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REPRODUCTIVE SOLUTIONS FOR THE G-NAVIER-STOKES AND G-KELVIN-VOIGHT EQUATIONS

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Abstract. This article presents the existence of reproductive solutions of g-Navier-Stokes and g-Kelvin-Voight equations. In this way, for weak solutions, we reach basically the same result as for classic Navier-Stokes equations.

1. Introduction

On one hand, in this work we consider the g-Navier-Stokes equation

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in }]0,T[\times\Omega, \\ &\frac{1}{g}(\nabla(g\mathbf{u})) = \frac{\nabla g}{g}\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0, \quad \text{in }]0,T[\times\Omega, \end{aligned} \tag{1.1}$$

defined on a domain $\Omega \subseteq \mathbb{R}^2$.

This system is derived in [10] from the 3-D Navier-Stokes equations

$$\begin{gathered} \frac{\partial \mathbf{U}}{\partial t} - \nu\Delta\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla\Phi = \mathbf{f}, \quad \text{in }]0,T[\times\Omega_g, \\ \nabla\cdot\mathbf{U} = 0, \quad \text{in }]0,T[\times\Omega_g, \end{gathered}$$

where $\Omega_g = \{(y_1,y_2,y_3) : (y_1,y_2)\in\Omega,\ 0\le y_3\le g(y_1,y_2)\}$, with the boundary conditions

$$\mathbf{U}\cdot\mathbf{n} = 0 \quad \text{on } \partial_{\rm top}\Omega_g \cap \partial_{\rm bottom}\Omega_g$$

being,

$$\begin{gathered} \partial_{\rm top}\Omega_g = \{(y_1,y_2,y_3)\in\Omega_g : y_3 = g(y_1,y_2)\}, \\ \partial_{\rm bottom}\Omega_g = \{(y_1,y_2,y_3)\in\Omega_g : y_3 = 0\}. \end{gathered}$$

More precisely, the authors assume that

$$\mathbf{U}(y_1,y_2,y_3) = (\mathbf{U}_1(y_1,y_2), \mathbf{U}_2(y_1,y_2), \mathbf{U}_3(y_1,y_2,y_3)),$$

and they define the following new variables and unknowns

$$\begin{gathered} y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 g(x_1,x_2), \\ \mathbf{U}_1(y_1,y_2) = \mathbf{u}_1(x_1,x_2), \quad \mathbf{U}_2(y_1,y_2) = \mathbf{u}_2(x_1,x_2), \quad \mathbf{U}_3(y_1,y_2,y_3) = \mathbf{u}_3(x_1,x_2,x_3) \end{gathered}$$

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Finally, they prove that $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is solution of the two equation of (1.1) and $\mathbf{u}_3 = x_3\nabla g \cdot \mathbf{u}$. The interested reader can also review [2], [8] and [9]. Although the g-Navier-Stokes system is defined in two dimension domain, we will also study the tridimensional case.

In this article, at first we seek a reproductive solution (or weak periodic solution) of (1.1), i.e. solutions satisfying

$$\mathbf{u}(0,x) = \mathbf{u}(T,x), \quad x \in \Omega, \tag{1.2}$$

instead of a initial condition. In the case of the Navier-Stokes equation, the study of the reproductive solutions was initiated by Kaniel and Shinbrot in [4], the reader can also see the classical textbook [6] by Lions. In [3] the authors review some results concerning the existence, uniqueness and regularity of reproductive and time periodic solutions of the Navier-Stokes equations and some variants defined in bounded domains. In order to obtain a reproductive solution, they introduce a Galerkin discretization of the problem, proving existence of approximate solution to certain initial conditions. Then, a Leray-Schauder argument, by means of fixed point process, permits to obtain a reproductive Galerkin solution, which converges towards a continuous reproductive solution.

To be more precise, in this work the first purpose is to solve the system

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in }]0,T[\times\Omega, \\
\frac{1}{g}(\nabla\cdot(g\mathbf{u})) = \frac{\nabla g}{g}\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0, \quad \text{in }]0,T[\times\Omega, \\
\mathbf{u}(0,x) = \mathbf{u}(T,x), \quad \text{in } \Omega, \\
\mathbf{u}(t,x) = \beta(t,x), \quad \text{on } [0,T]\times\partial\Omega.
\end{gathered} \tag{1.3}$$

Here $\beta \in C^1(\mathbb{R}, H^{1/2}(\partial\Omega)^n)$ is T-periodic function and satisfies the (g-SOC) condition

$$\int_{\partial\Omega} g\beta\cdot\mathbf{n}ds = 0. \tag{1.4}$$

This definition is inspired by that given in [7] when $g \equiv 1$, the so-called (SOC) condition,

$$\int_{\partial\Omega} \beta\cdot\mathbf{n}ds = 0. \tag{1.5}$$

Moreover, in a similar manner to the Navier-Stokes system, we can prove uniqueness of the solution in the bidimensional case.

On the other hand, in this paper we also consider the g-Kelvin-Voight equation

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} - \frac{\nu}{g}(\nabla\cdot g\nabla)\mathbf{u} + \frac{\nu}{g}(\nabla g\cdot\nabla)\mathbf{u} - \frac{\alpha}{g}(\nabla\cdot g\nabla)\mathbf{u}_t \\
+ \frac{\alpha}{g}(\nabla g\cdot\nabla)\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = f, \quad \text{in }]0,T[\times\Omega \\
\frac{1}{g}(\nabla\cdot(g\mathbf{u})) = \frac{\nabla g}{g}\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0, \quad \text{in }]0,T[\times\Omega
\end{gathered} \tag{1.6}$$

The derivation of this system is analogous to the g-Navier-Stokes. In fact, it is deduced from the Kelvin-Voight system

$$\begin{gathered}
\frac{\partial \mathbf{U}}{\partial t} - \nu\Delta\mathbf{U} - \alpha\Delta\mathbf{U}_t + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = \mathbf{F}, \quad \text{in }]0,T[\times\Omega_g, \\
\nabla\cdot\mathbf{U} = 0, \quad \text{in }]0,T[\times\Omega_g,
\end{gathered}$$

where $\Omega_g = \{(y_1, y_2, y_3) : (y_1, y_2) \in \Omega,\ 0 \le y_3 \le g(y_1, y_2)\}$. We refer interested readers to the article [5] and the reference given there.

The second purpose of this article is to solve the system

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} - \frac{\nu}{g}(\nabla \cdot g\nabla)\mathbf{u} + \frac{\nu}{g}(\nabla g \cdot \nabla)\mathbf{u} - \frac{\alpha}{g}(\nabla \cdot g\nabla)\mathbf{u}_t \\
+ \frac{\alpha}{g}(\nabla g \cdot \nabla)\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = f, \quad \text{in }]0, T[\times\Omega \\
\frac{1}{g}(\nabla \cdot (g\mathbf{u})) = \frac{\nabla g}{g} \cdot \mathbf{u} + \nabla \cdot \mathbf{u} = 0, \quad \text{in }]0, T[\times\Omega \\
\mathbf{u}(0, x) = \mathbf{u}(T, x), \quad \text{in } \Omega \\
\mathbf{u}(t, x) = 0, \quad \text{in }]0, T[\times\partial\Omega
\end{gathered} \tag{1.7}$$

in other words, we seek a reproductive solution for the g-Kelvin-Voight equation.

This article is organized as follows. In section 2 the basic definitions and results are introduced. Section 3 is devoted to proving the existence of the reproductive solution of the g-Navier-Stokes system, both for the case $\beta = 0$ and the case $\beta \neq 0$. Finally, in section 4 the existence of the reproductive solution of the g-Kelvin-Voight system is proved.

2. Preliminaries

In this section, we introduce notation and spaces to be used later. Let $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ be a bounded domain with smooth boundary $\partial\Omega$. We assume that $g \in W^{1,\infty}(\Omega)$ satisfies

$$0 < m_0 \le g(x) \le M_0, \quad \forall x \in \Omega, \quad \text{and} \quad \|\nabla g\|_\infty < \frac{m_0 \lambda_1^{1/2}}{2} \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (see [5]), i.e. the spectral problem

$$\begin{gathered}
-\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{w}^j + \nabla p^j = \lambda_j \mathbf{w}^j, \quad \text{in } \Omega, \\
\nabla \cdot g\mathbf{w}^j = 0 \quad \text{in } \Omega, \\
\mathbf{w}^j = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.2}$$

Problem (2.2) has eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_j \le \dots$ and corresponding eigenfunctions $\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^j, \dots$ form an orthonormal basis in $\mathbf{H}_g$ and total basis in $\mathbf{V}_g$, where $\mathbf{H}_g$ and $\mathbf{V}_g$ are defined in the following manner:

$$\begin{gathered}
\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega) : \nabla \cdot (g\mathbf{u}) = 0\}, \\
\mathbf{H}_g \text{ is the closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \\
\mathbf{V}_g \text{ is the closure of } \mathcal{V} \text{ in } \mathbf{H}_0^1(\Omega).
\end{gathered}$$

Where $\mathbf{H}_g$ is endowed with the scalar product

$$(\mathbf{u}, \mathbf{v})_g = \int_\Omega (\mathbf{u} \cdot \mathbf{v}) g dx \quad \text{and} \quad |\mathbf{u}|^2 = (\mathbf{u}, \mathbf{u})_g.$$

Notice that this inner product is equivalent to the usual inner product defined in $\mathbf{L}^2(\Omega)$. Similarly, we define in $\mathbf{V}_g$ the equivalent inner product:

$$((\mathbf{u}, \mathbf{v}))_g = \int_\Omega g \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx.$$

Let us recall that β satisfies condition (1.5) if

$$\int_{\partial\Omega} \beta \cdot \mathbf{n} ds = 0.$$

In this case, Morimoto [7, p. 636] proved the next Lemma.

Lemma 2.1. *Suppose $\beta \in C^1(\mathbb{R}, \mathbf{H}^{1/2}(\partial\Omega)^n)$ is T-periodic and satisfies (SOC). Then for every $\varepsilon > 0$, there exists a solenoidal and T-periodic function $\mathbf{b} \in C^1(\mathbb{R}; \mathbf{H}^1(\Omega))$ such that*

$$\begin{gathered}
\nabla_x \cdot \mathbf{b}(t,x) = 0 \quad a.e\ x \in \Omega,\ \forall t \in \mathbb{R}, \\
\mathbf{b}(t,x) = \beta(t,x), \quad x \in \partial\Omega,\ \forall t \in \mathbb{R}, \\
|((\mathbf{u}\cdot\nabla)\mathbf{b}, \mathbf{u})| \le \varepsilon \|\nabla \mathbf{u}\|^2,\ \forall \mathbf{u} \in V, \forall t \in \mathbb{R}.
\end{gathered}$$

Now, if $\beta \in C^1(\mathbb{R}, \mathbf{H}^{1/2}(\partial\Omega)^n)$ is T-periodic and satisfies the (1.4) condition:

$$\int_{\partial\Omega} g\beta \cdot \mathbf{n} ds = 0,$$

we have the following proposition.

Proposition 2.2. *Suppose $\beta \in C^1(\mathbb{R}, \mathbf{H}^{1/2}(\partial\Omega)^n)$ is T-periodic and satisfies* (1.4)*. Then for every $\varepsilon > 0$ there exists a T-periodic function $\Psi \in C^1(\mathbb{R}; \mathbf{H}^1(\Omega))$ such that:*

$$\begin{gathered}
\nabla_x \cdot (g(x)\Psi(t,x)) = 0 \quad a.e\ x \in \Omega, \quad \forall t \in \mathbb{R} \\
\Psi(t,x) = \beta(t,x), \quad a.e.\ x \in \partial\Omega,\ \forall t \in \mathbb{R}, \\
|((\mathbf{v}\cdot\nabla)\Psi, \mathbf{v})_g| \le C(\Omega, g)(\varepsilon + \|\nabla g\|_{L^\infty} |\nabla\Psi|)|\nabla \mathbf{v}|^2, \quad \forall t \in \mathbb{R},
\end{gathered}$$

for all $\mathbf{v} \in V_g$.

Proof. For $\varepsilon > 0$, define $\Psi(t,x) = \frac{\mathbf{b}(t,x)}{g(x)}$, where $\mathbf{b}(t,x) \in C^1(\mathbb{R}, \mathbf{H}^1(\Omega))$ is given by Lemma 2.1. It is clear that $\Psi \in \mathbf{V}_g$ is T-periodic and $\Psi = \beta$ on $[0,T] \times \partial\Omega$. We have

$$\begin{aligned}
((\mathbf{v}\cdot\nabla)\Psi, \mathbf{v})_g &= \sum_{i,j=1}^{3} \int_\Omega \mathbf{v}_i \frac{\partial \Psi_j}{\partial x_i} \mathbf{v}_j g dx \\
&= \int_\Omega \sum_{i,j=1}^{3} \big(\frac{1}{g^2}\big) g \mathbf{v}_i \frac{\partial (g\Psi_j)}{\partial x_i} g \mathbf{v}_j dx - \int_\Omega \sum_{i,j=1}^{3} \mathbf{v}_i \frac{\partial g}{\partial x_i} \Psi_j \mathbf{v}_j dx
\end{aligned}$$

Now, from Lemma 2.1

$$\begin{aligned}
\big|\int_\Omega \sum_{i,j=1}^{3} \big(\frac{1}{g^2}\big) g \mathbf{v}_i \frac{\partial (g\Psi_j)}{\partial x_i} g \mathbf{v}_j dx\big| &\le \frac{1}{m_0^2} |((g\mathbf{v}\cdot\nabla)(g\Psi), g\mathbf{v})| \\
&\le \frac{\varepsilon}{m_0^2} |\nabla(g\mathbf{v})|^2 \\
&\le \varepsilon C(\Omega, g)|\nabla \mathbf{v}|^2
\end{aligned}$$

moreover,

$$\begin{aligned}
\big|\int_\Omega \sum_{i,j=1}^{3} \mathbf{v}_i \frac{\partial g}{\partial x_i} \Psi_j \mathbf{v}_j\big| &\le \|\nabla g\|_{L^\infty} |\mathbf{v}|_{L^3} |\Psi|_{L^6} |\mathbf{v}| \\
&\le C(\Omega, g)\|\nabla g\|_{L^\infty} |\nabla\Psi||\nabla \mathbf{v}|^2
\end{aligned}$$

Therefore,

$$|((\mathbf{v}\cdot\nabla)\Psi,\mathbf{v})_g| \le C(\Omega,g)(\varepsilon+\|\nabla g\|_{L^\infty}|\nabla\Psi|)|\nabla\mathbf{v}|^2.$$

□

Remark 2.3. Similarly to the case of the Navier-Stokes equation, we can define the trilinear form $b_g:\mathbf{V}_g\times\mathbf{V}_g\times\mathbf{V}_g\to\mathbb{R}$ by

$$b_g(\mathbf{u},\mathbf{v},\mathbf{w})=\sum_{i,j=1}^{n}\int_\Omega \mathbf{u}_i\frac{\partial\mathbf{v}_j}{\partial x_i}\mathbf{w}_j g\,dx$$

for every $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}_g$. It is not difficult (see [12]) to prove that

$$b_g(\mathbf{u},\mathbf{v},\mathbf{v})=0,$$

for each $\mathbf{u},\mathbf{v}\in\mathbf{V}_g$, moreover (see [5]), if we further assume that $\Delta g=0$ we have

$$b_g\big(\frac{\nabla g}{g},\mathbf{v},\mathbf{v}\big)=0,$$

for all $\mathbf{v}\in\mathbf{V}_g$.

Define the g-Laplacian operator as

$$-\Delta_g\mathbf{u}=-\frac{1}{g}(\nabla\cdot g\nabla)\mathbf{u}=-\Delta\mathbf{u}-\frac{1}{g}\nabla g\cdot\nabla\mathbf{u}.$$

Now, we can rewrite the first equation of (1.3) as follows:

$$\frac{\partial\mathbf{u}}{\partial t}-\nu\Delta_g\mathbf{u}+\nu\frac{\nabla g}{g}\cdot\nabla\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}.$$

3. Existence of reproductive and periodic solutions for the g-Navier-Stokes system

The variational formulation of (1.3) is the following: given $\mathbf{f}\in L^2(0,T;\mathbf{V}_g')$ and $\mathbf{u}_0\in\mathbf{V}_g$ to find $\mathbf{u}-\Psi\in L^\infty(0,T;\mathbf{H}_g)\cap L^2(0,T;\mathbf{V}_g)$ such that

$$\begin{aligned}&\frac{d}{dt}(\mathbf{u}-\Psi,\mathbf{v})+\nu((\mathbf{u}-\Psi,\mathbf{v}))_g+b_g(\mathbf{u}-\Psi,\mathbf{u}-\Psi,\mathbf{v})\\&+b_g(\Psi,\mathbf{u}-\Psi,\mathbf{v})+b_g(\mathbf{u}-\Psi,\Psi,\mathbf{v})+\nu b_g\big(\frac{\nabla g}{g},\mathbf{u}-\Psi,\mathbf{v}\big)\\&=\langle f,\mathbf{v}\rangle-L(\Psi,\mathbf{v})\\&\qquad\mathbf{u}(0)=\mathbf{u}_0+\Psi(0)\end{aligned}\tag{3.1}$$

for all $\mathbf{v}\in\mathbf{V}_g$. Here Ψ is given in Proposition 2.2, b_g is the trilinear form given in Remark 2.3 and

$$L(\Psi,\mathbf{v})=\big(\frac{d\Psi}{dt},\mathbf{v}\big)+\nu((\Psi,\mathbf{v}))_g+b_g(\Psi,\Psi,\mathbf{v})+\nu b_g\big(\frac{\nabla g}{g},\Psi,\mathbf{v}\big).$$

Definition 3.1. Let $\mathbf{u}_0\in\mathbf{H}_g$ and $\mathbf{f}\in L^2(0,T;\mathbf{V}_g')$. A function $\mathbf{u}\in L^\infty(0,T;\mathbf{H}_g)\cap L^2(0,T;\mathbf{V}_g)$ is a weak solution of the problem (1.1) with initial data $\mathbf{u}(0)=\mathbf{u}_0$ and boundary data $\mathbf{u}=\beta$ on $[0,T]\times\partial\Omega$, if $\mathbf{u}$ verifies (3.1) for all $\mathbf{v}\in\mathbf{V}_g$.

In the case $\beta\equiv0$, we have the following theorem.

Theorem 3.2 ([2, thm 6.1])**.** *Assume* $\mathbf{f}\in L^2(0,T;\mathbf{V}_g')$ *and* $\mathbf{u}_0\in\mathbf{H}_g$*. Then there exists at least a weak solution of the problem* (1.1)*, in the sense of the Definition 3.1. Moreover,* $\mathbf{u}$ *is weakly continuous from* $[0,T]$ *into* $\mathbf{H}_g$*.*

Proposition 3.3. *If $\Omega \subseteq \mathbb{R}^2$, under the assumptions of Theorem 3.2, the weak solution of* (1.1) *with initial data $\mathbf{u}(0) = \mathbf{u}_0$ is unique.*

Proof. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of the problem (3.1) with initial data $\mathbf{u}_0$. If we define $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, then it satisfies the variational formulation

$$\frac{1}{2}\frac{d}{dt}(\mathbf{w},\mathbf{v})_g + \nu((\mathbf{w},\mathbf{v}))_g + \nu\Big(\big(\frac{\nabla g}{g}\cdot\nabla\big)\mathbf{w},\mathbf{v}\Big)_g = -b_g(\mathbf{u}_1,\mathbf{u}_1,\mathbf{v}) + b_g(\mathbf{u}_2,\mathbf{u}_2,\mathbf{v})$$

By replacing $\mathbf{v} = \mathbf{w}$ we get

$$\begin{aligned}\frac{d}{dt}|\mathbf{w}|^2 + 2\nu\|\mathbf{w}\|^2 &= -2\nu\Big(\big(\frac{\nabla g}{g}\cdot\nabla\big)\mathbf{w},\mathbf{w}\Big)_g - 2b_g(\mathbf{u}_1,\mathbf{u}_1,\mathbf{w}) + 2b_g(\mathbf{u}_2,\mathbf{u}_2,\mathbf{w})\\ &= -2\nu\Big(\big(\frac{\nabla g}{g}\cdot\nabla\big)\mathbf{w},\mathbf{w}\Big)_g - 2b_g(\mathbf{w},\mathbf{u}_1,\mathbf{w});\end{aligned}$$

therefore, since

$$-2\nu\Big(\big(\frac{\nabla g}{g}\cdot\nabla\big)\mathbf{w},\mathbf{w}\Big)_g \le 2\nu\frac{\|\nabla g\|_\infty}{m_0\lambda_1}\|\mathbf{w}\|^2,$$

by [1, Lemma 2.1], we also have

$$\begin{aligned}2b_g(\mathbf{w},\mathbf{u}_1,\mathbf{w}) &\le C\|\mathbf{u}_1\||\mathbf{w}|\|\mathbf{w}\|\\ &\le \varepsilon\|\mathbf{w}\|^2 + C_\varepsilon\|\mathbf{u}_1\|^2|\mathbf{w}|^2.\end{aligned}$$

Now, for ε small enough we can obtain

$$\frac{d}{dt}|\mathbf{w}|^2 \le C_\varepsilon\|\mathbf{u}_1\|^2|\mathbf{w}|^2;$$

then by using Gronwall's inequality, we conclude that $\mathbf{w} = \mathbf{0}$. □

Remark 3.4. After some tedious calculations, it is possible to see that Theorem 3.2 and Proposition 3.3 remain valid even if the β is not null.

Our main result is the following.

Theorem 3.5. *For any $\mathbf{f} \in L^2(0,T;\mathbf{V}_g')$ and $\|\nabla g\|_\infty$ small enough there exists a weak solution of* (1.3) *i.e. the weak solution $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$ has the so-called reproductive property, i.e. a solution of the variational problem* (3.1) *which satisfies $\mathbf{u}(0,x) = \mathbf{u}(t,x)$.*

Remark 3.6. Note that if $n = 2$ and the external force $\mathbf{f} \in L^2(\mathbb{R};\mathbf{V}_g')$ is a T-periodic in time function, the above Theorem 3.5 furnishes a T-periodic weak solution for (1.3). In fact, it is a strong solution and actually very regular. This is so because we can prove that $\mathbf{u} \in C^\infty(\Omega)$ for $t > 0$, where $\mathbf{u}$ is solution of the problem (1.1) with initial condition $\mathbf{u}_0 \in \mathbf{H}_g$. Thus $\mathbf{u}_p \in C^\infty(\Omega)$ for $t \in [T, 2T]$ and, by the T-periodicity, we conclude that $\mathbf{u}_p(t) = \mathbf{u}_p(t+T) \in C^\infty(\Omega)$, here $\mathbf{u}_p$ is the reproductive solution. In particular, $\mathbf{u}_p(0) \in C^\infty(\Omega)$.

3.1. **Proof of Theorem 3.5 when $\beta \equiv 0$.** Let $\{\mathbf{w}^i\}_{i=1}^\infty$ be orthonormal bases in $\mathbf{H}_g$ and total bases in $\mathbf{V}_g$ obtained in spectral problem (2.2). As kth-approximated solution of equation (3.1) we choose

$$\mathbf{u}^k(t,x) = \sum_{i=1}^k c_i^k(t)\mathbf{w}^i(x) \tag{3.2}$$

satisfying for all $i = 1, \dots, k$, and for all $t \in (0, T)$ the system of equations

$$\begin{gathered} \frac{d}{dt}(\mathbf{u}^k, \mathbf{v})_g + \nu((\mathbf{u}^k, \mathbf{v}))_g + b_g(\mathbf{u}^k, \mathbf{u}^k, \mathbf{v}) + \nu b_g\big(\frac{\nabla g}{g}, \mathbf{u}^k, \mathbf{v}\big) = \langle \mathbf{f}, \mathbf{v}\rangle \\ \mathbf{u}^k(0) = P_k \mathbf{u}_0 \end{gathered} \tag{3.3}$$

for all $\mathbf{v} \in \mathbf{V}^k = \langle \{\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^k\}\rangle$. Taking $\mathbf{v} = \mathbf{u}^k$, we have

$$\frac{d}{dt}|\mathbf{u}^k|^2 + 2\nu|\nabla \mathbf{u}^k|^2 = \langle f, \mathbf{u}^k\rangle - 2\nu\Big(\big(\frac{\nabla g}{g}\cdot\nabla\big)\mathbf{u}^k, \mathbf{u}^k\Big)_g$$

Therefore, by using the Poincaré inequality,

$$|\mathbf{v}|^2 \le \frac{1}{\lambda_1}|\nabla \mathbf{v}|^2 \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

we have

$$\frac{d}{dt}|\mathbf{u}^k|^2 + 2\nu|\nabla \mathbf{u}^k|^2 \le \frac{1}{\nu}\|\mathbf{f}\|_{V^*}^2 + \nu|\nabla \mathbf{u}^k|^2 + 2\nu\frac{\|\nabla g\|_\infty}{m_0\lambda_1^{1/2}}|\nabla \mathbf{u}^k|^2\,. \tag{3.4}$$

Finally, we obtain

$$\frac{d}{dt}|\mathbf{u}^k|^2 + \nu\lambda_1\gamma_0|\mathbf{u}^k|^2 \le \frac{1}{\nu}\|\mathbf{f}\|_{V^*}^2\,,$$

where $\gamma_0 = 1 - \frac{2\|\nabla g\|_\infty}{m_0\lambda_1^{1/2}} > 0$ for $\|\nabla g\|_\infty$ small. The above inequality implies

$$\frac{d}{dt}(e^{\nu\lambda_1\gamma_0 t}|\mathbf{u}^k|^2) \le \frac{e^{\nu\lambda_1\gamma_0 t}}{\nu}\|\mathbf{f}\|_{V^*}^2\,.$$

Integrating from 0 to T we have

$$e^{\nu\lambda_1\gamma_0 T}|\mathbf{u}^k(T)|^2 \le |\mathbf{u}^k(0)|^2 + \frac{1}{\nu}\int_0^T e^{\nu\lambda_1\gamma_0 t}\|\mathbf{f}(t)\|_{V^*}^2. \tag{3.5}$$

Next, we show that $\mathbf{u}^k$ is nothing but one fixed point of the operator Φ^k defined in what follows. Let $L^k : [0, T] \to \mathbb{R}^k$ the mapping defined by

$$L^k(t) = \mathbf{y}(t) = (c_1^k(t), \dots, c_k^k(t)),$$

where the time dependent functions $\{c_i^k(t)\}_{i=1}^k$ are the coefficients of the expansion of $\mathbf{u}^k$, as done in (3.2).

Since we have chosen the basis $\{\mathbf{w}^i(x)\}_{i=1}^\infty$ orthonormal in $\mathbf{H}_g$, we have

$$\|\mathbf{y}(t)\|_{\mathbb{R}^k} = |\mathbf{u}^k(t)| \quad \forall t \in\ [0, T]\,. \tag{3.6}$$

Next, we define the operator $\Phi^k : \mathbb{R}^k \to \mathbb{R}^k$ as

$$\Phi^k(\mathbf{x}) = \mathbf{y}(T)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_k)$ and $\mathbf{y}(T) = L^k(T)$ is the vector-coefficients at time T of the solution of (3.3) with initial condition

$$\mathbf{u}_0^k(x) = \sum_{i=1}^m x_i\ \mathbf{w}^i(x),$$

It is not difficult to see that Φ^k is continuous and we claim that Φ^k has at least one fixed point. It will be a consequence of Leray-Schauder's Homotopy Theorem. To prove this, it is enough to show that for any $\lambda \in [0, 1]$, a solution of the equation

$$\lambda\Phi^k(\mathbf{x}(\lambda)) = \mathbf{x}(\lambda) \tag{3.7}$$

has a bound independent of λ. Since $\mathbf{x}(0)=0$, we restrict the proof to $\lambda\in(0,1]$. In such case (3.7) may be rewritten as

$$\Phi^k(\mathbf{x}(\lambda))=\frac{1}{\lambda}\mathbf{x}(\lambda)\,.$$

By the definition of Φ^k and (3.6), we deduce from (3.5), that

$$e^{\nu\lambda_1\gamma_0T}\|\frac{1}{\lambda}\mathbf{x}(\lambda)\|^2_{\mathbb{R}^k}\le\|\mathbf{x}(\lambda)\|^2_{\mathbb{R}^k}+\int_0^T e^{\nu\lambda_1\gamma_0T}\|\mathbf{f}(t)\|^2_{\mathbf{V}^*}dt$$

Since we impose $\mathbf{u}^k(0)=\mathbf{u}^k(T)$, we obtain

$$\|\mathbf{x}(\lambda)\|^2_{\mathbb{R}^k}\le\frac{1}{e^{\nu\lambda_1\gamma_0T}-1}\int_0^T e^{\nu\lambda_1\gamma_0T}\|\mathbf{f}(t)\|_{V^*}dt\equiv M(T,\mathbf{f}),\tag{3.8}$$

for all $\lambda\in(0,1]$. Obviously, this upper bound do not depends on $\lambda\in[0,1]$ and so we have stated that the operator Φ^k has at least one fixed point, denoted by $\mathbf{x}(1)$ and then there exists a reproductive Galerkin solution $\mathbf{u}^k$, namely it satisfies $\mathbf{u}^k(0)=\mathbf{u}^k(T)$. Note that, from (3.8), we have that $\mathbf{u}^k\in L^\infty(0,T;\mathbf{H}_g)$, for every $k\in\mathbb{N}$ and it is uniformly bounded.

From (3.4) and by definition of γ_0 we can obtain the inequality

$$\frac{d}{dt}|\mathbf{u}^k|^2+\nu\gamma_0|\nabla\mathbf{u}^k|^2\le\frac{1}{\nu}\|\mathbf{f}\|^2\,.$$

Since $\mathbf{u}^k$ is a Galerkin reproductive solution and by integrating from 0 to T we have

$$\int_0^T|\nabla\mathbf{u}^k|^2dt\le\frac{1}{\gamma_0\nu^2}\int_0^T\|\mathbf{f}\|^2dt=\widetilde{M}(T,\mathbf{f}).\tag{3.9}$$

In other words, $\mathbf{u}^k\in L^2(0,T;\mathbf{V}_g)\cap L^\infty(0,T;\mathbf{H}_g)$, for each $k\in\mathbb{N}$ and it is uniformly bounded. It is not difficult to prove that $\frac{d}{dt}\mathbf{u}^k\in L^2(0,T;\mathbf{V}'_g)$ and it is uniformly bounded. By using compactness results (see [11]) with the triplets $\mathbf{H}_g\hookrightarrow\mathbf{V}'_g\hookrightarrow\mathbf{V}'_g$ and $\mathbf{V}_g\hookrightarrow\mathbf{H}_g\hookrightarrow\mathbf{V}'_g$, we have that $(\mathbf{u}^k)$ is relatively compact in $L^2(0,T;\mathbf{H}_g)\cap C([0,T];\mathbf{V}'_g)$. Thus, since $\mathbf{u}^k(0)=\mathbf{u}^k(T)$ and $\mathbf{u}^k(0)\to\mathbf{u}(0)$, we get that $\mathbf{u}(0)=\mathbf{u}(T)$ in $\mathbf{V}'_g$, but we also have that $\mathbf{u}\in C([0,T];\mathbf{H}_g)$, because $\mathbf{u}\in L^2(0,T;\mathbf{H}_g)$ and $\frac{d}{dt}\mathbf{u}\in L^2(0,T;\mathbf{V}'_g)$ (see [12]), therefore $\mathbf{u}(0)=\mathbf{u}(T)$ in $\mathbf{H}_g$.

3.2. **Proof of Theorem 3.5, general case.** Let us define $\hat{\mathbf{u}}=\mathbf{u}-\Psi$, where Ψ is given in Proposition 2.2, which satisfies

$$\begin{gathered}\frac{\partial\hat{\mathbf{u}}}{\partial t}-\nu\Delta\hat{\mathbf{u}}+(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}+(\hat{\mathbf{u}}\cdot\nabla)\Psi+(\Psi\cdot\nabla)\hat{\mathbf{u}}+\nabla p\\ =f-\frac{\partial\Psi}{\partial t}+\nu\Delta\Psi-(\Psi\cdot\nabla)\Psi\quad\text{in }]0,T[\times\Omega\,,\\ \frac{1}{g}(\nabla(g\hat{\mathbf{u}}))=\frac{\nabla g}{g}\cdot\hat{\mathbf{u}}+\nabla\cdot\hat{\mathbf{u}}=0\quad\text{in }]0,T[\times\Omega\,,\\ \hat{\mathbf{u}}(0,x)=\hat{\mathbf{u}}_0(x)\quad\text{in }]0,T[\times\Omega,\\ \hat{\mathbf{u}}(t,x)=0\quad\text{on }[0,T]\times\partial\Omega\,.\end{gathered}\tag{3.10}$$

Since Ψ is a T-periodic function it is only necessary to prove that there exists a reproductive solution of the problem (3.10).

The variational formulation is as follows: Find $\hat{\mathbf{u}} \in L^\infty(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$ such that for all $\mathbf{v} \in \mathbf{V}_g$ we have

$$\begin{aligned} &\frac{d}{dt}(\hat{\mathbf{u}},\mathbf{v})_g + \nu((\hat{\mathbf{u}},\mathbf{v}))_g + b_g(\hat{\mathbf{u}},\hat{\mathbf{u}},\mathbf{v}) + b_g(\hat{\mathbf{u}},\Psi,\mathbf{v}) \\ &+ b_g(\Psi,\hat{\mathbf{u}},\mathbf{v}) + \nu b_g\big(\frac{\nabla g}{g},\hat{\mathbf{u}},\mathbf{v}\big) \\ &= \langle \mathbf{f},\mathbf{v}\rangle - L(\Psi,\mathbf{v}), \end{aligned} \tag{3.11}$$

where

$$L(\Psi,\mathbf{v}) = \Big(\frac{d\Psi}{dt},\mathbf{v}\Big)_g + \nu((\Psi,\mathbf{v}))_g + b_g(\Psi,\Psi,\mathbf{v}) + \nu b_g\big(\frac{\nabla g}{g},\Psi,\mathbf{v}\big).$$

After some calculations, we can write

$$\begin{aligned} |L(\Psi,\mathbf{v})| &\le \Big(|\frac{d\Psi}{dt}| + \frac{\nu\|\nabla g\|_\infty}{m_0}|\nabla\Psi|\Big)|\mathbf{v}| + (\nu|\nabla\Psi| + |\nabla\Psi|^2)|\nabla\mathbf{v}| \\ &\le \frac{1}{2\varepsilon_1}\Big(|\frac{d\Psi}{dt}| + \frac{\nu\|\nabla g\|_\infty}{m_0}|\nabla\Psi|\Big)^2 + \frac{\varepsilon_1}{2}|\mathbf{v}|^2 \\ &\quad + \frac{1}{2\varepsilon_1}(|\nabla\Psi|^2 + \nu|\nabla\Psi|)^2 + \frac{\varepsilon_1}{2}|\nabla\mathbf{v}|^2 . \end{aligned}$$

Let us put

$$F = \frac{1}{2\varepsilon_1}\Big(|\frac{d\Psi}{dt}| + \frac{\nu\|\nabla g\|_\infty}{m_0}|\nabla\Psi|\Big)^2 + \frac{1}{2\varepsilon_1}(|\nabla\Psi|^2 + \nu|\nabla\Psi|)^2 + \frac{1}{2\varepsilon_1}\|\mathbf{f}\|_{V_g^*} .$$

By replacing $\mathbf{v}$ by $\hat{\mathbf{u}}$ in (3.11) we obtain

$$\begin{aligned} \frac{d}{dt}|\hat{\mathbf{u}}|^2 + 2\nu|\nabla\hat{\mathbf{u}}|^2 &\le \frac{\varepsilon_1}{2}|\hat{\mathbf{u}}|^2 + F + \Big(\varepsilon_1 C(\Omega,g) + \varepsilon_1 \\ &\quad + C(\Omega,g)\|\nabla g\|_\infty|\nabla\Psi| + 2\nu\frac{\|\nabla g\|_\infty}{m_0\lambda^{1/2}}\Big)|\nabla\hat{\mathbf{u}}|^2 . \end{aligned}$$

By choosing ε_1 and $\|\nabla g\|_\infty$ small enough, we obtain

$$\frac{d}{dt}|\hat{\mathbf{u}}(t)|^2 + C|\hat{\mathbf{u}}(t)|^2 \le F(t), \tag{3.12}$$

where $C > 0$, we can obtain a reproductive solution by following the same argument as in the proof of the case $\beta \equiv 0$.

4. Existence of reproductive solutions for the g-Kelvin-Voight system

The variational formulation of problem (1.6) is: Given $\mathbf{f} \in L^2(0,T;\mathbf{V}'_g)$ and $\mathbf{u}_0 \in \mathbf{H}_g$, find $\mathbf{u} \in \mathbf{V}_g$ such that

$$\begin{gathered} \frac{d}{dt}(\mathbf{u},\mathbf{v})_g + \nu((\mathbf{u},\mathbf{v})) + \alpha((\mathbf{u}_t,\mathbf{v})) + \nu b_g\big(\frac{\nabla g}{g},\mathbf{u},\mathbf{v}\big) \\ + \alpha b_g\big(\frac{\nabla g}{g},\mathbf{u}_t,\mathbf{v}\big) + b_g(\mathbf{u},\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle \\ \mathbf{u}(0) = \mathbf{u}_0 , \end{gathered} \tag{4.1}$$

for all $\mathbf{v} \in \mathbf{V}_g$.

Definition 4.1. Let $\mathbf{u}_0 \in \mathbf{H}_g$ and $\mathbf{f} \in L^2(0,T;\mathbf{V}_g')$. A function $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$ is a weak solution of the problem (1.6) with initial condition $\mathbf{u}(0) = \mathbf{u}_0$ if $\mathbf{u}$ verifies (4.1) for all $\mathbf{v} \in \mathbf{V}_g$.

Theorem 4.2 ([5]). *If $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ $\mathbf{u}_0 \in \mathbf{V}_g$ and g satisfying* (2.1) *and $\Delta g = 0$, then there exists a unique weak solution of* (1.6).

Remark 4.3. It is possible to prove that the hypothesis that $\mathbf{f}$ does not depend on time t can be removed and replaced by $\mathbf{f} \in L^2(0,T;\mathbf{V}_g')$, and the theorem is still valid.

The main result of this section is the following.

Theorem 4.4. *For $\|\mathbf{f}\|_{L^2(0,T;\mathbf{V}_g')}$ and $\|\nabla g\|_\infty$ small enough there exists a weak solution of* (1.7) *i.e. the weak solution $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$ has the so-called reproductive property, i.e. a solution of the variational problem* (4.1) *which satisfies $\mathbf{u}(0,x) = \mathbf{u}(T,x)$.*

4.1. **Proof of Theorem 4.4.** In the same manner as in the proof of Theorem 3.5, we define

$$\mathbf{u}^k(t,x) = \sum_{i=1}^{k} c_i^k(t)\mathbf{w}^i(x) \tag{4.2}$$

as the solution of the variational problem

$$\begin{aligned}
&\frac{d}{dt}(\mathbf{u}^k,\mathbf{v})_g + \nu((\mathbf{u}^k,\mathbf{v})) + \alpha((\mathbf{u}_t^k,\mathbf{v})) + \nu b_g\big(\frac{\nabla g}{g},\mathbf{u}^k,\mathbf{v}\big) \\
&+ \alpha b_g\big(\frac{\nabla g}{g},\mathbf{u}_t^k,\mathbf{v}\big) + b_g(\mathbf{u}^k,\mathbf{u}^k,\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}^k = \langle\{\mathbf{w}^1,\dots,\mathbf{w}^k\}\rangle$. The proof of the following lemma can be found in [5, pp 499-501]. For simplicity, we denote

$$y(t) = \|\mathbf{u}^k(t)\|_g^2 + (\alpha+\nu)\|\nabla\mathbf{u}^k(t)\|_g^2.$$

Lemma 4.5. *For $\|\nabla g\|_\infty$ small enough there exist positive constants β and δ such that the function $y(t)$ satisfies*

$$\frac{dy}{dt} + \beta y \le \delta y^2 + C\|\mathbf{f}(t)\|_g^2.$$

Proposition 4.6. *Let $M_1 > 0$ be such that*

$$\delta s < \frac{\beta}{2}, \quad \forall s \in]0, M_1].$$

Let us suppose that δ satisfies $\|\mathbf{f}\|^2_{L^\infty(0,T;\mathbf{V}_g')} \le \frac{\beta}{2}M_1$. If $y(0) \le M_1$, then $y(t) \le M_1$, for all $t \in [0,T]$.

Proof. From Lemma 4.5, y satisfies the differential inequality

$$y' + (\beta - \delta y)y \le \|\mathbf{f}(t)\|_g^2. \tag{4.3}$$

By hypothesis, there exists $\sigma > 0$ such that

$$\delta s \le \frac{\beta}{2}, \quad \forall s \in [M_1, M_1+\sigma]. \tag{4.4}$$

At first, we will prove that

$$y(t) < M_1 + \sigma, \quad \forall t \in [0,T].$$

By contradiction, let $T^* \in]0,T]$ be the first value so that $y(T^*) = M_1 + \sigma$ and $y(t) < M_1 + \sigma$, for all $t \in [0,T^*[$. By (4.4), we have that $\delta y(t) \le \frac{\beta}{2}$, for all $t \in [0,T^*]$. From (4.3) and the hypothesis

$$y' + \frac{\beta}{2} y \le \frac{\beta}{2} M_1, \tag{4.5}$$

by multiplying by $e^{\frac{\beta}{2}t}$ and integrating in time in $[0,T^*]$ we obtain

$$\begin{gathered}
e^{\frac{\beta}{2}T^*} y(T^*) - y(0) \le M_1 (e^{\frac{\beta}{2}T^*} - 1)\,, \\
e^{\frac{\beta}{2}T^*} y(T^*) \le y(0) + M_1 e^{\frac{\beta}{2}T^*} - M_1\,, \\
e^{\frac{\beta}{2}T^*} y(T^*) \le M_1 + M_1 e^{\frac{\beta}{2}T^*} - M_1 \le M_1\,.
\end{gathered}$$

In other words, $y(T^*) \le M_1$ which is a contradiction and, therefore, $y(t) \le M_1 + \sigma$ for all $t \in [0,T]$. Furthermore, the inequality (4.5) holds for every $t \in [0,T]$, hence by repeating the same arguments in each interval $[0,t]$, for all $t \in]0,T]$, we get $y(t) \le M_1$, which completes the proof. □

Now, for $(\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$ and $\mathbf{u} = \xi_1 \mathbf{w}^1 + \xi_2 \mathbf{w}^2 + \dots + \xi_m \mathbf{w}^m$, we define the norm

$$\|(\xi_1, \xi_2, \dots, \xi_m)\|_{\mathbb{R}^m} = \|\mathbf{u}(t)\|_g^2 + (\alpha + \nu)\|\nabla \mathbf{u}(t)\|_g^2\,.$$

Given $(\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, define $\Phi^m : \mathbb{R}^m \to \mathbb{R}^m$ in as

$$\Phi^m(\xi_1, \xi_2, \dots, \xi_m) = (c_1^m(T), c_2^m(T), \dots, c_m^m(T)),$$

where $(c_1^m(t), c_2^m(t), \dots, c_m^m(t))$ are the coefficients of the Galerkin solution (4.2) with initial condition $\mathbf{u}_0 = \xi_1 \mathbf{w}^1 + \xi_2 \mathbf{w}^2 + \dots + \xi_m \mathbf{w}^m$. If we define,

$$\overline{B} = \{(\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m : \|(\xi_1, \xi_2, \dots, \xi_m)\| < M_1\}$$

where M_1 is given in Proposition 4.6, from Proposition 4.6, Φ^m maps $\overline{B}$ into $\overline{B}$; therefore, by the Brower Fixed-Point Theorem Φ^m has a fixed point and, consequently there exists a reproductive Galerkin solution $\mathbf{u}^m$. The Theorem follows from the standard compact arguments.

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