine, quality control of industrial products and etc. (see [5, 9]). Similar problems studied in [2, 7]. In this work we investigate direct and inverse problems for fourth-order mixed type equations.

Let $\Omega = \{(x,t) : 0 < x < 1,\, -p < t < q\}$, $\Omega^+ = \Omega \cap (t > 0)$, $\Omega^- = \Omega \cap (t < 0)$, where $p, q > 0$ are real numbers. In Ω we consider the equations

$$\begin{gathered}\frac{\partial^4 u}{\partial x^4} + {}_C D_{0t}^{\alpha} u = f(x,t), \quad t > 0, \\ \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = f(x,t), \quad t < 0,\end{gathered} \tag{1.1}$$

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and

$$\begin{gathered}\frac{\partial^4 u}{\partial x^4} + {}_C D_{0t}^{\alpha} u = f(x), \quad t > 0, \\ \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = f(x), \quad t < 0,\end{gathered} \tag{1.2}$$

where ${}_C D_{0t}^{\alpha}$ is the Caputo fractional operator of the order $\alpha \in (0, 1]$ with respect to variable t [6, p. 92],

$${}_C D_{0t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \quad t > 0.$$

We study the following problems.

Direct problem. Find a function $u(x, t)$ such that:

1. u is continuous in $\bar{\Omega}$, together with its derivatives appearing in the boundary conditions,
2. u satisfies equation (1.1) in $\Omega^+ \cup \Omega^-$,
3. u satisfies the boundary conditions
$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad -p \le t \le q, \tag{1.3}$$
$$u(x,-p) = 0, \quad 0 \le x \le 1, \tag{1.4}$$
4. u satisfies the matching condition
$${}_C D_{0t}^{\alpha} u(x,+0) = \frac{\partial u(x,-0)}{\partial t}, \quad 0 < x < 1. \tag{1.5}$$

Inverse problem. Find a pair of functions $\{u(x,t), f(x)\}$ with the following properties:

1. u is continuous in $\bar{\Omega}$, with its derivatives appearing in the boundary conditions, $f(x) \in C(0, 1)$,
2. u satisfies equation (1.2) in $\Omega^+ \cup \Omega^-$,
3. u satisfies the boundary conditions (1.3) and
$$u(x,-p) = \psi(x),\ u(x,q) = \varphi(x), 0 \le x \le 1,$$
where $\varphi(x), \psi(x)$ are given functions,
4. u satisfies the matching condition (1.5).

2. Uniqueness and existence of a solution for the direct problem

Theorem 2.1. *Let p be a number such that*

$$\Delta_n = (\pi n)^2 \sin((\pi n)^2 p) + \cos((\pi n)^2 p) \neq 0.$$

Then, if there exists a regular solution of the direct problem, it is unique.

Proof. Let $f(x,t) \equiv 0$. We will show that the homogeneous problem has only the trivial solution. Let $u(x,t)$ be a solution of the homogeneous problem. Consider the ortho-normal system of functions, which is complete in $L_2(0,1)$,

$$X_n(x) = \sqrt{2} \sin(\lambda_n x), \quad \lambda_n = \pi n, \quad n \in \mathbb{N}. \tag{2.1}$$

Let

$$c_n(t) = \int_0^1 u(x,t) X_n(x) dx, \quad t \ge 0,\ n \in \mathbb{N}. \tag{2.2}$$

Acting with operator ${}_C D_{0t}^{\alpha}$ to both sides of (2.2) and considering equation (1.1), boundary conditions (1.3), we have that $c_k(t)$ satisfies the equation

$$ {}_C D_{0t}^{\alpha} c_n(t) + \lambda_n^4 c_n(t) = 0, $$

whose solution can be represented as [6, p. 231]

$$ c_n(t) = A_n E_{\alpha}(-\lambda_n^4 t^{\alpha}), \quad t \geq 0, \tag{2.3} $$

where $E_{\alpha}(z)$ is the Mittag-Leffler function [6, p. 40], defined as

$$ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha\, n+1)}, \quad z \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0. $$

For negative values of t, $-p \leq t \leq 0$, we set

$$ d_n(t) = \int_0^1 u(x,t) X_n(x) dx. \tag{2.4} $$

Differentiating twice with respect to t and considering (1.1), (1.3), we obtain

$$ d''_n(t) + \lambda_n^4 d_n(t) = 0, $$

whose solution is

$$ d_n(t) = B_n \sin(\lambda_n^2 t) + L_n \cos(\lambda_n^2 t), \quad t \leq 0. \tag{2.5} $$

For determining the unknown coefficients of (2.3) and (2.5) we use the continuity property of the function in $\overline{\Omega}$ as well as the matching condition (1.5). We obtain

$$ A_n = L_n, \quad B_n + \lambda_n^2 A_n = 0, \quad B_n \sin(\lambda_n^2 p) - L_n \cos(\lambda_n^2 p) = 0. $$

This system has only the trivial solution, since its determinant Δ_n is not equal to zero because of condition (2.1). Hence, $c_n(t) = d_n(t) = 0$. Then from (2.2) and (2.4) it follows that

$$ \int_0^1 u(x,t) X_n(x) dx \equiv 0, \quad t \in [-p; q], \quad n \in \mathbb{N}. $$

Based on the completeness of system (2.1) we conclude that $u(x,t) = 0$ in $\overline{\Omega}$. The proof is complete. $\square$

Remark 2.2. The set of numbers p satisfying condition (2.1) is not empty. For instance, if $p = \frac{1}{\pi}$, then $\Delta_n = \pm 1 \neq 0$.

Now we prove the existence of a solution for the direct problem.

Theorem 2.3. *Let* $f(x,t) \in C_{x,t}^{4,0}(\overline{\Omega})$, $\frac{\partial^5 f(x,t)}{\partial x^5} \in L_2(\Omega)$,

$$ f(0,t) = f_{xx}(0,t) = f_{xxxx}(0,t) = 0, \quad f(1,t) = f_{xx}(1,t) = f_{xxxx}(1,t) = 0, $$

and $\Delta_n \neq 0$ *at* $n \in \mathbb{N}$. *Then the solution of direct problem exists.*

Proof. We use the method of separation of variables for $f \equiv 0$ in Ω, we set $u(x,t) = X(x)T(t) \neq 0$. Substituting this into (1.1), taking boundary conditions (1.3) into account with respect to $X(x)$ we get the spectral problem

$$ X^{IV}(x) - \lambda^4 X(x) = 0, \quad X(0) = X(1) = X''(0) = X''(1) = 0. \tag{2.6} $$

This problem is selfadjoint and has complete system of eigenfunctions represented by (2.1). Let us set

$$u(x,t)=\begin{cases} u^+(x,t), & (x,t)\in\Omega^+,\\ u^-(x,t), & (x,t)\in\Omega^-.\end{cases}$$

For the solution of the non-homogeneous equation (1.1) we set

$$u^+(x,t)=\sum_{n=1}^{\infty}u_n^+(t)X_n(x),\quad \text{in } \Omega^+, \tag{2.7}$$

$$u^-(x,t)=\sum_{n=1}^{\infty}u_n^-(t)X_n(x),\quad \text{in } \Omega^-, \tag{2.8}$$

where $u_n^+(t)$ and $u_n^-(t)$ are unknown functions.

Solutions of (2.7) and (2.8) satisfy conditions (1.3). Let us expand $f(x,t)$ into the series

$$f(x,t)=\sum_{n=1}^{\infty}f_n(t)X_n(x), \tag{2.9}$$

where

$$f_n(t)=\int_0^1 f(x,t)X_n(x)dx.$$

Substiting (2.7)–(2.9) in (1.1) lead to the equations

$$\begin{gathered} {}_CD_{0t}^{\alpha}u_n^+(t)+\lambda_n^4u_n^+(t)=f_n(t),\quad t>0,\\ \frac{d^2}{dt^2}u_n^-(t)+\lambda_n^4u_n^-(t)=f_n(t),\quad t<0, \end{gathered}$$

whose solutions are [6, p. 231]

$$u_n^+(t)=A_nE_{\alpha}(-\lambda_n^4t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^4(t-\tau)^{\alpha})f_n(\tau)d\tau, \tag{2.10}$$

$$u_n^-(t)=B_n\sin(\lambda_n^2t)+L_n\cos(\lambda_n^2t)+\frac{1}{\lambda_n^2}\int_t^0 f_n(\tau)\sin(\lambda_n^2(\tau-t))d\tau, \tag{2.11}$$

where A_n, B_n, L_n are unknown constants, $E_{\alpha,\beta}(z)$ is the Mittag-Leffler type function [6, p. 42],

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad z,\beta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ E_{\alpha,1}(z)=E_{\alpha}(z).$$

To determine the unknown constants of (2.10) and (2.11) we use the continuity of the looked for function in $\bar{\Omega}$ and condition (1.5) of direct problem. We obtain

$$u_n^+(0)=u_n^-(0),\quad {}_CD_{0t}^{\alpha}u_n^+(0)=\frac{du_n^-(0)}{dt},\quad u_n^-(-p)=0.$$

Then concerning the unknowns A_n, B_n, L_n, we obtain the system of equations

$$\begin{gathered} A_n=L_n,\\ f_n(0)-\lambda_n^4A_n=\lambda_n^2B_n,\\ B_n\sin(\lambda_n^2p)-L_n\cos(\lambda_n^2p)=\frac{1}{\lambda_n^2}\int_{-p}^0 f_n(\tau)\sin(\lambda_n^2(\tau+p))d\tau. \end{gathered}$$

Since $\Delta_n \neq 0$ the coefficients A_n, B_n, L_n can be uniquely determined. Substituting A_n, B_n, L_n in (2.10) and (2.11) we find that

$$
\begin{aligned}
u_n^+(t) = {} & \frac{f_n(0)\sin(\lambda_n^2 p)}{\lambda_n^2 \Delta_n} E_\alpha(-\lambda_n^4 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^4 (t-\tau)^\alpha) f_n(\tau) d\tau \\
& - \frac{E_\alpha(-\lambda_n^4 t^\alpha)}{\lambda_n^2 \Delta_n} \int_{-p}^0 f_n(\tau) \sin(\lambda_n^2(\tau+p)) d\tau,
\end{aligned}
\tag{2.12}
$$

$$
u_n^-(t) = \frac{f_n(0)\sin(\lambda_n^2(t+p))}{\lambda_n^2 \Delta_n} + \int_{-p}^0 K_n(t,\tau) f_n(\tau) d\tau, \tag{2.13}
$$

where

$$
K_n(t,\tau) = \begin{cases} \frac{(\lambda_n^2 \sin(\lambda_n^2 t) - \cos(\lambda_n^2 t)) \sin(\lambda_n^2(\tau+p))}{\lambda_n^2 \Delta_n}, & -p \le \tau \le t, \\ \frac{(\lambda_n^2 \sin(\lambda_n^2 \tau) - \cos(\lambda_n^2 \tau)) \sin(\lambda_n^2(t+p))}{\lambda_n^2 \Delta_n}, & t \le \tau \le 0. \end{cases} \tag{2.14}
$$

Thus we find a formal solution of direct problem in Ω^+ and Ω^-, given by formulas (2.7) and (2.8), respectively, where $u_n^+(t)$, $u_n^-(t)$ are defined by formulas (2.12), (2.13).

We need to prove that this formal solution is a true solution. For this aim, we prove the convergence of series (2.7), (2.8) and

$$
\sum_{n=1}^\infty \lambda_n^2 u_n^+(t) X_n(x), \quad \sum_{n=1}^\infty \lambda_n^2 u_n^-(t) X_n(x), \tag{2.15}
$$

$$
\sum_{n=1}^\infty \lambda_n^4 u_n^+(t) X_n(x), \quad \sum_{n=1}^\infty {}_C D_{0t}^\alpha u_n^+(t) X_n(x), \tag{2.16}
$$

$$
\sum_{n=1}^\infty \lambda_n^4 u_n^-(t) X_n(x), \quad \sum_{n=1}^\infty \frac{d^2 u_n^-(t)}{dt^2} X_n(x). \tag{2.17}
$$

First we prove convergence of series (2.16) and (2.17). Convergence of other series can be done similarly.

Let us prove convergence of the first series of (2.16). It can be majorized by

$$
\sum_{n=1}^\infty \lambda_n^4 |u_n^+(t)|. \tag{2.18}
$$

We have $\Delta_n \neq 0$, hence there exists $\delta > 0$ such that $|\Delta_n| \ge \delta > 0$.

Further, we use the following properties of the Mittag-Leffler function $E_{\alpha,\beta}(z)$:

(1) at $\mu > 0$, $\alpha, \beta \in (0,1]$, $\alpha \le \beta$, $t^{\alpha-1} E_{\alpha,\beta}(-\mu t^\alpha)$ is completely monotone [7], i.e. $(-1)^n [t^{\beta-1} E_{\alpha,\beta}(-\mu t^\alpha)]^{(n)} \ge 0$, $n \in \mathbb{N} \cup \{0\}$;

(2) at $\alpha \in (0,2)$, $\gamma \le |\operatorname{arg} z| \le \pi$, $\beta \in R$, $\gamma \in (\pi\alpha/2; \min\{\pi; \pi\alpha\})$ we have for

$$
|E_{\alpha,\beta}(z)| \le \frac{M}{1+|z|},
$$

where M is constant which does not depend on argument z [1, p. 136];

(3) the following formulas are valid [1, pp. 118, 120],

$$
E_{\alpha,\mu}(z) = \frac{1}{\Gamma(\mu)} + z E_{\alpha,\alpha+\mu}(z), \quad \int_0^z t^{\mu-1} E_{\alpha,\mu}(\lambda t^\alpha) dt = z^\mu E_{\alpha,\mu+1}(\lambda z^\alpha). \tag{2.19}
$$

Set

$$I_1(t) = \frac{f_n(0)\sin(\lambda_n^2 p)}{\lambda_n^2 \Delta_n} E_\alpha(-\lambda_n^4 t^\alpha),$$
$$I_2(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^4 (t-\tau)^\alpha) f_n(\tau) d\tau,$$
$$I_3(t) = \frac{E_\alpha(-\lambda_n^4 t^\alpha)}{\lambda_n^2 \Delta_n} \int_{-p}^0 f_n(\tau)\sin(\lambda_n^2(\tau+p)) d\tau.$$

We estimate the function $I_1(t)$:

$$|I_1(t)| = \Big|\frac{f_n(0)\sin(\lambda_n^2 p)}{\lambda_n^2 \Delta_n} E_{\alpha,1}(-\lambda_n^4 t^\alpha)\Big| \le \frac{M}{\lambda_n^2 \delta}|f_n(0)|.$$

Based on the condition imposed on $f(x,t)$ in the theorem, from (2) we have

$$\begin{gathered} f_n(t) = \frac{1}{\lambda_n^4}\int_0^1 \frac{\partial^4 f}{\partial x^4}\sqrt{2}\sin(\lambda_n x)\, dx = \frac{1}{\lambda_n^4} f_{n4}(t), \\ |f_n(t)| \le \frac{1}{\lambda_n^4} N, N = \max_{[0;q]} |f_{n4}(t)|. \end{gathered} \tag{2.20}$$

Then

$$\lambda_n^4 |I_1(t)| \le \frac{MN}{\lambda_n^2 \delta} = \frac{C}{\lambda_n^2}. \tag{2.21}$$

Here and further, C is a positive that may change from line to line. Considering (2.20), we have the estimate

$$|I_2(t)| \le \frac{N}{\lambda_n^4}\int_0^t \big|(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^4 (t-\tau)^\alpha)\big| d\tau.$$

From here, applying formula (27) we obtain

$$|I_2(t)| \le \frac{N}{\lambda_n^4}\left|1 - E_\alpha(-\lambda_n^4 t^\alpha)\right| \le \frac{C}{\lambda_n^4}. \tag{2.22}$$

Further, taking (2.20) into account we have the estimate

$$|I_3(t)| \le \frac{M}{\lambda_n^2 \delta}\int_{-p}^0 |f_{n4}(\tau)| d\tau \le \frac{C}{\lambda_n^2}. \tag{2.23}$$

From (2.21)–(2.23) it follows the convergence of series (2.18). Then based on Weierstrass' theorem [3, p. 20], first series of (2.16) converges absolutely and uniformly.

Since

$${}_C D_{0t}^\alpha u_n^+(t) = f_n(t) - \lambda_n^4 u_n^+(t),$$

the absolute and uniform convergence of the second series of (2.16) can be proved similarly.

Now we prove the absolute and uniform convergence of the first series of (2.17). Let

$$J_1(t) = \frac{f_k(0)\sin(\lambda_n^2(t+p))}{\lambda_n^2 \Delta_n}, \quad J_2(t) = \int_{-p}^0 K_n(t,\tau) f_n(\tau) d\tau.$$

Considering (2.20), which is valid as well in the case $t \le 0$, we obtain

$$\lambda_n^4 |J_1(t)| = \lambda_n^4 \Big|\frac{f_n(0)\sin(\lambda_n^2(t+p))}{\lambda_n^2 \Delta_n}\Big| \le \frac{\lambda_n^2}{\delta}|f_n(0)| \le \frac{C}{\lambda_n^2}. \tag{2.24}$$

Note that the function $K_n(t,\tau)$ defined by (2.14) is bounded, i.e., $|K_n(t,\tau)| \le C$, $-p \le t,\tau \le 0$. From (2) it follows that

$$f_n(t) = -\frac{1}{\lambda_n^5}\int_0^1 \frac{\partial^5 f}{\partial x^5}\sqrt{2}\cos(\lambda_n x)dx = -\frac{1}{\lambda_n^5} f_{n5}(t).$$

Further, applying the inequality $2ab \le a^2 + b^2$, and the Cauchy-Schwarz inequality, we have the estimate

$$\begin{aligned}
\lambda_n^4|J_2(t)| &= \lambda_n^4\Big|\int_{-p}^0 K_n(t,\tau)f_n(\tau)d\tau\Big| \\
&\le \frac{C}{\lambda_n}\int_{-p}^0 |f_{n5}(\tau)|d\tau \\
&\le C\Big[\frac{1}{\lambda_n^2} + \Big(\int_{-p}^0 |f_{n5}(\tau)|d\tau\Big)^2\Big] \\
&\le C\Big[\frac{1}{\lambda_n^2} + \int_{-p}^0 |f_{n5}(\tau)|^2 d\tau\Big] \\
&= C\Big[\frac{1}{\lambda_n^2} + \|f_{n5}(t)\|\|_{L_2(-p,0)}^2\Big].
\end{aligned}$$

Since

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^2} = \frac{1}{6}, \quad \sum_{n=1}^\infty \|f_{n5}(t)\|\|_{L_2(-p,0)}^2 \le \|\frac{\partial^5 f(x,t)}{\partial x^5}\|\|_{L_2(\Omega)}^2,$$

from this and (2.24) , it follows the absolute and uniform convergence of the series for $t \le 0$. The convergence of the second series can be obtained similarly. The proof is complete. □

3. Inverse problem

For solving the inverse problem, we set

$$u(x,t) = \begin{cases} \sum_{n=1}^\infty u_n^+(t)X_n(x), & (x,t) \in \Omega^+ \\ \sum_{n=1}^\infty u_n^-(t)X_n(x), & (x,t) \in \Omega^-, \end{cases} \tag{3.1}$$

$$f(x) = \sum_{n=1}^\infty f_n X_n(x), \tag{3.2}$$

where $u_n^+(t), u_n^-(t)$ are unknown functions, f_n is unknown coefficient.

Substitution (3.1) and (3.2) in (1.2), leads to equations

$${}_C D_{0t}^\alpha u_n^+(t) + \lambda_n^4 u_n^+(t) = f_n, \quad t > 0, \tag{3.3}$$

$$\frac{d^2}{dt^2}u_n^-(t) + \lambda_n^4 u_k^-(t) = f_n, \quad t < 0, \tag{3.4}$$

whose solutions are (2.10) and (2.11). From here, considering f_n constant, we obtain

$$u_n^+(t) = A_n E_\alpha(-\lambda_n^4 t^\alpha) + f_n t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^4 t^\alpha), \tag{3.5}$$

$$u_n^-(t) = B_n \sin(\lambda_n^2 t) + L_n \cos(\lambda_n^2 t) + \frac{f_n}{\lambda_n^4}(1 - cos\lambda_n^2 t), \tag{3.6}$$

where A_n, B_n, L_n, f_n are unknown coefficients.

To determine the unknown coefficients we use the continuity of the unkown function in $\bar{\Omega}$ and conditions (1.5) and (1). We obtain

$$u_n^+(0) = u_n^-(0), \quad {}_C D_{0t}^{\alpha} u_n^+(0) = \frac{du_n^-(0)}{dt}, \quad u_n^+(q) = \varphi_n,\ u_n^-(-p) = \psi_n,$$

where

$$\varphi_n = \int_0^1 \varphi(x) X_n(x) dx, \quad \psi_n = \int_0^1 \psi(x) X_n(x) dx. \tag{3.7}$$

Then concerning the unknowns $A_n, B_n, L_n,\ f_n$ we obtain system of equations

$$\begin{gathered} L_n = A_n, \\ \lambda_n^4 A_n + \lambda_n^2 B_n - f_n = 0, \\ A_n E_\alpha(-\lambda_n^4 q^\alpha) + f_n q^\alpha E_{\alpha,\alpha+1}(-\lambda_n^4 q^\alpha) = \varphi_n, \\ \cos(\lambda_n^2 p) \cdot A_n - \sin(\lambda_n^2 p) \cdot B_n + \frac{1}{\lambda_n^4}(1 - \cos(\lambda_n^2 p)) f_n = \psi_n, \end{gathered} \tag{3.8}$$

whose solutions are

$$A_n = \varphi_n + \frac{1 - E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}}(\psi_n - \varphi_n), \quad B_n = \frac{\lambda_n^2}{\Delta_{n*}}(\varphi_n - \psi_n),$$

$$L_n = A_n, \quad f_n = \frac{\lambda_n^4 E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}}(\varphi_n - \psi_n) + \lambda_n^4 \varphi_n,$$

where

$$\Delta_{n*} = \lambda_n^2 \sin(\lambda_n^2 p) + \cos(\lambda_n^2 p) - E_\alpha(-\lambda_n^4 q^\alpha). \tag{3.9}$$

Substituting of A_n, B_n, L_n into (3.5) and (3.6), we find

$$u_n^+(t) = \frac{(\varphi_n - \psi_n)}{\Delta_{n*}} \big(E_\alpha(-\lambda_n^4 q^\alpha) - E_\alpha(-\lambda_n^4 t^\alpha)\big) + \varphi_n, \tag{3.10}$$

$$u_n^-(t) = \frac{\lambda_n^2 \sin(\lambda_n^2 t) - \cos(\lambda_n^2 t) + E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}}(\varphi_n - \psi_n) + \varphi_n. \tag{3.11}$$

Further, from (3.1), (3.2), (3.10), (3.11) we obtain formal solution of the inverse problem, which is given by formulas

$$u(x,t) = \sum_{n=1}^{\infty} \Big(\frac{\varphi_n - \psi_n}{\Delta_{n*}} \big(E_\alpha(-\lambda_n^4 q^\alpha) - E_\alpha(-\lambda_n^4 t^\alpha)\big) + \varphi_n \Big) X_n(x), \quad t \geq 0, \tag{3.12}$$

$$u(x,t) = \sum_{n=1}^{\infty} \Big(\frac{\lambda_n^2 \sin(\lambda_n^2 t) - \cos(\lambda_n^2 t) + E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}}(\varphi_n - \psi_n) + \varphi_n \Big) X_n(x), \quad \text{for } t \leq 0, \tag{3.13}$$

$$f(x) = \sum_{n=1}^{\infty} \Big(\frac{\lambda_n^4 E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}}(\varphi_n - \psi_n) + \lambda_n^4 \varphi_n \Big) X_n(x). \tag{3.14}$$

Theorem 3.1. *Let $\varphi(x) \in C^6[0,1]$, $\varphi^{(7)}(x) \in L_2(0,1)$,*

$$\frac{d^{2k}\varphi(x)}{dx^{2k}}\Big|_{x=0} = \frac{d^{2k}\varphi(x)}{dx^{2k}}\Big|_{x=1} = 0, \quad k = \overline{0,3},$$

$\psi(x) \in C^6[0,1]$, $\psi^{(7)}(x) \in L_2(0,1)$,

$$\frac{d^{2k}\psi(x)}{dx^{2k}}\Big|_{x=0} = \frac{d^{2k}\psi(x)}{dx^{2k}}\Big|_{x=1} = 0, \quad k = \overline{0,3},$$

and $\Delta_{n*} \neq 0$ at $n \in \mathbb{N}$. Then there exists the unique solution for the inverse problem.

Proof. Since $\Delta_{n*} \neq 0$, the uniqueness of the solution follows from (3.12)–(3.14), and from the completeness of (2.1). Therefore, we prove only the existence of the solution.

Solutions (3.12)–(3.14) satisfy (1.2) and conditions (1.3), (1.5), (1). We need to prove that this formal solution is a true solution. For this aim we prove the convergence of series (3.12)–(3.14) and

$$\sum_{n=1}^{\infty} \lambda_n^2 u_n^+(t) X_n(x), \quad \sum_{n=1}^{\infty} \lambda_n^2 u_n^-(t) X_n(x), \tag{3.15}$$

$$\sum_{n=1}^{\infty} \lambda_n^4 u_n^+(t) X_n(x), \quad \sum_{n=1}^{\infty} {}_C D_{0t}^{\alpha} u_n^+(t) X_n(x), \tag{3.16}$$

$$\sum_{n=1}^{\infty} \lambda_n^4 u_n^-(t) X_n(x), \quad \sum_{n=1}^{\infty} \frac{d^2 u_n^-(t)}{dt^2} X_n(x). \tag{3.17}$$

Let us prove convergence of the series of (3.17). We have $\Delta_{n*} \neq 0$, hence there exists $\delta > 0$ such that $|\Delta_{n*}| \geq \delta > 0$. From here, taking properties of Mittag-Leffler function we get

$$|u_n^-(t)| = \Big| \frac{\lambda_n^2 \sin(\lambda_n^2 t) - \cos(\lambda_n^2 t) + E_\alpha(-\lambda_n^4 q^\alpha)}{\Delta_{n*}} (\varphi_n - \psi_n) + \varphi_n \Big| \leq C\lambda_n^2 (|\varphi_n| + |\psi_n|).$$

Based on the conditions imposed on $\varphi(x)$ and $\psi(x)$ in theorem 3.1, from (3.7) we obtain

$$\varphi_n = -\frac{1}{\lambda_n^7} \int_0^1 \frac{d^7\varphi}{\partial x^7} \sqrt{2} \cos \lambda_n x dx = -\frac{1}{\lambda_n^7} \varphi_{n7}, \tag{3.18}$$

$$\psi_n = -\frac{1}{\lambda_n^7} \int_0^1 \frac{d^7\psi}{\partial x^7} \sqrt{2} \cos \lambda_n x \, dx = -\frac{1}{\lambda_n^7} \psi_{n7}. \tag{3.19}$$

From here it follows that given series can be majorized by

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(|\varphi_{n7}| + |\psi_{n7}|\right).$$

Since

$$\frac{1}{\lambda_n} \left(|\varphi_{n7}| + |\psi_{n7}|\right) \leq \frac{1}{2} \Big(\frac{1}{\lambda_n^2} + 2|\varphi_{n7}|^2 + 2|\psi_{n7}|^2 \Big), \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{1}{6},$$

$$\sum_{n=1}^{\infty} |\varphi_{n7}|^2 \leq \|\varphi^{(7)}(x)\|\|_{L_2(0,1)}^2, \quad \sum_{n=1}^{\infty} |\psi_{n7}|^2 \leq \|\psi^{(7)}(x)\|\|_{L_2(0,1)}^2,$$

it follows the absolute and uniform convergence of the first series of (3.17).

Now we consider the second series of (3.17). Since

$$\frac{d^2 u_n^-(t)}{dt^2} = \frac{\varphi_n - \psi_n}{\Delta_{n*}} (\lambda_n^6 \sin(\lambda_n^2 t) - \lambda_n^4 \cos(\lambda_n^2 t)),$$

$$\Big| \frac{d^2 u_n^-(t)}{dt^2} \Big| \leq C \cdot \lambda_n^6 \left(|\varphi_n| + |\psi_n|\right),$$

from this (3.18) and (3.19), it follows the absolute and uniform convergence of the second series of (3.17). This completes the proof. □

Remark 3.2. Set of numbers p, satisfying condition $\Delta_{n*} \neq 0$ is not empty. For instance, if $p = 2/\pi$, then $\Delta_{n*} = 1 - E_\alpha(-\lambda_n^4 q^\alpha)$. Since $\lambda_n \neq 0$, $n \in \mathbb{N}$, $0 < E_\alpha(-\lambda_n^4 q^\alpha) < 1$, it follows that $\Delta_{n*} \neq 0$.

Remark 3.3. If condition $\Delta_{n*} \neq 0$ is not valid for $n = k$ and for some p and q, i.e.

$$\Delta_{k*} = \lambda_k^2 \sin(\lambda_k^2 p) + \cos(\lambda_k^2 p) - E_\alpha(-\lambda_k^4 q^\alpha) = 0,$$

homogeneous system (3.8) has nontrivial solution ($\varphi_n = \psi_n = 0$). Then homogeneous inverse problem also has nontrivial solution. For instance, when

$$A_k = L_k = 1, \quad B_k = \frac{\lambda_k^2}{E_\alpha(-\lambda_k^4 q^\alpha) - 1}, \quad f_k = \frac{\lambda_k^4 E_\alpha(-\lambda_k^4 q^\alpha)}{E_\alpha(-\lambda_k^4 q^\alpha) - 1}$$

, the functions

$$u(x,t) = \begin{cases} \left(\frac{E_\alpha(-\lambda_k^4 q^\alpha) - E_\alpha(-\lambda_k^4 t^\alpha)}{E_\alpha(-\lambda_n^4 q^\alpha) - 1}\right)\sqrt{2}\sin(\lambda_k^2 x), & t > 0, \\ \left(\frac{\lambda_k^2 \sin(\lambda_k^2 t) - \cos(\lambda_k^2 t) - E_\alpha(-\lambda_k^4 q^\alpha)}{E_\alpha(-\lambda_n^4 q^\alpha) - 1}\right)\sqrt{2}\sin(\lambda_k^2 x), & t < 0, \end{cases}$$

$$f(x) = \frac{\lambda_k^4 E_\alpha(-\lambda_k^4 q^\alpha)}{E_\alpha(-\lambda_k^4 q^\alpha) - 1}\sqrt{2}\sin(\lambda_k^2 x)$$

form a solution of the homogeneous problem.

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