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COMPARISON THEOREMS FOR THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. We establish comparison theorems for the oscillation of solutions to third-order neutral differential equations via linear ordinary and delay differential equations. Several applications illustrate the role of the deviating argument in the differential operator.

1. Introduction

The recent monograph [19] is devoted to the various aspects of differential equations of third order. In particular, Chapter 6 concerns the oscillation of delay differential equations. Motivated by these results and recent ones for delay and neutral differential equations [1, 3, 4, 9, 18]), we study the relationship between ordinary, delay and neutral differential equations.

In this article we study the third-order neutral differential equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{r(t)}\big[x(t)+a(t)x\big(\gamma(t)\big)\big]'\Big)'\Big)'+q(t)f\big(x\big(\delta(t)\big)\big)=0, \tag{1.1}$$

where $t \ge t_0$.

We make the following assumptions:

(i) $p(t)$, $r(t)$, $q(t)$, $a(t)$, $\gamma(t)$, $\delta(t) \in C[t_0,\infty)$, $p(t)$, $r(t)$, $q(t)$, $\gamma(t)$, $\delta(t)$ are positive for $t \ge t_0$,
(ii) $\int_{t_0}^\infty p(t)\,\mathrm{d}t = \int_{t_0}^\infty r(t)\,\mathrm{d}t = \infty$,
(iii) $\gamma(t) \le t$, $\lim_{t\to\infty}\gamma(t) = \infty$,
(iv) $\lim_{t\to\infty}\delta(t) = \infty$,
(v) $0 \le a(t) \le a_0 < 1$ for $t \ge t_0$,
(vi) $f \in C(\mathbb{R},\mathbb{R})$, $f(0) = 0$ and $f(v)v > 0$ for $v \ne 0$.

It is convenient to set, for each solution x of (1.1),

$$u(t) = x(t) + a(t)x\big(\gamma(t)\big)\,. \tag{1.2}$$

For this function we define the functions

$$u^{[0]} = u, \quad u^{[1]} = \frac{1}{r(t)}u', \quad u^{[2]} = \frac{1}{p(t)}\Big(\frac{1}{r(t)}u'\Big)' = \frac{1}{p(t)}\big(u^{[1]}\big)'$$

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that are called quasiderivatives of u. To simplify notation, we set

$$L_3(\cdot) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r(t)}\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)\,.$$

Assumption (ii) implies that operator L_3 is in the so-called canonical form.

A solution x of (1.1) is said to be *proper* if it is defined on the interval $[t_0, \infty)$ and satisfies the condition

$$\sup\{|x(s)|\colon t \le s < \infty\} > 0 \quad \text{for all } t \ge t_0.$$

A proper solution is called *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros.

Equation (1.1) covers not only the linear ordinary differential equations (ODE when $a(t) = 0$, $\delta(t) = t$) but also the functional differential equations (FDE when $a(t) = 0$). It is natural to try to investigate the relationship between (1.1) and the corresponding linear ODE or FDE. The oscillation theory of these equations was deeply studied by many authors; in the case of the ODEs we refer reader to [6, 7, 8, 10, 13] and the monograph [15], in the case of the FDEs we refer to [1, 11, 17] and the monograph [12]. Recently, a considerable attention has been paid to the asymptotic theory of the neutral differential equations, see e.g. [2, 3, 4, 9, 18] and the monograph [12, Section 10.4–10.6].

Oscillatory properties of the third-order neutral differential equations are usually described [2, 3, 9, 18] in the sense corresponding to the so-called property A. Therefore, motivated by the classical definition of property A for the higher order ordinary differential equations by Kiguradze [15] and its extension for the functional equations by Kusano and Naito [16], we introduce the following definition of property A for equation (1.1).

Definition 1.1. Equation (1.1) is said to have *property A* if any proper solution x of (1.1) is oscillatory or satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Some authors (e.g. [18]) use a different terminology and instead of using property A, they say that equation (1.1) is *almost oscillatory*.

Our aim here is to give comparison theorems for (1.1) via the linear ordinary or functional differential equations of the form

$$L_3 y(t) + k q(t) y\big(\delta(t)\big) = 0, \tag{1.3}$$

where $\delta(t) \le t$ and k is a suitable constant. These results enable us to obtain oscillation criteria for (1.1) from those given for (1.3). We refer to [19, Sections 6.2-6.3], where numerous criteria for the oscillation of (1.3) can be found.

We will give a special attention to the case when the differential operator L_3 is symmetric, i.e. $p(t) = r(t)$, prototype of that is the linear neutral equation

$$\Big(x(t) + a(t)x\big(\gamma(t)\big)\Big)''' + q(t)x\big(\delta(t)\big) = 0.$$

Our main tool for the comparison method is the linearization technique. Therefore in Sections 2 and 3 we recall basic properties of linear equation (1.3). Section 3 also contains some new results for the FDEs. In Section 4 properties of nonoscillatory solutions of (1.1) are given. Our main results are stated in Section 5. Section 6 presents some applications.

2. Preliminaries: Linear ODE

Consider the third-order linear differential equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{r(t)}x'(t)\Big)'\Big)' + q(t)x(t) = 0. \tag{2.1}$$

For completeness, we summarize basic results concerning the oscillatory behaviour of (2.1), which we will need in our later consideration.

It is well-known (see for instance [16]) that all nonoscillatory solutions x of (2.1) can be divided into the two classes:

$$\mathcal{N}_0 = \big\{x \text{ solution of (2.1)}, \exists T_x \colon x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \ge T_x\big\}$$
$$\mathcal{N}_2 = \{x \text{ solution of (2.1)}, \exists T_x \colon x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \ge T_x\big\}.$$

Definition 2.1. Equation (2.1) is said to have *property A* if every proper solution x of (2.1) is oscillatory or satisfies

$$\big|x^{[i]}(t)\big| \downarrow 0 \quad \text{as } t \to \infty, \quad i = 0, 1, 2.$$

Equation (2.1) is said to have *property* $\bar{A}$ if any proper solution x of (2.1) is oscillatory or belongs to $\mathcal{N}_0$.

Theorem 2.2 ([8, Theorem 5]). *If*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} r(s) \int_{t_0}^{s} p(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty, \tag{2.2}$$

then all solutions of (2.1) *are nonoscillatory.*

Theorem 2.3 ([7, Lemma 2.2]). *Equation* (2.1) *has property* $\bar{A}$ *if and only if it has at least one oscillatory solution.*

Theorem 2.4 ([7, Theorem 2.2]). *Equation* (2.1) *has property A if and only if it has at least one oscillatory solution and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t = \infty. \tag{2.3}$$

From Theorems 2.2–2.4 we obtain the following results.

Proposition 2.5. *The class* $\mathcal{N}_0$ *is not empty for* (2.1). *If* (2.2) *holds, then* $\mathcal{N}_2$ *is not empty for* (2.1).

Proof. The first part follows from results of Hartman and Wintner [14, p. 506]. The second part follows from Theorems 2.2 and 2.3. □

Proposition 2.6. *Consider equation* (2.1), *where* $p(t) = r(t)$ *for large* t. *Then* (2.1) *has property A if and only if it has property* $\bar{A}$.

3. Functional differential equations

Consider the linear functional differential equation

$$L_3x(t) + q(t)x\big(\delta(t)\big) = 0. \tag{3.1}$$

The classification of nonoscillatory solutions of (3.1) and definitions of property A and Ā are the same as for equation (2.1).

We recall the comparison theorem for the functional differential equations stated in [16, Theorem 2]. We reformulate it in a slightly different form, which will be useful for our purpose.

Consider the third-order linear functional differential equations

$$L_3 y(t) + q_1(t) y\big(\delta_1(t)\big) = 0 \tag{3.2}$$

and

$$L_3 z(t) + q_2(t) z\big(\delta_2(t)\big) = 0 \tag{3.3}$$

where $q_1(t) \ge q_2(t) > 0$ and $\lim_{t\to\infty} \delta_1(t) = \lim_{t\to\infty} \delta_2(t) = \infty$.

Proposition 3.1. *Assume*

$$\delta_1(t) \ge \delta_2(t) \quad \text{and} \quad q_1(t) \ge q_2(t) \quad \text{for } t \ge t_1.$$

(a) *If there exists a solution $y \in \mathcal{N}_2$ of* (3.2), *then there exists a solution $z \in \mathcal{N}_2$ of* (3.3).
(b) *If there exists a solution $y \in \mathcal{N}_0$ of* (3.2) *such that $\lim_{t\to\infty} |y(t)| > 0$, then there exists a solution $z \in \mathcal{N}_0$ of* (3.3) *such that $\lim_{t\to\infty} |z(t)| > 0$.*

Proposition 3.2. *If $\delta(t) \le t$ and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t < \infty, \tag{3.4}$$

then equation (3.1) *has a solution $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} |x(t)| > 0$.*

Proof. By Theorem 2.4 and Proposition 2.5, equation (2.1) has a solution x in the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} |x(t)| > 0$. Now the conclusion follows from Proposition 3.1-b). □

By Proposition 3.2 we have that if the delay equation (3.1) has property A, then equation (2.1) has also property A. Under the additional conditions the delay equations can be compared with ODE (without delay).

Proposition 3.3 ([16, Theorem 8]). *Let $|t - \delta(t)|$ be bounded and let functions $p(t)$, $r(t)$ be non-increasing for $t \in [t_0, \infty)$. Then equation* (3.1) *has property A if and only if equation* (2.1) *has property A.*

Our next theorem extends Proposition 2.6 for the functional differential equations with the symmetrical operator

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)} x'(t)\Big)'\Big)' + q(t) x\big(\delta(t)\big) = 0 \tag{3.5}$$

and complements some results from [19, Chapter 6].

Theorem 3.4. *Consider equation* (3.5) *and assume that $\delta(t) \le t$. Then the following statements are equivalent:*

(a) *$\mathcal{N}_2 = \emptyset$, i.e.* (3.5) *has property $\bar{A}$;*
(b) *every solution is oscillatory or tends to zero as $t \to \infty$, i.e.* (3.5) *has property A.*

Proof. "(b)⇒ (a)": It is immediate.

"(a)⇒ (b)": Assume by contradiction that there exists a solution $x \in \mathcal{N}_0$ of (3.5) such that $\lim_{t\to\infty} x(t) = c > 0$. Consider the linear equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)} y'(t)\Big)'\Big)' + q(t) \frac{x(\delta(t))}{x(t)} y(t) = 0. \tag{3.6}$$

Then $y = x$ is a solution of (3.6). By Theorem 2.4, we have

$$\int_{t_0}^{\infty} q(t)\frac{x(\delta(t))}{x(t)} \int_{t_0}^{t} p(s) \int_{t_0}^{s} p(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Obviously, $\lim_{t\to\infty} \frac{x(\delta(t))}{x(t)} = 1$, so

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} p(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

By Theorem 2.2, the linear equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)}z'(t)\Big)'\Big)' + q(t)z(t) = 0 \tag{3.7}$$

does not have oscillatory solutions. Therefore it has a solution $z \in \mathcal{N}_2$ by Proposition 2.5.

Consider the linear equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)}v'(t)\Big)'\Big)' + q(t)\frac{z(t)}{z(\delta(t))}v\big(\delta(t)\big) = 0. \tag{3.8}$$

Then $v = z$ is a solution of (3.8). Since z is increasing and $\delta(t) \le t$, we have

$$\frac{z(t)}{z(\delta(t))} \ge 1 \quad \text{for large } t.$$

By the comparison theorem for the functional differential equation (Proposition 3.1), equation (3.5) has a solution $x \in \mathcal{N}_2$, a contradiction. □

4. Neutral nonlinear equation - basic properties

In this section we study properties of nonoscillatory solutions of (1.1).

Lemma 4.1. *Let x be a nonoscillatory solution of* (1.1) *and let u be defined by* (1.2)*. Then u, $u^{[1]}$, $u^{[2]}$ are monotone for large t.*

Proof. Set $y = u^{[1]}$ and $z = u^{[2]}$. Then x is a solution of (1.1) if and only if (u, y, z) is a solution of the system

$$\begin{aligned} u'(t) &= r(t)y(t)\\ y'(t) &= p(t)z(t)\\ z'(t) &= -q(t)f\big(x\big(\delta(t)\big)\big). \end{aligned}$$

From the last equation we see that z' is of one sign for large t and so z is of one sign as well. Using this fact we obtain from the second equation that the same is true for y'. Similarly, we obtain from the first equation that u' is also of one sign. Therefore u, $u^{[1]}$ and $u^{[2]}$ are monotone. □

Lemma 4.2. *Let x be a solution of* (1.1) *and let u be defined by* (1.2)*. If either $u(t) > 0$ and $u^{[1]}(t) > 0$ or $u(t) < 0$ and $u^{[1]}(t) < 0$ for $t \ge T$, then*

$$(1 - a_0)|u(t)| \le |x(t)| \le |u(t)| \tag{4.1}$$

for $t \ge T$.

Proof. Assume that $u(t) > 0$ and $u^{[1]}(t) > 0$ for $t \geq T$. Since $\gamma(t) \leq t$ and u is an increasing function, we have $x\big(\gamma(t)\big) \leq u\big(\gamma(t)\big) \leq u(t)$. Hence

$$x(t) = u(t) - a(t)x\big(\gamma(t)\big) \geq u(t) - a_0 x\big(\gamma(t)\big) \geq u(t) - a_0 u\big(\gamma(t)\big) \geq u(t)(1-a_0).$$

The proof for $u(t) < 0$ and $u^{[1]}(t) < 0$ for $t \geq T$ is similar and is omitted. □

Lemma 4.3. *Let x be a nonoscillatory solution of* (1.1) *and let u be defined by* (1.2)*. Then there are only two possible classes of solutions*

$$\mathcal{N}_0 = \big\{x \text{ solution}, \exists T_x \colon u(t)u^{[1]}(t) < 0,\ u(t)u^{[2]}(t) > 0 \text{ for } t \geq T_x\big\},$$
$$\mathcal{N}_2 = \big\{x \text{ solution}, \exists T_x \colon u(t)u^{[1]}(t) > 0,\ u(t)u^{[2]}(t) > 0 \text{ for } t \geq T_x\big\}.$$

Proof. Without loss of generality we may assume that there exists t_1 such that $x\big(\delta(t)\big) > 0$, $x(t) > 0$ for $t \geq t_1$. Then $u(t) \geq x(t) > 0$ and from (1.1),

$$\big(u^{[2]}(t)\big)' = -q(t)f\big(x\big(\delta(t)\big)\big) < 0, \quad t \geq t_1.$$

Therefore $u^{[2]}$ is decreasing and there exists $t_2 \geq t_1$ such that there are two possibilities, either $u^{[2]}(t) < 0$ or $u^{[2]}(t) > 0$ for $t \geq t_2$. Assume that $u^{[2]}(t) < 0$ for $t \geq t_2$. Then there exists a constant $M > 0$ such that

$$u^{[2]}(t) \leq -M < 0.$$

Integrating this inequality from t_2 to t we obtain

$$u^{[1]}(t) \leq u^{[1]}(t_2) - M\int_{t_2}^{t} p(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \infty$, we obtain $u^{[1]}(t) \to -\infty$, i.e. $u^{[1]}(t) < 0$ eventually. Proceeding by the same way and using the fact that $\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$, we obtain $u(t) \to -\infty$, a contradiction. Thus $u^{[2]}(t) > 0$ and $u^{[1]}$ is increasing for $t \geq t_2$. Therefore there are two possibilities, either $u(t) > 0$, $u^{[1]}(t) < 0$, $u^{[2]}(t) > 0$, or $u(t) > 0$, $u^{[1]}(t) > 0$, $u^{[2]}(t) > 0$. □

Lemma 4.4. *Let x be a solution of* (1.1) *from the class $\mathcal{N}_2$. Then*

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |u(t)| = \infty.$$

Proof. Let $x \in \mathcal{N}_2$. Without loss of generality we may assume that x is eventually positive, i.e. there exists $T \geq t_0$ such that $x(t) > 0$, $u(t) > 0$, $u^{[1]}(t) > 0$ and $u^{[2]}(t) > 0$ for $t \geq T$. As $u^{[1]}$ is positive and increasing function there exists $K > 0$ such that $u^{[1]}(t) \geq K$ for large t. Integrating this inequality from T to t we obtain

$$u(t) \geq u(T) + K\int_{T}^{t} r(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$, we obtain $u(t) \to \infty$. By Lemma 4.2, $x(t) \geq (1-a_0)u(t)$. From this it follows that $x(t) \to \infty$. □

Proposition 4.5. *Let x be a solution of* (1.1) *from the class $\mathcal{N}_0$. Then*

$$\lim_{t\to\infty} u^{[i]}(t) = 0 \quad \text{for } i = 1, 2$$

and

$$\liminf_{t\to\infty} |x(t)| > 0 \iff \lim_{t\to\infty} |u(t)| > 0. \tag{4.2}$$

Moreover, if (2.3) *holds, then*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} u(t) = 0. \tag{4.3}$$

Proof. Assume that $x \in \mathcal{N}_0$. Without loss of generality we may assume that x is eventually positive, i.e. $u(t) > 0$, $u^{[1]}(t) < 0$, $u^{[2]}(t) > 0$ for $t \geq T_x$. Since u is positive, there exists $\lim_{t\to\infty} u^{[i]}(t) = \ell_i$, $i = 0, 1, 2$.

First, assume that $\ell_1 < 0$. Then $u'(t) \leq \ell_1 r(t)$. Integrating from T_x to t and letting $t \to \infty$ we obtain a contradiction with the positivity of u. In the similar manner we can see that $\ell_2 = 0$.

If $\ell = \ell_0 > 0$, then for any $\varepsilon > 0$ we have $l + \varepsilon > u\big(\gamma(t)\big) > l$ for large t, and choosing $0 < \varepsilon < \frac{l(1-a_0)}{a_0}$ we obtain the lower estimate

$$x(t) = u(t) - a(t)x\big(\gamma(t)\big) > l - a_0 u\big(\gamma(t)\big) > l - a_0(l+\varepsilon) = k(l+\varepsilon) > kl, \tag{4.4}$$

where $k = \frac{l-a_0(l+\varepsilon)}{l+\varepsilon} > 0$, i.e. $\liminf_{t\to\infty} |x(t)| > 0$. The vice versa in (4.2) follows from (1.2).

To prove (4.3), assume by contradiction that $\ell = \ell_0 > 0$. From (4.4) and in view of the fact that f is continuous, there exists K such that

$$f\big(x\big(\delta(t)\big)\big) \geq K$$

for large t. Hence from equation (1.1) it follows that

$$\Big(u^{[2]}(t)\Big)' \leq -q(t)K.$$

Integrating this inequality two times from t to ∞ we obtain

$$-u^{[1]}(t) \geq K \int_t^\infty p(v) \int_v^\infty q(s)\,\mathrm{d}s\,\mathrm{d}v.$$

Integrating from t_1 to t we obtain

$$-u(t) + u(t_1) \geq K \int_{t_1}^t r(w) \int_w^\infty p(v) \int_v^\infty q(s)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}w.$$

Letting $t \to \infty$ we obtain

$$\int_{t_1}^\infty r(w) \int_w^\infty p(v) \int_v^\infty q(s)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}w < \infty.$$

Changing the order of the integration we obtain the contradiction with condition (2.3). Therefore $l = 0$ and the inequality $0 \leq x(t) \leq u(t)$ implies that $\lim_{t\to\infty} x(t) = 0$. □

5. Main results: Comparison theorems

We state comparison theorems under the assumption that

$$\limsup_{|v|\to\infty} \frac{v}{f(v)} < \infty. \tag{5.1}$$

Set

$$S_f = \limsup_{v\to\infty} \frac{v}{f(v)}.$$

Our first theorem is based on the comparison with the linear ordinary differential equations and holds for the advanced argument $\delta(t) \geq t$.

Theorem 5.1. *Assume that* (5.1) *holds and* $\delta(t) \geq t$.

(i) *If* $S_f > 0$ *and the linear ODE*

$$L_3 y(t) + \frac{1-a_0}{S_f}\, q(t)\, y(t) = 0 \tag{5.2}$$

has property A, then equation (1.1) *has also property A.*

(ii) *If* $S_f = 0$, *i.e.* $\lim_{|v|\to\infty} \frac{f(v)}{v} = \infty$, *and for some* $K > 0$ *the linear ODE*

$$L_3 y(t) + K q(t) y(t) = 0 \tag{5.3}$$

has property A, then equation (1.1) *has also property A.*

Proof. (i) Let (5.2) have property A and let x be a solution of (1.1) such that $x(t) > 0$ for $t \ge t_1$, $t_1 \ge t_0$ and $u(t)$ be defined by (1.2). Assume by contradiction that $x \in \mathcal{N}_2$. Then u is nondecreasing and so $u(t) \le u\big(\delta(t)\big)$. Using Lemma 4.2 we obtain the following estimate

$$1 - a_0 \le \frac{x\big(\delta(t)\big)}{u\big(\delta(t)\big)} \le \frac{x\big(\delta(t)\big)}{u(t)}. \tag{5.4}$$

Consider the equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{r(t)} y'(t)\Big)'\Big)' + q(t)\frac{f\big(x\big(\delta(t)\big)\big)}{u(t)} y(t) = 0. \tag{5.5}$$

This equation has a solution $y = u$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) > 0$ for large t, i.e. y is a solution of (5.5) from the class $\mathcal{N}_2$. Since $S_f > 0$, we can make the following estimate

$$f(v) \ge \frac{v}{S_f} \quad \text{for large } v.$$

By Lemma 4.4, we have that $x(t) \to \infty$ as $t \to \infty$, so from here and (5.4) there exists $T \ge t_1$ such that

$$q(t)\frac{f\big(x\big(\delta(t)\big)\big)}{u(t)} \ge q(t)\frac{x\big(\delta(t)\big)}{S_f u(t)} \ge q(t)\frac{1-a_0}{S_f}.$$

Since (5.2) has property A, $\mathcal{N}_2 = \emptyset$ for (5.2). Consequently, by Proposition 3.1, $\mathcal{N}_2 = \emptyset$ for (5.5), a contradiction.

Now assume that $x \in \mathcal{N}_0$. Since (5.2) has property A, we have according to Theorem 2.4 that (2.3) holds. By Proposition 4.5, $\lim_{t\to\infty} x(t) = 0$.

(ii). We proceed by a similar way as before. Let (5.3) have property A for some $K > 0$. First, assume that equation (1.1) has a solution $x \in \mathcal{N}_2$ such that $x(\delta(t)) > 0$ for $t \ge t_1$ and u is defined by (1.2). Consider the linear delay equation

$$L_3 y(t) + q(t)\frac{f\big(x\big(\delta(t)\big)\big)}{u(t)} y(t) = 0. \tag{5.6}$$

This equation has a solution $y = u$ from the class $\mathcal{N}_2$.

By Lemma 4.4, $\lim_{t\to\infty} x(t) = \infty$. Since $S_f = 0$, we have

$$\frac{f\big(x\big(\delta(t)\big)\big)}{x\big(\delta(t)\big)} \ge \frac{K}{(1-a_0)}$$

for large t. From here and (5.4)

$$\frac{f\big(x\big(\delta(t)\big)\big)}{u(t)} = \frac{f\big(x\big(\delta(t)\big)\big)}{x\big(\delta(t)\big)}\frac{x\big(\delta(t)\big)}{u(t)} \ge \frac{K}{1-a_0}(1-a_0) = K.$$

Thus equation (5.6) is a majorant of (5.3). Since $\mathcal{N}_2 = \emptyset$ for (5.3), we have by Proposition 3.1 that $\mathcal{N}_2 = \emptyset$ for (5.6), a contradiction.

If $x \in \mathcal{N}_0$, then by the same argument as in the proof of (i) we obtain (2.3), which implies that $\lim_{t\to\infty} x(t) = 0$. $\square$

Our second theorem is established for the delay argument $\delta(t) \le t$.

Theorem 5.2. *Assume that* (5.1) *holds and* $\delta(t) \le t$.

(i) *If* $S_f > 0$ *and the linear delay equation*

$$L_3 y(t) + \frac{1-a_0}{S_f} q(t) y\big(\delta(t)\big) = 0 \tag{5.7}$$

has property A, then equation (1.1) *has also property A.*

(ii) *If* $S_f = 0$, *i.e.* $\lim_{|v|\to\infty} f(v)/v = \infty$, *and for some* $K > 0$ *the linear delay equation*

$$L_3 y(t) + K q(t) y(t) = 0$$

has property A, then equation (1.1) *has also property A.*

Proof. (i) Let (5.7) have property A and let x be a solution of (1.1) such that $x(t) > 0$ for $t \ge t_1$, $t_1 \ge t_0$ and $u(t)$ be defined by (1.2).

Assume by contradiction that $x \in \mathcal{N}_2$, and consider the delay equation

$$L_3 y(t) + q(t) \frac{f\big(x(\delta(t))\big)}{u\big(\delta(t)\big)} y\big(\delta(t)\big) = 0. \tag{5.8}$$

This equation has a solution $y = u$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) > 0$ for large t, i.e. y is the solution of (5.8) from the class $\mathcal{N}_2$. By the same argument as in the proof of Theorem 5.1-(i) we obtain

$$\frac{f\big(x(\delta(t))\big)}{u\big(\delta(t)\big)} \ge \frac{1-a_0}{S_f}\,.$$

Now by Proposition 3.1, $\mathcal{N}_2 = \emptyset$ for (5.8), a contradiction.

Assume that $x \in \mathcal{N}_0$, $x(t) > 0$ for large t and assume by contradiction that $\lim_{t\to\infty} u(t) = \ell > 0$. Then there exists $c_1 > 0$ such that $x\big(\delta(t)\big) \ge c_1$ for large t. Now, f being continuous, we can assume that there exists $c_2 > 0$ such that

$$\frac{f\big(x(\delta(t))\big)}{u(t)} \ge c_2 \tag{5.9}$$

for large t. Consider the linear equation

$$L_3 z(t) + q(t) \frac{f\big(x(\delta(t))\big)}{u(t)} z(t) = 0.$$

This equation has a solution $z = u$ which tends to a nonzero constant. Hence by Theorem 2.4,

$$\int_{t_0}^{\infty} q(t) \frac{f\big(x(\delta(t))\big)}{u(t)} \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

From (5.9) we conclude that

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Applying Proposition 3.2 to (5.7) we obtain that equation (5.7) has a solution $y \in \mathcal{N}_0$ such that $\lim_{t\to\infty} |y(t)| > 0$. This is a contradiction with the fact that (5.7) has property A.

(ii) The proof is similar to the proof of Theorem 5.1-(ii) and is omitted. □

Now we complete Theorem 5.2 for the neutral equation with the symmetric operator.

Corollary 5.3. *If the linear ODE*

$$y'''(t) + (1 - a_0)q(t)y(t) = 0 \tag{5.10}$$

has an oscillatory solution, then the neutral equation

$$\big(x(t) + a(t)x\big(\gamma(t)\big)\big)''' + q(t)x(t-\sigma) = 0, \quad \sigma > 0 \tag{5.11}$$

has property A.

Proof. By Theorem 2.3 equation (5.10) has property $\bar{\text{A}}$ and by Proposition 2.6 it has property A. Therefore by Proposition 3.3 the delay equation

$$y'''(t) + (1 - a_0)q(t)y(t-\sigma) = 0$$

has property A. Using Theorem 5.2 with $S_f = 1$ we obtain the assertion. □

Remark 5.4. Equation (5.11) with $\gamma(t) = t - \tau$, $\tau > 0$, has been considered in [12]. Corollary 5.3 extends [12, Theorem 10.4.1] for $n = 3$, where it was proved that (5.11) has property A provided $\int^\infty q(t)\,\mathrm{d}t = \infty$.

Corollary 5.5. *Let $\delta(t) \le t$. If the linear delay equation*

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)}y'\Big)'\Big)' + (1 - a_0)q(t)y\big(\delta(t)\big) = 0 \tag{5.12}$$

has property $\bar{A}$, then the neutral equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{p(t)}\big[x(t) + a(t)x\big(\gamma(t)\big)\big]'\Big)'\Big)' + q(t)x\big(\delta(t)\big) = 0. \tag{5.13}$$

has property A.

Proof. By Theorem 3.4, we have that (5.12) has property A and using Theorem 5.2 with $S_f = 1$ we obtain the assertion. □

Open problem. As far as the class $\mathcal{N}_0$ is concerned, it is always nonempty for equation (2.1), while it can be empty for equation (3.1) with $\delta(t) < t$. Thus it is possible that all solutions are oscillatory for equation (3.1) with delay argument. The oscillation of (3.1) in the case that all solutions are oscillatory has been studied in [11], see also [19, Corollary 3].

We conjecture that Theorem 3.4 holds for oscillations, in the sense that all solutions are oscillatory. More precisely, if $S_f > 0$, $\delta(t) < t$ and all solutions to (5.7) are oscillatory, then all solutions to (1.1) are oscillatory.

6. Applications and examples

In this section we illustrate Theorems 3.4, 5.1, 5.2.

Example 6.1. Consider the linear neutral equation

$$(x(t)+a(t)x(\gamma(t)))''' + \frac{k}{t^3}x(\delta(t)) = 0,$$

where $\delta(t) \geq t$. We show that this equation has the property A for

$$k > \frac{2}{3(1-a_0)\sqrt{3}} .$$

Consider the corresponding linear ODE

$$y'''(t) + (1-a_0)\frac{k}{t^3}y(t) = 0.$$

It is well-known [13] that if $(1-a_0)k > \frac{2}{3\sqrt{3}}$ then this equation has an oscillatory solution, and it has property A. Applying Theorem 5.1 we obtain the conclusion.

Example 6.2. Consider the neutral equation

$$\Big(x(t)+\frac{1}{2}x(\gamma(t))\Big)''' + \frac{k}{t^3}x(t-c) = 0, \quad c \in \mathbb{R}.$$

This equation has the property A for every $k > 4/(3\sqrt{3})$. Indeed, the case $c \leq 0$ follows from Example 6.1 and the case $c > 0$ follows from Corollary 5.3.

If we apply [9, Theorem 2.7] we obtain that this equation has property A for $k > 1$. Hence we can say our result improves the one mentioned there.

Now consider the linear neutral equation

$$\Big(\frac{1}{p(t)}\Big(\frac{1}{r(t)}\big[x(t)+a(t)x(\gamma(t))\big]'\Big)'\Big)' + q(t)x(t) = 0. \tag{6.1}$$

Corollary 6.3. *Let* (2.3) *and at least one of the following conditions hold:*

(i)

$$\int_{t_0}^{\infty} q(t)\int_{t_0}^{t} r(s)\,\mathrm{d}s = \infty,$$

(ii)

$$\limsup_{t\to\infty}\int_{t_0}^{t} p(s)\,\mathrm{d}s\int_{t}^{\infty} q(s)\frac{\int_{t_0}^{s} r(u)\int_{t_0}^{u} p(v)\,\mathrm{d}v\,\mathrm{d}u}{\int_{t_0}^{s} p(u)\,\mathrm{d}u}\mathrm{d}s > \frac{1}{1-a_0}.$$

Then equation (6.1) *has property A.*

Proof. Either condition (i) or (ii) ensures that the corresponding linear equation

$$L_3y(t) + \frac{1}{1-a_0}q(t)y(t) = 0 \tag{6.2}$$

has an oscillatory solution, see [8, Theorem 8] or [15, Lemma 2.2], respectively. Moreover, (2.3) ensures that (6.2) has property A. Applying Theorem 5.1 we obtain the conclusion. □

Example 6.4 ([18, Example 3.1])**.** Consider the neutral equation

$$\Big(t\Big(x(t)+a_0\,x\Big(\frac{t}{2}\Big)\Big)''\Big)'+\frac{k}{t^2}x(t)=0, \tag{6.3}$$

where $a_0 \in [0,1)$. Applying Corollary 6.3-(i) we obtain that this equation has property A for any $k>0$.

Observe that applying [18, Theorem 2.1] or [3, Corollary 3] we obtain that (6.3) has property A for $k > (4l(1-a_0))$ for some $l \in (1/4,1)$, or $k > 2/(1-a_0)$, respectively.

Now consider the neutral delay equation

$$\Big(x(t)+a(t)x\big(\gamma(t)\big)\Big)'''+q(t)x\big(\delta(t)\big)=0, \quad \delta(t)<t, \tag{6.4}$$

and the corresponding functional equation

$$y'''(t)+(1-a_0)q(t)y\big(\delta(t)\big)=0. \tag{6.5}$$

To apply Corollary 5.5, we can use results in [19, Sections 6.2–6.3] ensuring that (6.5) has property $\bar{\text{A}}$. For instance, we obtain the following oscillation criteria.

Corollary 6.5. *Equation* (6.4) *has property A if any of the following conditions hold:*

(i) $\delta(t)<t$, $t-\delta(t)\to\infty$ *as* $t\to\infty$ *and*

$$\limsup_{t\to\infty}\big(\delta(t)\big)^2\int_{\delta^{-1}(t)}^{\infty}q(s)\,\mathrm{d}s>\frac{2}{1-a_0},$$

(ii) $\delta(t)<t$, $t-\delta(t)\to\infty$ *as* $t\to\infty$ *and*

$$\limsup_{t\to\infty}\int_{t-\delta(t)}^{t}(t-s)\int_{\delta^{-1}(\delta^{-1}(t))}^{\infty}q(u)\,\mathrm{d}u\,\mathrm{d}s>\frac{1}{1-a_0},$$

Example 6.6. Consider the equation

$$\Big(x(t)+a(t)x\big(\gamma(t)\big)\Big)'''+\frac{k}{t^3}x(\mu t)=0.$$

where $0<\mu<1$. By Corollary 6.5-(i), this equation has property A for

$$k>\frac{4}{(1-a_0)\mu^4}.$$

Example 6.7. Consider the equation

$$\Big(x(t)+a(t)x\big(\gamma(t)\big)\Big)'''+\frac{k}{t^3}x^{\lambda}(\mu t)=0,$$

where $\lambda>1$ is a quotient of odd positive integers and $0<\mu<1$. Using Example 6.6 with $a_0=0$ and Theorem 5.2-(ii) we obtain that this equation has property A for any $k>0$.

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