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INVERSE PROBLEMS ASSOCIATED WITH THE HILL OPERATOR

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Abstract. Let ℓ_n be the length of the n-th instability interval of the Hill operator $Ly = -y'' + q(x)y$. We prove that if $\ell_n = o(n^{-2})$ and the set $\{(n\pi)^2 : n \text{ is even and } n > n_0\}$ is a subset of the periodic spectrum of the Hill operator, then $q = 0$ a.e., where n_0 is a sufficiently large positive integer such that $\ell_n < \varepsilon n^{-2}$ for all $n > n_0(\varepsilon)$ with some $\varepsilon > 0$. A similar result holds for the anti-periodic case.

1. Introduction

Consider the Hill operator

$$Ly = -y'' + q(x)y, \tag{1.1}$$

where $q(x)$ is a real-valued summable function on $[0,1]$ and $q(x+1) = q(x)$. Let λ_n and μ_n $(n = 0, 1, \dots)$ denote, respectively, the n-th periodic and anti-periodic eigenvalues of the Hill operator (1.1) on $[0,1]$ with the periodic boundary conditions

$$y(0) = y(1), \quad y'(0) = y'(1), \tag{1.2}$$

and the anti-periodic boundary conditions

$$y(0) = -y(1), \quad y'(0) = -y'(1).$$

It is well-known [5, 7] that

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \dots \to \infty.$$

The intervals (μ_{2m}, μ_{2m+1}) and $(\lambda_{2m+1}, \lambda_{2m+2})$ are respectively referred to as the $(2m+1)$-th and $(2m+2)$-th finite instability intervals of the operator L, while $(-\infty, \lambda_0)$ is called the zero-th instability interval. The length of the n-th instability interval of (1.1) will be denoted by ℓ_n $(n = 2m+1,\ 2m+2)$. For further background see [15, 16, 14].

Borg [2], Ungar [23] and Hochstadt [14] proved independently of each other the following statement:

> If $q(x)$ is real and integrable, and if all finite instability intervals vanish then $q(x) = 0$ a.e.

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Hochstadt [14] showed that if precisely one of the finite instability intervals does not vanish, then $q(x)$ is the elliptic function which satisfies

$$q'' = 3q^2 + Aq + B \quad \text{a.e.},$$

where A and B are suitable constants. Hochstadt [14] also proved that $q(x)$ is infinitely differentiable a.e. when n finite instability intervals fail to vanish. For more results see [9, 10, 11, 12].

Furthermore, Hochstadt [13] proved that the lengths of the instability intervals ℓ_n vanish faster than any power of $(1/n)$ for q in C_1^∞. McKean and Trubowitz [17] established the converse: if q is in L_1^2, the space of 1-periodic square integrable functions in $[0,1]$, and the length of the n-th instability interval for $n \geq 1$ is rapidly decreasing, then q is in C_1^∞. Later Trubowitz [22] proved the following: if q is real analytic, the lengths of the instability intervals are exponentially decreasing. Conversely if q is in L_1^2 and the lengths of the instability intervals are exponentially decreasing, q is real analytic. Denoting the Fourier coefficients of q by

$$c_n =: (q, \exp(i2n\pi\cdot))_{L^2([0,1];dx)}, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.3}$$

Coskun [6] showed that

$$\text{if } \ell_n = O(n^{-2}), \text{ then } c_n = O(n^{-2}) \text{ as } n \to \infty. \tag{1.4}$$

At this point, we refer to some Ambarzumyan-type theorems in [1, 4, 27, 3]. In 1929, Ambarzumyan [1] obtained the following first theorem in inverse spectral theory: If $\{n^2 : n = 0, 1, \dots\}$ is the spectrum of the Sturm-Liouville operator (1.1) on $[0,1]$ with the Neumann boundary conditions, then $q = 0$ a.e. In [4], they extended the classical Ambarzumyan's theorem for the Sturm-Liouville equation to the general separated boundary conditions, by imposing an additional condition on the potential function, and their result supplements the Pöschel-Trubowitz inverse spectral theory (see [18]). In [27], based on the well-known extremal property of the first eigenvalue, they find two analogs of Ambarzumyan's theorem to the Sturm-Liouville systems of n dimension under periodic or anti-periodic boundary conditions. In [3], by using the Rayleigh-Ritz inequality and imposing a condition on the second term in the Fourier cosine series (see (1.5)), they proved the following Ambarzumyan-type theorem:

(a) If all periodic eigenvalues of Hill's equation (1.1) are nonnegative and they include $\{(2m\pi)^2 : m \in \mathbb{N}\}$, then $q = 0$ a.e.
(b) If all anti-periodic eigenvalues of Hill's equation (1.1) are not less than π^2 and they include $\{(2m-1)^2\pi^2 : m \in \mathbb{N}\}$, and

$$\int_0^1 q(x)\cos(2\pi x)\,dx \geq 0, \tag{1.5}$$

then $q = 0$ a.e.

Recently, in [21], we obtain the classical Ambarzumyan's theorem for the Sturm-Liouville operators with $q \in L^1[0,1]$ and quasi-periodic boundary conditions in cases when there is not any additional condition on the potential q such as (1.5).

In this paper, we prove the following inverse spectral result, more precisely, a uniqueness-type result of the following form:

Theorem 1.1. *Denote the nth instability interval by ℓ_n, and suppose that $\ell_n = o(n^{-2})$ as $n \to \infty$. Then the following two assertions hold:*

(i) *If* $\{(n\pi)^2 : n$ *even and* $n > n_0\}$ *is a subset of the periodic spectrum of the Hill operator then* $q = 0$ *a.e. on* $(0, 1)$*;*
(ii) *If* $\{(n\pi)^2 : n$ *odd and* $n > n_0\}$ *is a subset of the anti-periodic spectrum of the Hill operator then* $q = 0$ *a.e. on* $(0, 1)$*.*

Given $\varepsilon > 0$*, thee exists* $n_0 = n_0(\varepsilon) \in \mathbb{N}$*, a sufficiently large positive integer such that*

$$\ell_n < \varepsilon n^{-2} \quad \text{for all } n > n_0(\varepsilon).$$

Theorem 1.1 is deduced from the following result.

Theorem 1.2. *Denote the Fourier coefficients of* q *by* c_n *(see* (1.3)*), and assume* $\ell_n = o(n^{-2})$*. Then* $c_n = o(n^{-2})$ *as* $n \to \infty$*.*

Note that, from Theorem 1.2, the assertion in (1.4) holds with the improved o-terms $o(n^{-2})$. In Ambarzumyan-type theorems, it is necessary to specify the whole spectrum. In [8], they proved that it is enough to know the first eigenvalue only. Unlike the above works, to prove of Theorem 1.1, we have information only on the sufficiently large eigenvalues of the spectrum of the Hill operator. Also, the proof does not depend on multiplicities of the given eigenvalues.

2. Preliminaries and proof of main results

We shall consider only the periodic (for even n) eigenvalues of the Hill operator. The anti-periodic (for odd n) problem is completely similar. It is well known [7, Theorem 4.2.3] that the periodic eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ are asymptotically located in pairs such that

$$\lambda_{2m+1} = \lambda_{2m+2} + o(1) = (2m + 2)^2\pi^2 + o(1) \tag{2.1}$$

for sufficiently large m. From this formula, for all $k \neq 0, (2m + 2)$ and $k \in \mathbb{Z}$, the inequality

$$|\lambda - (2(m - k) + 2)^2\pi^2| > |k||(2m + 2) - k| > C\, m, \tag{2.2}$$

is satisfied by both eigenvalues λ_{2m+1} and λ_{2m+2} for large m, where, here and in subsequent relations, C denotes a positive constant whose exact value is not essential. Note that, when $q = 0$, the system $\{e^{-i(2m+2)\pi x}, e^{i(2m+2)\pi x}\}$ is a basis of the eigenspace corresponding to the double eigenvalues $(2m+2)^2\pi^2$ of the problem (1.1)-(1.2).

To obtain the asymptotic formulas for the periodic eigenvalues $\lambda_{2m+1}, \lambda_{2m+2}$ corresponding to the normalized eigenfunctions $\Psi_{m,1}(x), \Psi_{m,2}(x)$ respectively, let us consider the well-known relation, for sufficiently large m,

$$\Lambda_{m,j,m-k}(\Psi_{m,j}, e^{i(2(m-k)+2)\pi x}) = (q\, \Psi_{m,j}, e^{i(2(m-k)+2)\pi x}), \tag{2.3}$$

where $\Lambda_{m,j,m-k} = (\lambda_{2m+j} - (2(m - k) + 2)^2\pi^2)$, $j = 1, 2$. The relation (2.3) can be obtained from the equation (1.1) by multiplying $e^{i(2(m-k)+2)\pi x}$ and using integration by parts. From [25, Lemma 1], to iterate (2.3) for $k = 0$, in the right hand-side of formula (2.3), we use the following relations

$$(q\, \Psi_{m,j}, e^{i(2m+2)\pi x}) = \sum_{m_1=-\infty}^{\infty} c_{m_1}(\Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x}), \tag{2.4}$$

$$|(q\, \Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x})| < 3M \tag{2.5}$$

for all large m, where $j = 1, 2$ and $M = \sup_{m\in\mathbb{Z}} |c_m|$.

First, we fix the terms with indices $m_1 = 0, (2m+2)$. Then all the other terms in the right hand-side of (2.4) are replaced, in view of (2.2) and (2.3) for $k = m_1$, by

$$c_{m_1} \frac{(q\,\Psi_{m,j}, e^{i(2(m-m_1)+2)\pi x})}{\Lambda_{m,j,m-m_1}}.$$

In the same way, by applying the above process for the eigenfunction $e^{-i(2m+2)\pi x}$ corresponding to the eigenvalue $(2m+2)^2\pi^2$ of the problem (1.1)-(1.2) for $q = 0$, we obtain the following lemma (see also Section 2 in [20, 19]).

Lemma 2.1. *The following relations hold for sufficiently large m: (i)*

$$[\Lambda_{m,j,m} - c_0 - \sum_{i=1}^{2} a_i(\lambda_{2m+j})]u_{m,j} = [c_{2m+2} + \sum_{i=1}^{2} b_i(\lambda_{2m+j})]v_{m,j} + R_2, \tag{2.6}$$

where $j = 1, 2$,

$$\begin{gathered} u_{m,j} = (\Psi_{m,j}, e^{i(2m+2)\pi x}), \quad v_{m,j} = (\Psi_{m,j}, e^{-i(2m+2)\pi x}), \\ a_1(\lambda_{2m+j}) = \sum_{m_1} \frac{c_{m_1} c_{-m_1}}{\Lambda_{m,j,m-m_1}}, \\ a_2(\lambda_{2m+j}) = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2} c_{-m_1-m_2}}{\Lambda_{m,j,m-m_1}\Lambda_{m,j,m-m_1-m_2}}, \\ b_1(\lambda_{2m+j}) = \sum_{m_1} \frac{c_{m_1} c_{2m+2-m_1}}{\Lambda_{m,j,m-m_1}}, \\ b_2(\lambda_{2m+j}) = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2} c_{2m+2-m_1-m_2}}{\Lambda_{m,j,m-m_1}\Lambda_{m,j,m-m_1-m_2}}, \\ R_2 = \sum_{m_1,m_2,m_3} \frac{c_{m_1} c_{m_2} c_{m_3} (q\Psi_{m,j}(x), e^{i(2(m-m_1-m_2-m_3)+2)\pi x})}{\Lambda_{m,j,m-m_1}\,\Lambda_{m,j,m-m_1-m_2}\Lambda_{m,j,m-m_1-m_2-m_3}}. \end{gathered} \tag{2.7}$$

The summations in these formulas are taken over all integers m_1, m_2, m_3 such that $m_1, m_1+m_2, m_1+m_2+m_3 \neq 0,\ 2m+2$.

(ii)

$$[\Lambda_{m,j,m} - c_0 - \sum_{i=1}^{2} a_i'(\lambda_{2m+j})]v_{m,j} = [c_{-2m-2} + \sum_{i=1}^{2} b_i'(\lambda_{2m+j})]u_{m,j} + R_2', \tag{2.8}$$

where $j = 1, 2$,

$$\begin{gathered} a_1'(\lambda_{2m+j}) = \sum_{m_1} \frac{c_{m_1} c_{-m_1}}{\Lambda_{m,j,m+m_1}}, \\ a_2'(\lambda_{2m+j}) = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2} c_{-m_1-m_2}}{\Lambda_{m,j,m+m_1}\,\Lambda_{m,j,m+m_1+m_2}}, \\ b_1'(\lambda_{2m+j}) = \sum_{m_1} \frac{c_{m_1} c_{-2m-2-m_1}}{\Lambda_{m,j,m+m_1}}, \\ b_2'(\lambda_{2m+j}) = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2} c_{-2m-2-m_1-m_2}}{\Lambda_{m,j,m+m_1}\,\Lambda_{m,j,m+m_1+m_2}}, \end{gathered}$$

$$R_2' = \sum_{m_1,m_2,m_3} \frac{c_{m_1} c_{m_2} c_{m_3} (q\Psi_{m,j}(x), e^{i(2(m+m_1+m_2+m_3)+2)\pi x})}{\Lambda_{m,j,m+m_1}\Lambda_{m,j,m+m_1+m_2}\,\Lambda_{m,j,m+m_1+m_2+m_3}} \tag{2.9}$$

and the sums in these formulas are taken over all integers m_1, m_2, m_3 such that $m_1, m_1+m_2, m_1+m_2+m_3 \neq 0, -2m-2$.

Note that, by substituting $m_1 = -k_1$ and $m_1 + m_2 = -k_1$, $m_2 = k_2$ into the relations $a_1'(\lambda_{2m+j})$ and $a_2'(\lambda_{2m+j})$ respectively, we have

$$a_i(\lambda_{2m+j}) = a_i'(\lambda_{2m+j}) \quad \text{for } i = 1, 2. \tag{2.10}$$

Here, using the equality

$$\frac{1}{m_1(2m+2-m_1)} = \frac{1}{2m+2}\Big(\frac{1}{m_1} + \frac{1}{2m+2-m_1}\Big),$$

we obtain the relation

$$\sum_{m_1 \neq 0, (2m+2)} \frac{1}{|m_1(2m+2-m_1)|} = O\Big(\frac{\ln|m|}{m}\Big).$$

This, together with (2.2), (2.3) and (2.5), gives the following estimates (see, respectively, (2.1) and (2.9) for R_2 and R_2')

$$R_2,\ R_2' = O\Big(\big(\frac{\ln|m|}{m}\big)^3\Big). \tag{2.11}$$

Moreover, in view of (2.2), (2.3) and (2.5), we obtain (see [25, Theorem 2], [19])

$$\sum_{k\in\mathbb{Z};\, k\neq \pm(m+1)} \big|(\Psi_{m,j}, e^{i2k\pi x})\big|^2 = O\Big(\frac{1}{m^2}\Big). \tag{2.12}$$

Therefore, the expansion of the normalized eigenfunctions $\Psi_{m,j}(x)$ by the orthonormal basis $\{e^{i2k\pi x} : k \in \mathbb{Z}\}$ on $[0,1]$ has the form

$$\Psi_{m,j}(x) = u_{m,j}\, e^{i(2m+2)\pi x} + v_{m,j}\, e^{-i(2m+2)\pi x} + h_m(x), \tag{2.13}$$

where

$$\begin{gathered} (h_m, e^{\mp i(2m+2)\pi x}) = 0, \quad \|h_m\| = O(m^{-1}), \\ \sup_{x\in[0,1]} |h_m(x)| = O\Big(\frac{\ln|m|}{m}\Big), \quad |u_{m,j}|^2 + |v_{m,j}|^2 = 1 + O\big(m^{-2}\big). \end{gathered} \tag{2.14}$$

Proof of Theorem 1.2. First, let us estimate the expressions in (2.6) and (2.8). From (2.1), (2.2) and (2.12), one can readily see that

$$\begin{aligned} &\sum_{m_1\neq 0, \pm(2m+2)} \Big|\frac{1}{\Lambda_{m,j,m\mp m_1}} - \frac{1}{\Lambda_{m,0,m\mp m_1}}\Big| \\ &\leq C|\Lambda_{m,j,m}| \sum_{m_1\neq 0,\pm(2m+2)} |m_1|^{-2}|2m+2\mp m_1|^{-2} = o\big(m^{-2}\big), \end{aligned} \tag{2.15}$$

where $\Lambda_{m,0,m\mp m_1} = ((2m+2)^2\pi^2 - (2(m\mp m_1)+2)^2\pi^2)$. Thus, we obtain

$$a_i(\lambda_{2m+j}) = a_i((2m+2)^2\pi^2) + o\big(m^{-2}\big) \quad \text{for } i = 1, 2. \tag{2.16}$$

Here, by (2.15), we also have, arguing as in [19, Lemma 3] (see also [26, Lemma 6]),

$$\begin{aligned} b_1(\lambda_{2m+j}) &= \frac{1}{4\pi^2} \sum_{m_1 \neq 0,(2m+2)} \frac{c_{m_1} c_{2m+2-m_1}}{m_1(2m+2-m_1)} + o\big(m^{-2}\big) \\ &= -\int_0^1 (Q(x)-Q_0)^2 \, e^{-i2(2m+2)\pi x} dx + o\big(m^{-2}\big) \\ &= \frac{-1}{i2\pi(2m+2)} \int_0^1 2(Q(x)-Q_0)\, q(x)\, e^{-i2(2m+2)\pi x} dx + o\big(m^{-2}\big), \end{aligned} \tag{2.17}$$

where

$$\begin{gathered} Q(x) - Q_0 = \sum_{m_1 \neq 0} Q_{m_1} \, e^{i2m_1\pi x}, \\ Q_{m_1} =: (Q(x), e^{i2m_1\pi x}) = \frac{c_{m_1}}{i2\pi m_1}, \quad m_1 \neq 0, \end{gathered} \tag{2.18}$$

are the Fourier coefficients with respect to the system $\{e^{i2m_1\pi x} : m_1 \in \mathbb{Z}\}$ of the function $Q(x) = \int_0^x q(t)\, dt$. For the proof of Theorem 1.2, we suppose without loss of generality that $c_0 = 0$, so that $Q(1) = c_0 = 0$.

Now using the assumption $\ell_n = o(n^{-2})$ of the theorem it is also true that $\ell_n = O(n^{-2})$. In view of (1.4), we obtain $c_n = O(n^{-2})$ as $n \to \infty$. Thus, from [14, Lemma 5], we obtain that $q(x)$ is absolutely continuous a.e. Hence integration by parts, together with $Q(1) = 0$, gives

$$\begin{aligned} &b_1(\lambda_{2m+j}) \\ &= \frac{1}{2\pi^2(2m+2)^2} \int_0^1 \big(q^2(x) + (Q(x)-Q_0)q'(x)\big)\, e^{-i2(2m+2)\pi x} dx + o\big(m^{-2}\big). \end{aligned} \tag{2.19}$$

Since $q(x)$ is absolutely continuous a.e., $\big(q^2(x) + (Q(x)-Q_0)q'(x)\big) \in L^1[0,1]$. By the Riemann-Lebesgue lemma, we find that

$$b_1(\lambda_{2m+j}) = o\big(m^{-2}\big). \tag{2.20}$$

Similarly

$$b_1'(\lambda_{2m+j}) = o\big(m^{-2}\big). \tag{2.21}$$

Let us prove that

$$b_2(\lambda_{2m+j}),\ b_2'(\lambda_{2m+j}) = o\big(m^{-2}\big). \tag{2.22}$$

Taking into account that $q(x)$ is absolutely continuous a.e. and periodic, we obtain $c_{m_1} c_{m_2} c_{\pm(2m+2)-m_1-m_2} = o\left(m^{-1}\right)$ (see [26, p. 665]). Using this and arguing as in the proof of (2.11), we obtain

$$\begin{aligned} |b_2(\lambda_{2m+j})| &= o\left(m^{-1}\right) \sum_{m_1,m_2} \frac{1}{|m_1(2m+2-m_1)(m_1+m_2)(2m+2-m_1-m_2)|} \\ &= o\left(m^{-1}\right) O\Big(\big(\frac{\ln|m|}{m}\big)^2\Big) = o\big(m^{-2}\big). \end{aligned}$$

Thus, the first estimate of (2.22) is proved. Similarly $b_2'(\lambda_{2m+j}) = o\big(m^{-2}\big)$. Substituting the estimates given by (2.10), (2.11), (2.16) and (2.20)-(2.22) into the relations (2.6) and (2.8), we find that

$$[\Lambda_{m,j,m}-\sum_{i=1}^{2}a_i((2m+2)^2\pi^2)]u_{m,j}=c_{2m+2}v_{m,j}+o\big(m^{-2}\big), \tag{2.23}$$

$$[\Lambda_{m,j,m}-\sum_{i=1}^{2}a_i((2m+2)^2\pi^2)]v_{m,j}=c_{-2m-2}\,u_{m,j}+o\big(m^{-2}\big) \tag{2.24}$$

for $j=1,2$.

Now suppose that, contrary to what we want to prove, there exists an increasing sequence $\{m_k\}$ $(k=1,2,\dots)$ such that

$$|c_{2m_k+2}|>Cm_k^{-2}\quad\text{for some } C>0. \tag{2.25}$$

Further, the formula (2.14) for $m=m_k$ implies that either $|u_{m_k,j}|>1/2$ or $|v_{m_k,j}|>1/2$ for sufficiently large m_k. Without loss of generality, we assume that $|u_{m_k,j}|>1/2$. Then it follows from both (2.23) and (2.24) for $m=m_k$ that

$$[\Lambda_{m_k,j,m_k}-\sum_{i=1}^{2}a_i((2m_k+2)^2\pi^2)]\sim c_{2m_k+2}, \tag{2.26}$$

where the notation $a_m\sim b_m$ means that there exist constants c_1, c_2 such that $0<c_1<c_2$ and $c_1<|a_m/b_m|<c_2$ for all sufficiently large m. This, together with (2.24) for $m=m_k$, (2.25) and $|u_{m_k,j}|>1/2$, implies that

$$u_{m_k,j}\sim v_{m_k,j}\sim 1. \tag{2.27}$$

Multiplying (2.24) for $m=m_k$ by c_{2m_k+2}, and then using (2.23) for $m=m_k$ in (2.24), we arrive at the relation

$$\begin{aligned}&[\Lambda_{m_k,j,m_k}-\sum_{i=1}^{2}a_i((2m_k+2)^2\pi^2)]\Big([\Lambda_{m_k,j,m_k}\\&-\sum_{i=1}^{2}a_i((2m_k+2)^2\pi^2)]u_{m_k,j}+o\big(m_k^{-2}\big)\Big)\\&=|c_{2m_k+2}|^2\,u_{m_k,j}+c_{2m_k+2}\,o\big(m_k^{-2}\big),\end{aligned} \tag{2.28}$$

which, by (2.26) and (2.27), implies

$$\Lambda_{m_k,j,m_k}-\sum_{i=1}^{2}a_i((2m_k+2)^2\pi^2)=\pm|c_{2m_k+2}|+o\big(m_k^{-2}\big) \tag{2.29}$$

for $j=1,2$.

Let us prove that the periodic eigenvalues for large m_k are simple. Assume that there exist two orthogonal eigenfunctions $\Psi_{m_k,1}(x)$ and $\Psi_{m_k,2}(x)$ corresponding to $\lambda_{2m_k+1}=\lambda_{2m_k+2}$. From the argument of [26, Lemma 4], using the relation (2.13) with $\|h_{m_k}\|=O(m_k^{-1})$ for the eigenfunctions $\Psi_{m_k,j}(x)$ and the orthogonality of eigenfunctions, we can choose these eigenfunctions such that either $u_{m_k,j}=0$ or $v_{m_k,j}=0$, which contradicts (2.27).

Since the eigenfunctions $\Psi_{m_k,1}$ and $\overline{\Psi_{m_k,2}}$ of the self-adjoint problem corresponding to the different eigenvalues $\lambda_{2m_k+1}\neq\lambda_{2m_k+2}$ are orthogonal we find, by (2.13), that

$$0=(\Psi_{m_k,1},\overline{\Psi_{m_k,2}})=u_{m_k,2}v_{m_k,1}+u_{m_k,1}v_{m_k,2}+O(m_k^{-1}). \tag{2.30}$$

Note that, for the simple eigenvalues in (2.29), there are two cases. First case: The simple eigenvalues λ_{2m_k+1} and λ_{2m_k+2} in (2.29) corresponds respectively to the lower sign $-$ and upper sign $+$. Then

$$\ell_{2m_k+2} = \lambda_{m_k,2,m_k} - \lambda_{m_k,1,m_k} = 2|c_{2m_k+2}| + o\big(m_k^{-2}\big),$$

which implies that (see (2.25)) $\ell_{2m_k+2} > Cm_k^{-2}$ for some C. This contradicts the hypothesis that $\ell_{2m_k+2} = o(m_k^{-2})$. Now let us consider the second case: We assume that both the simple eigenvalues correspond to the lower sign $-$ (the proof corresponding to the upper sign $+$ is similar). Then $\Lambda_{m_k,2,m_k} - \Lambda_{m_k,1,m_k} = o\big(m_k^{-2}\big)$. Using this, (2.23) and (2.29), we have

$$o\big(m_k^{-2}\big)u_{m_k,2} = c_{2m_k+2}v_{m_k,2} + |c_{2m_k+2}|u_{m_k,2} + o\big(m_k^{-2}\big), \tag{2.31}$$

$$o\big(m_k^{-2}\big)u_{m_k,1} = -c_{2m_k+2}v_{m_k,1} - |c_{2m_k+2}|u_{m_k,1} + o\big(m_k^{-2}\big). \tag{2.32}$$

Therefore, multiplying both sides of (2.31) and (2.32) by $v_{m_k,1}$ and $v_{m_k,2}$ respectively and adding the results, we have, in view of (2.25),

$$u_{m_k,2}v_{m_k,1} - u_{m_k,1}v_{m_k,2} = o(1).$$

This, together with (2.30), gives $u_{m_k,2}v_{m_k,1} = o(1)$ which contradicts (2.27). Thus the assumption (2.25) is false, that is, $c_{2m+2} = o\big(m^{-2}\big)$. A similar result holds for the anti-periodic problem, that is, $c_{2m+1} = o\big(m^{-2}\big)$. The theorem is proved.

For the proof of Theorem 1.1, we need the sharper estimates in the following lemma.

Lemma 2.2. *Let $q(x)$ be absolutely continuous a.e. and $c_0 = 0$. Then, for all sufficiently large m, we have the following estimates*

$$\begin{gathered} a_1(\lambda_{2m+j}) = \frac{-1}{(2\pi(2m+2))^2}\int_0^1 q^2(x)dx + o\big(m^{-2}\big), \\ a_2(\lambda_{2m+j}) = o\big(m^{-2}\big). \end{gathered} \tag{2.33}$$

Proof. First, let us consider $a_1(\lambda_{2m+j})$. By (2.15) we obtain

$$a_1(\lambda_{2m+j}) = \frac{1}{4\pi^2}\sum_{m_1\neq 0,(2m+2)} \frac{c_{m_1}c_{-m_1}}{m_1(2m+2-m_1)} + o\big(m^{-2}\big).$$

Arguing as in [19, Lemma 3] (see also [24, Lemma 2.3(a)]), we obtain, in our notation,

$$\begin{aligned} &a_1(\lambda_{2m+j}) \\ &= \frac{1}{2\pi^2}\sum_{m_1>0,m_1\neq(2m+2)} \frac{c_{m_1}c_{-m_1}}{(2m+2+m_1)(2m+2-m_1)} + o\big(m^{-2}\big) \\ &= \int_0^1 (G^+(x,m) - G_0^+(m))^2\, e^{i2(4m+4)\pi x}\,dx + o\big(m^{-2}\big) \\ &= \frac{-2}{i2\pi(4m+4)}\int_0^1 \big(G^+(x,m) - G_0^+(m)\big) \\ &\quad\times \big(q(x)e^{-i2(2m+2)\pi x} - c_{2m+2}\big)e^{i2(4m+4)\pi x}dx + o\big(m^{-2}\big) \end{aligned} \tag{2.34}$$

where

$$G^{\pm}_{m_1}(m) =: (G^{\pm}(x,m), e^{i2m_1\pi x}) = \frac{c_{m_1\pm(2m+2)}}{i2\pi m_1}, \tag{2.35}$$

for $m_1 \neq 0$, are the Fourier coefficients with respect to $\{e^{i2m_1\pi x} : m_1 \in \mathbb{Z}\}$ of the functions

$$G^{\pm}(x,m) = \int_0^x q(t)\, e^{\mp i2(2m+2)\pi t} dt - c_{\pm(2m+2)}x \tag{2.36}$$

and

$$G^{\pm}(x,m) - G_0^{\pm}(m) = \sum_{m_1 \neq (2m+2)} \frac{c_{m_1}}{i2\pi(m_1 \mp (2m+2))}\, e^{i2(m_1 \mp (2m+2))\pi x}.$$

Here, taking into account the [19, Lemma 1] and (2.36), we have the estimate

$$G^{\pm}(x,m) - G_0^{\pm}(m) = G^{\pm}(x,m) - \int_0^1 G^{\pm}(x,m)\, dx = o(1) \quad \text{as } m \to \infty \tag{2.37}$$

uniformly in x.

From the equalities (see (2.36))

$$G^{\pm}(1,m) = G^{\pm}(0,m) = 0, \tag{2.38}$$

and since $q(x)$ is absolutely continuous a.e., integration by parts gives, for the right hand-side of (2.34), the value

$$\begin{aligned} a_1(\lambda_{2m+j}) = \frac{-1}{(2\pi(2m+2))^2} \Big[\int_0^1 q^2 + \int_0^1 (G^+(x,m) - G_0^+(m)) q'(x) e^{i2(2m+2)\pi x} dx \Big] \\ + \frac{|c_{2m+2}|^2}{(2\pi(2m+2))^2} + o\big(m^{-2}\big) \end{aligned}$$

for sufficiently large m. Thus, by using the Riemann-Lebesgue lemma, this with $(G^+(x,m) - G_0^+(m))q'(x) \in L^1[0,1]$ implies the first equality of (2.33).

Now, it remains to prove that $a_2(\lambda_{2m+j}) = o\big(m^{-2}\big)$. Similarly, by (2.16) for $i = 2$, we obtain

$$\begin{aligned} a_2(\lambda_{2m+j}) = \sum_{m_1,m_2} \frac{(2\pi)^{-4}\, c_{m_1} c_{m_2} c_{-m_1-m_2}}{m_1(2m+2-m_1)(m_1+m_2)(2m+2-m_1-m_2)} \\ + o\big(m^{-2}\big). \end{aligned} \tag{2.39}$$

As in [19, Lemma 4], using the summation variable m_2 to represent the previous $m_1 + m_2$ in (2.39), we write (2.39) in the form

$$a_2(\lambda_{2m+j}) = \frac{1}{(2\pi)^4} \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_1(2m+2-m_1)m_2(2m+2-m_2)} + o\big(m^{-2}\big),$$

where the forbidden indices in the sums take the form of $m_1, m_2 \neq 0,\ 2m+2$. Here the equality

$$\frac{1}{k(2m+2-k)} = \frac{1}{2m+2}\Big(\frac{1}{k} + \frac{1}{2m+2-k}\Big)$$

gives

$$a_2(\lambda_{2m+j}) = \frac{1}{(2\pi)^4(2m+2)^2} \sum_{j=1}^4 S_j, \tag{2.40}$$

where

$$S_1 = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_1 m_2}, \quad S_2 = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_2(2m+2-m_1)},$$

$$S_3 = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{m_1(2m+2-m_2)}, \quad S_4 = \sum_{m_1,m_2} \frac{c_{m_1} c_{m_2-m_1} c_{-m_2}}{(2m+2-m_1)(2m+2-m_2)}.$$

Using (2.18), integration by parts and the assumption $c_0 = 0$ which implies $Q(1) = 0$, we deduce that

$$S_1 = 4\pi^2 \int_0^1 (Q(x) - Q_0)^2 q(x)\, dx = 0. \tag{2.41}$$

Similarly, in view of (2.18) and (2.35)-(2.38), we obtain, by the Riemann-Lebesgue lemma,

$$S_2 = -4\pi^2 \int_0^1 (Q(x) - Q_0)(G^+(x,m) - G_0^+(m))q(x)e^{i2(2m+2)\pi x} dx = o(1),$$
$$S_3 = -4\pi^2 \int_0^1 (Q(x) - Q_0)(G^-(x,m) - G_0^-(m))q(x)e^{-i2(2m+2)\pi x} dx = o(1)$$

and, by (2.37),

$$S_4 = 4\pi^2 \int_0^1 (G^+(x,m) - G_0^+(m))(G^-(x,m) - G_0^-(m))\, q(x)\, dx = o(1).$$

Thus, (2.40) implies that $a_2(\lambda_{2m+j}) = o\big(m^{-2}\big)$. The proof is complete. □

Proof of Theorem 1.1. (i) First let us prove that $c_0 = 0$. Considering the first step of the procedure in Lemma 2.1, and using the estimate in (2.11), we may rewrite the relations (2.6) and (2.8) as follows:

$$\begin{gathered} u_{m,j} = c_{2m+2} v_{m,j} + O\Big(\frac{\ln|m|}{m}\Big), \\ [\Lambda_{m,j,m} - c_0] v_{m,j} = c_{-2m-2}\, u_{m,j} + O\Big(\frac{\ln|m|}{m}\Big) \end{gathered} \tag{2.42}$$

for $j = 1, 2$ and sufficiently large m. By using the assumption $\ell_{2m+2} = o(m^{-2})$, namely, $\ell_n = o(n^{-2})$ for even $n = 2m+2$ and Theorem 1.2 which implies $c_{\mp(2m+2)} = o(m^{-2})$, we obtain the relations (see (2.42)) in the form

$$[\Lambda_{m,j,m} - c_0] u_{m,j} = O\big(\frac{\ln|m|}{m}\big), \tag{2.43}$$
$$[\Lambda_{m,j,m} - c_0] v_{m,j} = O\big(\frac{\ln|m|}{m}\big). \tag{2.44}$$

Again by (2.14), we have either $|u_{m,j}| > 1/2$ or $|v_{m,j}| > 1/2$ for large m. In either case, in view of (2.43) and (2.44), there exists a sufficiently large positive integer N_0 such that both the eigenvalues λ_{2m+j} (see definition of (2.3)) satisfy the estimate

$$\lambda_{2m+j} = (2m+2)^2\pi^2 + c_0 + O\big(\frac{\ln|m|}{m}\big) \tag{2.45}$$

for all $m > N_0$ and $j = 1, 2$. Under the assumption of Theorem 1.1 (i), when $m > \max\{(n_0-2)/2, N_0\}$, the eigenvalue $(2m+2)^2\pi^2$ corresponds to the eigenvalue λ_{2m+1} or λ_{2m+2}. In either case we obtain $c_0 = 0$ by (2.45).

Finally, for sufficiently large m, substituting the estimates of $a_i(\lambda_{2m+j})$, $a_i'(\lambda_{2m+j})$, $b_i(\lambda_{2m+j})$, $b_i'(\lambda_{2m+j})$, R_2, R_2' for $i = 1, 2$, given by Lemma 2.2 with the equalities $a_i(\lambda_{2m+j}) = a_i'(\lambda_{2m+j})$ (see (2.10)), (2.20)-(2.22) and (2.11) into the relations (2.6) and (2.8) and using $c_0 = 0$, we find the relations in the form

$$\begin{aligned} &\Big[\Lambda_{m,j,m} + \frac{1}{(2\pi(2m+2))^2}\int_0^1 q^2\Big]u_{m,j} = c_{2m+2}v_{m,j} + o\big(m^{-2}\big),\\ &\Big[\Lambda_{m,j,m} + \frac{1}{(2\pi(2m+2))^2}\int_0^1 q^2\Big]v_{m,j} = c_{-2m-2}\,u_{m,j} + o\big(m^{-2}\big) \end{aligned} \tag{2.46}$$

for $j = 1, 2$. In the same way, by using the assumption $\ell_{2m+2} = o(m^{-2})$ and Theorem 1.2, we write (2.46) in the form

$$\begin{aligned} &\Big[\Lambda_{m,j,m} + \frac{1}{(2\pi(2m+2))^2}\int_0^1 q^2\Big]u_{m,j} = o\big(m^{-2}\big),\\ &\Big[\Lambda_{m,j,m} + \frac{1}{(2\pi(2m+2))^2}\int_0^1 q^2\Big]v_{m,j} = o\big(m^{-2}\big). \end{aligned}$$

Thus, arguing as in the proof of (2.45), there exists a positive large number N_1 such that the eigenvalues λ_{2m+j} satisfy the following estimate

$$\lambda_{2m+j} = (2m+2)^2\pi^2 - \frac{1}{(2\pi(2m+2))^2}\int_0^1 q^2 + o\big(m^{-2}\big) \tag{2.47}$$

for all $m > N_1$ and $j = 1, 2$. Let $m > \max\{(n_0 - 2)/2, N_1\}$. Using the same argument as above, by (2.47), we obtain $\int_0^1 q^2 = 0$ which implies that $q = 0$ a.e.

(ii) The procedure in Section 2 works for the anti-periodic boundary conditions

$$y(0) = -y(a), \quad y'(0) = -y'(a).$$

Thus, it is readily seen that the corresponding results for the anti-periodic eigenvalues μ_{2m}, μ_{2m+1} hold, replacing $(2m + 2)$ in (2.1)-(2.3) by $(2m + 1)$.

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