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PERIODIC SOLUTIONS OF A MULTI-DIMENSIONAL CAHN-HILLIARD EQUATION

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Abstract. This article concerns a multi-dimensional Cahn-Hilliard equation subject to Neumann boundary condition. We show existence of the periodic solutions by using the viscosity approach. By applying the Schauder fixed point theorem, we show existence of the solutions to the suitable approximate problem and then obtain the solutions of the considered periodic problem using a priori estimates. Our results extend those in [20].

1. Introduction

In 1958, Cahn and Hilliard [3] derived the Cahn-Hilliard equation

$$u_\tau - \Delta(-\kappa\Delta u + g(u)) = f, \tag{1.1}$$

which is a model of phase separation in binary material. Here $g(u)$ is the derivative of free energy $F(u)$. If $F(u)$ is a smooth function, (1.1) can be used to characterize the spread of populations and the diffusion of an oil film over a solid surface, see [4, 16]. While $F(u)$ is not smooth, (1.1) is used to describe the phase separation with constraints, see for example [2].

Because of the applications of Cahn-Hilliard equation (1.1) in physics, there has been a great interest in studying the qualitative properties of solutions to the Cahn-Hilliard equation. For example, we can refer to [6, 19] for existence, uniqueness and regularity of the solutions, and [7, 13] for asymptotic behavior of the solutions. In addition, using the techniques of subdifferential operator, Kenmochi et al [9] investigated the Cahn-Hilliard equation with constraints. More recently, Kubo [11] considered the strong solution and weak solution to the Cahn-Hilliard equation with a time-dependent constraint and also discussed the relation between these solutions.

It is well known that one of the most interesting topics of the higher-order parabolic equations, from a theoretical and practical point of view, is existence of the periodic solutions, which has been considered in several works [12, 14, 18, 20, 22]. Zhao et al [22] studied existence and uniqueness of the time-periodic generalized solutions to a fourth-order parabolic equation by the Galerkin method. Moreover,

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[12, 14] are concerned with the existence, uniqueness and attractivity of the time-periodic solutions to the Cahn-Hilliard equations with periodic gradient-dependent potentials and sources. It should be remarked that [12, 14, 22] are all in the case of one spatial dimension. Also in one spatial dimension, Yin et al [20] used the qualitative theory of parabolic equations to prove existence of the periodic solutions in the classical sense to the following equation

$$u_\tau + \kappa u_{xxxx} = (A(\tau)u^3 - B(\tau)u)_{xx} + f(x,\tau),$$

where $A(\tau)$ and $B(\tau)$ are positive, continuous and periodic functions with the period $\omega > 0$, and $f(\tau)$ is also a smooth ω-periodic function satisfying $\int_0^1 f(x,\tau)dx = 0$ for any $\tau \in [0,\omega]$. As for the case of higher dimensions, Wang and Zheng [18] recently showed the existence of periodic solutions to the Cahn-Hillard equation with a constraint by applying the viscosity approach.

Motivated by the above works, the purpose of this paper is to show existence of the periodic solutions to the problem

$$u_\tau(x,\tau) - \Delta(-\kappa\Delta u(x,\tau) + g(u(x,\tau))) = f(x,\tau) \quad \text{in } Q_\omega := \Omega \times (0,\omega), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}(x,\tau) = \frac{\partial}{\partial \nu}(-\kappa\Delta u(x,\tau) + g(u(x,\tau))) = 0 \quad \text{on } \Sigma_\omega := \partial\Omega \times (0,\omega), \tag{1.3}$$

$$u(x,0) = u(x,\omega) \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N (1 \le N \le 3)$ with smooth boundary, $\frac{\partial}{\partial\nu}$ stands for the outward normal derivative on $\partial\Omega$, f is a ω-periodic function and $g(u) = a_3u^3 + a_2u^2 + a_1u + a_0$ with constants $a_3 > 0$ and $a_i \in \mathbb{R}$ $(i = 0,1,2)$. In this case, the free energy $F(u) = \frac{a_3}{4}u^4 + \frac{a_2}{3}u^3 + \frac{a_1}{2}u^2 + a_0u + C$, where C is a constant. Particularly, if $a_2 = 0$ and $a_1 < 0$, $F(u)$ is called double-well form potential. Since the principle part of (1.2) is a fourth-order operator, we take the viscosity approach in order to use the standard theory of the second order parabolic equations. More precisely, we study the approximate problem

$$\begin{gathered} u_\tau(x,\tau) - \Delta(\varepsilon u_\tau(x,\tau) - \kappa\Delta u(x,\tau) + g(u(x,\tau))) = f(x,\tau) \quad \text{in } Q_\omega, \\ \frac{\partial u}{\partial \nu}(x,\tau) = \frac{\partial}{\partial \nu}(-\kappa\Delta u(x,\tau) + g(u(x,\tau))) = 0 \quad \text{on } \Sigma_\omega, \\ u(x,0) = u(x,\omega) \quad \text{in } \Omega, \end{gathered} \tag{1.5}$$

where $0 < \varepsilon < 1$. In order to apply the Schauder fixed point theorem to show existence of the periodic solutions of (1.5), we need to establish some a priori estimates on the solutions of (1.5) (cf. Lemma 3.4 below).

The plan of this article is as follows. In Section 2, we state some basic results in functional analysis and give the main results. In Section 3, we first establish some estimates of the solutions for (1.5), and then obtain existence of the periodic solutions for (1.5) by the Schauder fixed point theorem. In Section 4, based on the a priori estimates in Section 3, we can take the limit as $\varepsilon \to 0$ and then obtain the periodic solutions of (1.2)–(1.4).

2. Preliminaries

The notation and the basic results that we will use here are stated as follows.

(1) We denote by $(\cdot,\cdot)$ and $|\cdot|_2$ the usual inner product and the norm in $L^2(\Omega)$, respectively. Also, we denote the Hilbert space $L^2(\Omega)$ by H.

(2) We denote $H^1(\Omega)$ by V and its inner product by $(\cdot,\cdot)_V$, where $(\eta_1,\eta_2)_V = (\eta_1,\eta_2)+(\nabla\eta_1,\nabla\eta_2)$ for any $\eta_1,\eta_2 \in H^1(\Omega)$. As a result, the norm in $H^1(\Omega)$ can be denoted by $|\eta|_V = (\eta,\eta)_V^{1/2}$. V^* denotes the dual space of V and $\langle\cdot,\cdot\rangle_{V^*,V}$ stands for the duality pairing between V^* and V.

(3) We define $H_0 := \{\eta \in H | \int_\Omega \eta(x)dx = 0\}$ which is the closed subspace of H. We choose the notation π_0 to denote the projection operator from H onto H_0, that is, $\pi_0[\eta](x) = \eta(x) - \frac{1}{|\Omega|}\int_\Omega \eta(y)dy$. Also, we denote the inner product on H_0 by $(\cdot,\cdot)_0$.

(4) We denote by V_0 the space $V \cap H_0$ with the inner product $(\cdot,\cdot)_{V_0}$ and the norm $|\cdot|_{V_0}$, where $(\eta_1,\eta_2)_{V_0} = (\nabla\eta_1,\nabla\eta_2)$ for any $\eta_1,\eta_2 \in V_0$. Furthermore, F_0^{-1} and $\langle\cdot,\cdot\rangle_{V_0^*,V_0}$ denote the duality mapping from V_0^* onto V_0 and the duality pairing between V_0^* and V_0, respectively. Thus, we see that V_0^* is a Hilbert space and its inner product can be defined as

$$(\eta_1,\eta_2)_{V_0^*} = \langle \eta_1, F_0^{-1}\eta_2\rangle_{V_0^*,V_0} = \langle F_0^{-1}\eta_1,\eta_2\rangle_{V_0,V_0^*} \quad \text{for any } \eta_1,\eta_2 \in V_0^*. \tag{2.1}$$

It is observed that the Hilbert spaces stated above satisfy the following relations

$$V \subset H \subset V^*, \quad V_0 \subset H_0 \subset V_0^*,$$

where all the injections are compact and densely defined. Throughout this article, we denote by $C_j > 0 (j = 1,2,\dots)$ the constants induced by injection. Therefore, from the above injections, we have

$$\begin{aligned} &|\eta|_{V^*} \le C_1|\eta|_2 \quad \text{for any } \eta \in H,\\ &|\eta|_2 \le C_2|\eta|_{V_0} \quad \text{for any } \eta \in V_0. \end{aligned} \tag{2.2}$$

(5) Let Δ_N be the Laplace operator with homogeneous Neumann boundary condition in H_0 with its domain

$$D(\Delta_N) = \big\{\eta \in H^2(\Omega) \cap H_0 : \frac{\partial\eta}{\partial\nu} = 0 \text{ a.e. on } \partial\Omega\big\}.$$

Specially, $\Delta_N\eta = \Delta\eta$ a.e. on Ω for any $\eta \in D(\Delta_N)$. We note that $-\Delta_N$ is invertible in H_0 and the inverse $(-\Delta_N)^{-1}$ is linear, continuous, positive and selfadjoint in H_0 as well as its fractional power $(-\Delta_N)^{1/2}$ [21, Chapter 9, Section 11]. In addition, we have

$$|(-\Delta_N)^{1/2}\eta|_{H_0} = |(-\Delta_N)^{-1}\eta|_{V_0} = |\eta|_{V_0^*}, \quad \forall \eta \in H_0. \tag{2.3}$$

In this article, we always assume that the following condition holds

- (H1) $f \in L^\infty(0,\omega;H)$ is a $\omega-$periodic function and satisfies $\int_0^\omega\int_\Omega f(x,\tau)\,dx\,d\tau = 0$.

Now, we give the notion of the solution for (1.2)–(1.4).

Definition 2.1. A function u is called a solution of (1.2)–(1.4), if the conditions below hold:

- (H2) $u \in L^2(0,\omega;H^2(\Omega)) \cap L^\infty(0,\omega;V) \cap W^{1,2}(0,\omega;V^*)$, $\frac{\partial u}{\partial\nu} = 0$ a.e. on Σ_ω.
- (H3) For all $\eta \in H^2(\Omega)$ with $\frac{\partial\eta}{\partial\nu}\Big|_{\partial\Omega} = 0$,
$$\begin{aligned}&\int_0^\omega \langle u_\tau(\tau),\eta\rangle_{V^*,V}d\tau + \kappa\int_0^\omega(\Delta u(\tau),\Delta\eta)d\tau - \int_0^\omega (g(u(\tau)),\Delta\eta)d\tau\\ &= \int_0^\omega (f(\tau),\eta)d\tau.\end{aligned}$$
- (H4) $u(0) = u(\omega)$ in H.

Now, we subtract $\frac{1}{|\Omega|}\int_\Omega f(x,\tau)dx$ from (1.2) and obtain

$$\begin{aligned}&\frac{d}{d\tau}\Big[u(x,\tau)-\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds\Big]-\Delta(-\kappa\Delta u(x,\tau)+g(u(x,\tau)))\\&=\pi_0[f(x,\tau)].\end{aligned}\tag{2.4}$$

Let

$$w(x,\tau)=u(x,\tau)-\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds.$$

Then (2.4) can be rewritten as

$$\begin{aligned}&w_\tau(x,\tau)-\Delta\Big[-\kappa\Delta w(x,\tau)+g\Big(w(x,\tau)+\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds\Big)\Big]\\&=\pi_0[f(x,\tau)].\end{aligned}\tag{2.5}$$

Therefore $\frac{1}{|\Omega|}\int_\Omega w(x,\tau)dx=m_0$ for some constant m_0. Further, putting $v(x,\tau)=w(x,\tau)-m_0$, we can rewrite (2.5) as

$$v_\tau(x,\tau)-\Delta_N(-\kappa\Delta_N v(x,\tau))-\Delta_N\pi_0[g\left(v(x,\tau)+m(\tau)\right)]=\pi_0[f(x,\tau)],\tag{2.6}$$

with $\int_\Omega v(x,\tau)dx=0$ for all $\tau>0$, where $m(\tau)=m_0+\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds$.

Now for any function $z\in H_0$, we can take $(-\Delta_N)^{-1}z$ as η in $(H3)$. Hence by the arguments in[5, Proposition 1.1], for any $z\in H_0$, it holds that

$$\begin{aligned}&\int_0^\omega((-\Delta_N)^{-1}v_\tau(\tau),z)_0d\tau+\kappa\int_0^\omega(-\Delta_N v(\tau),z)_0d\tau\\&+\int_0^\omega(\pi_0[g(v(\tau)+m(\tau))],z)_0d\tau\\&=\int_0^\omega((-\Delta_N)^{-1}\pi_0[f(\tau)],z)_0d\tau.\end{aligned}\tag{2.7}$$

From (2.3), (2.7) and the definition of F_0^{-1}, we obtain an equivalent form of (1.2), that is,

$$F_0^{-1}v_\tau(\tau)-\kappa\Delta_N v(\tau)+\pi_0[g((v(\tau)+m(\tau)))]=F_0^{-1}\pi_0[f(\tau)].\tag{2.8}$$

Similarly, (1.5) is equivalent to

$$\begin{gathered}(F_0^{-1}+\varepsilon I)v_\varepsilon'(\tau)-\kappa\Delta_N v_\varepsilon(\tau)+\pi_0[g(v_\varepsilon(\tau)+m(\tau))]=F_0^{-1}\pi_0[f(\tau)]\quad\text{in } Q_\omega,\\\frac{\partial v_\varepsilon}{\partial\nu}(x,\tau)=0\quad\text{on }\Sigma_\omega,\\v_\varepsilon(x,0)=v_\varepsilon(x,\omega)\quad\text{in }\Omega,\end{gathered}\tag{2.9}$$

where $\varepsilon\in(0,1)$, $v_\varepsilon'(\tau)=\frac{d}{d\tau}v_\varepsilon(\tau)$ and I is identity operator in H_0.

The main result of this article can be stated as follows.

Theorem 2.2. *Assume that* (H1) *holds. Then for any constant m_0,* (1.2)–(1.4) *admits a solution $u(x,\tau)$ with*

$$\frac{1}{|\Omega|}\int_\Omega u(x,\tau)dx=m_0+\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds.$$

To prove this theorem, we use the viscosity approach. Therefore, we need to investigate (2.9) first. We have the following result which is proved in next section.

Theorem 2.3. *Under the hypothesis of Theorem 2.2,* (2.9) *admits a solution which has the following properties:*

(H2') $v_\varepsilon \in L^2(0,\omega;H^2(\Omega)\cap H_0)\cap L^\infty(0,\omega;V_0)\cap W^{1,2}(0,\omega;H_0)$, $\frac{\partial v_\varepsilon}{\partial\nu}=0$ *a.e. on* Σ_ω.

(H3') *For any* $\eta\in D(\Delta_N)$ *and* $0<\tau<\omega$,
$$\begin{aligned}&\int_0^\omega((F_0^{-1}+\varepsilon I)v_\varepsilon'(\tau)-\kappa\Delta_N v_\varepsilon(\tau)+\pi_0[g(v_\varepsilon(\tau)+m(\tau))],\eta)_0d\tau\\&=\int_0^\omega(F_0^{-1}\pi_0[f(\tau)],\eta)_0d\tau\quad\text{in } H_0.\end{aligned}$$

(H4') $v_\varepsilon(0)=v_\varepsilon(\omega)$ *in* H_0.

3. Proof of Theorem 2.3

For this purpose we use the Schauder fixed point theorem. Firstly, we study the system
$$\begin{gathered}(F_0^{-1}+\varepsilon I)v'(\tau)-\kappa\Delta_N v(\tau)=\widehat{f}\quad\text{in } H_0,\\ v(0)=v(\omega)\quad\text{in } H_0,\end{gathered}\tag{3.1}$$
where $\hat{f}\in L^\infty(0,\omega;H_0)$.

Theorem 3.1. *Let* $\hat{f}\in L^\infty(0,\omega;H_0)$. *Then there exists a unique solution* $v(x,t)$ *to problem* (3.1).

We prove this theorem using Poincaré's mapping. Thus, we first introduce the corresponding Cauchy problem
$$\begin{gathered}(F_0^{-1}+\varepsilon I)v'(\tau)-\kappa\Delta_N v(\tau)=\widehat{f},\quad 0<\tau<\omega,\\ v(0)=v_0\in H_0.\end{gathered}\tag{3.2}$$
With the help of the results in [8, 10], we can see that (3.2) admits one and only one solution $v\in C([0,\omega];H_0)\cap L^\infty_{loc}(0,\omega;V_0)$. Consequently, with the unique solution $v(\tau)$, we can define a single-valued mapping $P: v(0)\in H_0\to v(\omega)\in H_0$.

Define $\phi: H_0\to\mathbb{R}\bigcup\{+\infty\}$ by
$$\phi(v)=\begin{cases}\frac{\kappa}{2}|\nabla v|_2^2, & \text{if } v\in V_0,\\ +\infty, & \text{otherwise.}\end{cases}$$
We see that ϕ is a proper, lower semicontinuous, and convex functional on H_0. Now, we give two lemmas which play an important role in the proof of Theorem 3.1.

Lemma 3.2. *There exists a constant* $R>0$ *such that* P *is a self-mapping on the set*
$$B_R:=\{v\in D(\phi);\ \phi(v)\le R\},$$
that is $P(B_R)\subset B_R$.

Proof. Multiplying the equation in (3.2) by v', we have
$$|v'|_{V_0^*}^2+\varepsilon|v'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v|_2^2=(\widehat{f},v')_0\le|\widehat{f}|_2|v'|_2\le\frac{1}{2\varepsilon}|\widehat{f}|_2^2+\frac{\varepsilon}{2}|v'|_2^2,$$
i.e.,
$$|v'|_{V_0^*}^2+\frac{\varepsilon}{2}|v'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v|_2^2\le\frac{1}{2\varepsilon}|\widehat{f}|_2^2.\tag{3.3}$$

We also multiply the equation by v and obtain

$$\begin{aligned}\kappa|\nabla v|_2^2 &= (\widehat{f}, v)_0 - (\varepsilon v', v)_0 - \langle F_0^{-1} v', v\rangle_{V_0, V_0^*}\\ &\le |\widehat{f}|_2|v|_2 + \varepsilon|v'|_2|v|_2 + |v'|_{V_0^*}|v|_{V_0^*}\\ &\le \frac{2C_2^2}{\kappa}|\widehat{f}|_2^2 + \frac{\kappa}{8}|\nabla v|_2^2 + \frac{2\varepsilon^2 C_2^2}{\kappa}|v'|_2^2 + \frac{\kappa}{8}|\nabla v|_2^2 + \frac{C_1^2 C_2^2}{\kappa}|v'|_{V_0^*}^2 + \frac{\kappa}{4}|\nabla v|_2^2\\ &= \frac{2C_2^2}{\kappa}|\widehat{f}|_2^2 + \frac{2\varepsilon^2 C_2^2}{\kappa}|v'|_2^2 + \frac{C_1^2 C_2^2}{\kappa}|v'|_{V_0^*}^2 + \frac{\kappa}{2}|\nabla v|_2^2,\end{aligned}$$

which implies

$$\frac{\kappa}{2}|\nabla v|_2^2 \le \frac{2C_2^2}{\kappa}|\widehat{f}|_2^2 + \frac{2\varepsilon^2 C_2^2}{\kappa}|v'|_2^2 + \frac{C_1^2 C_2^2}{\kappa}|v'|_{V_0^*}^2 . \tag{3.4}$$

Letting $\mu > 0$ and performing (3.3) $\times\, \mu +$ (3.4), we obtain

$$\begin{aligned}&\mu\frac{d}{dt}(\frac{\kappa}{2}|\nabla v|_2^2) + \frac{\kappa}{2}|\nabla v|_2^2\\ &\le (\frac{\mu}{2\varepsilon} + \frac{2C_2^2}{\kappa})|\widehat{f}|_2^2 + (\frac{C_1^2 C_2^2}{\kappa} - \mu)|v'|_{V_0^*}^2 + (\frac{2\varepsilon^2 C_2^2}{\kappa} - \frac{\mu\varepsilon}{2})|v'|_2^2.\end{aligned}$$

Choosing $\mu = \max\big\{\frac{C_1^2 C_2^2}{\kappa}, \frac{4C_2^2}{\kappa}\big\}$, from $0 < \varepsilon < 1$ we have

$$\frac{d}{dt}\phi(v) + \frac{1}{\mu}\phi(v) \le \big(\frac{1}{2\varepsilon} + \frac{2C_2^2}{\kappa\mu}\big)|\widehat{f}|_2^2.$$

It follows from the Gronwall inequality that

$$\phi(v(\omega)) \le e^{-\frac{\omega}{\mu}}\phi(v(0)) + (1 - e^{-\frac{\omega}{\mu}})\big(\frac{\mu}{2\varepsilon} + \frac{2C_2^2}{\kappa}\big)\|\widehat{f}\|_{L^\infty(0,\omega;H_0)}^2.$$

Set $R = (\frac{\mu}{2\varepsilon} + \frac{2C_2^2}{\kappa})\|\widehat{f}\|_{L^\infty(0,\omega;H_0)}^2$. Then $\phi(v(\omega)) \le R$ provided that $\phi(v(0)) \le R$. The proof is complete. $\square$

Lemma 3.3. *The mapping P is continuous in H_0.*

Proof. Let $v_{0,n} \in H_0$ be such that $v_{0,n} \to v_0$ in H_0. We denote the unique solution of (3.2) by v_n and v corresponding to the initial data $v_{0,n}$ and v_0, respectively. Then we have

$$F_0^{-1}(v_n' - v') + \varepsilon(v_n' - v') - \kappa\Delta_N(v_n - v) = 0. \tag{3.5}$$

Multiplying (3.5) by $v_n - v$ and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}|v_n - v|_{V_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{dt}|v_n - v|_2^2 + \kappa|\nabla(v_n - v)|_2^2 = 0.$$

It can be easy to see that

$$\frac{1}{2}\frac{d}{dt}|v_n - v|_{V_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{dt}|v_n - v|_2^2 \le 0.$$

Therefore,

$$\frac{1}{2}|v_n(\omega) - v(\omega)|_{V_0^*}^2 + \frac{\varepsilon}{2}|v_n(\omega) - v(\omega)|_2^2 \le \frac{1}{2}|v_{n0} - v_0|_{V_0^*}^2 + \frac{\varepsilon}{2}|v_{n0} - v_0|_2^2,$$

which implies $v_n(w) \to v(\omega)$ in H_0 as $n \to \infty$. Hence, P is continuous in H_0. $\square$

Proof of Theorem 3.1. On the one hand, it follows from the definition of B_R and the convexity of ϕ that B_R is compact and convex in H_0. On the other hand, Lemmas 3.2 and 3.3 ensure that P maps B_R to B_R and is continuous in H_0. Thus, the Schauder fixed point theorem admits a fixed point $v_0^*\in B_R$ such that $Pv_0^*=v_0^*$, which implies that the solution $v(x,t)$ of (3.2) with $v_0=v_0^*$ is the desired solution of (3.1).

Now, we prove that the solution for (3.1) is unique. To this end, let v_1 and v_2 be two solutions of (3.1). Then we have

$$F_0^{-1}(v_1'-v_2')+\varepsilon(v_1'-v_2')-\kappa\Delta_N(v_1-v_2)=0. \tag{3.6}$$

We multiply (3.6) by v_1-v_2 and then get that

$$\frac{1}{2}\frac{d}{dt}|v_1-v_2|_{V_0^*}^2+\frac{\varepsilon}{2}\frac{d}{dt}|v_1-v_2|_2^2+\kappa|\nabla(v_1-v_2)|_2^2=0.$$

Integrating the equation over $(0,\omega)$ and by the periodic property, we obtain

$$\int_0^\omega|\nabla(v_1(\tau)-v_2(\tau))|_2^2d\tau\le 0,$$

which, together with (2.2), implies that

$$\int_0^\omega\int_\Omega|v_1-v_2|^2\,dx\,d\tau\le 0.$$

Hence, $v_1=v_2$ and the proof is complete. $\square$

To apply the Schauder fixed point theorem to (2.9), we need to establish a priori estimates for v_ε.

Lemma 3.4. *Let v_ε be a solution of (2.9). Then*

$$\varepsilon\int_0^\omega|v_\varepsilon'(\tau)|_2^2d\tau+\int_0^\omega|v_\varepsilon'(\tau)|_{V_0^*}^2d\tau\le\omega C_1^2\|f\|_{L^\infty(0,\omega;H)}^2, \tag{3.7}$$

$$\sup_{\tau\in[0,\omega]}|v_\varepsilon(\tau)|_{V_0}^2\le\frac{2}{\kappa}\Big(3A_1+4A_2+\frac{3C_1^2+4\omega}{2}C_1^2\|f\|_{L^\infty(0,\omega;H)}^2\Big), \tag{3.8}$$

$$\begin{aligned}
&\|-\Delta_Nv_\varepsilon\|_{L^2(0,\omega;H_0)}^2\\
&\le\frac{4\omega}{\kappa^2}\Big(\frac{a_2^2}{2a_3}+|a_1|\Big)(3A_1+4A_2)\\
&\quad+\frac{\omega}{\kappa^2}\Big[(3C_1^2+4\omega)C_1^2\Big(\frac{a_2^2}{a_3}+2|a_1|\Big)+C_1^4\Big]\|f\|_{L^\infty(0,\omega;H)}^2
\end{aligned} \tag{3.9}$$

$$\|v_\varepsilon(\tau)+m(\tau)\|_{C([0,\omega];L^6(\Omega))}^6\le A_3^3, \tag{3.10}$$

where a_i $(i=0,1,2,3)$ are the coefficients of $g(\cdot)$,

$$\begin{aligned}
A_1:=|\Omega|\Big[&\frac{9^3}{4}a_3\Big(|m_0|+\frac{\omega}{|\Omega|^{1/2}}\|f\|_{L^\infty(0,\omega;H)}\Big)^4+\Big(\frac{3a_1^2}{2a_3}+\frac{3a_2^2}{a_3}+1\Big)\\
&\times\Big(|m_0|+\frac{\omega}{|\Omega|^{1/2}}\|f\|_{L^\infty(0,\omega;H)}\Big)^2\\
&+\frac{a_2^4}{4}\Big(\frac{9}{a_3}\Big)^3+\frac{a_3}{12}+\frac{3a_1^2+3}{a_3}\Big],
\end{aligned}$$

$$A_2:=6|\Omega|\Big[\frac{6a_2^4}{a_3^3}+\frac{a_1^2}{2a_3}+\frac{a_0^{4/3}}{4}\Big(\frac{6}{a_3}\Big)^{1/3}\Big],$$

$$
\begin{aligned}
A_3 := C_3\Big\{ & \frac{2}{\kappa}(2\omega C_1^2C_2^2+2C_2^2+1)(3A_1+4A_2)+4m_0^2|\Omega|(\omega C_1^2+1) \\
& + \Big[4\omega^3C_1^2+4\omega^2\Big(\frac{2C_1^4C_2^2}{\kappa}+1\Big)+2C_1^2\omega\Big(\frac{3C_1^4C_2^2+4C_2^2+2}{\kappa}+2\Big) \\
& + \frac{3C_1^4(2C_2^2+1)}{\kappa}\Big]\|f\|^2_{L^\infty(0,\omega;H)}\Big\}.
\end{aligned}
$$

Proof. From (2.2), the definition of π_0 and $(H1)$, we know that

$$
\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)} \le C_1^2\|\pi_0[f]\|^2_{L^2(0,\omega;H_0)} \le \omega C_1^2\|f\|^2_{L^\infty(0,\omega;H)}. \tag{3.11}
$$

It follows from the Hölder inequality and (H1) that for any $\tau\in[0,\omega]$

$$
\begin{aligned}
|m(\tau)| &= \Big|m_0+\frac{1}{|\Omega|}\int_0^\tau\int_\Omega f(x,s)\,dx\,ds\Big| \\
&\le |m_0|+\frac{\omega}{|\Omega|^{1/2}}\|f\|_{L^\infty(0,\omega;H)}.
\end{aligned} \tag{3.12}
$$

We multiply the equation in (2.9) by v_ε', and obtain

$$
\begin{aligned}
&|v_\varepsilon'|^2_{V_0^*}+\varepsilon|v_\varepsilon'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v_\varepsilon|_2^2+\int_\Omega v_\varepsilon'\pi_0[g(v_\varepsilon(\tau)+m(\tau))]dx \\
&= \langle F_0^{-1}\pi_0[f],v_\varepsilon'\rangle_{V_0,V_0^*} \\
&\le |\pi_0[f]|_{V_0^*}|v_\varepsilon'|_{V_0^*} \\
&\le \frac12|\pi_0[f]|^2_{V_0^*}+\frac12|v_\varepsilon'|^2_{V_0^*}.
\end{aligned} \tag{3.13}
$$

By the definition of π_0 and $g(\cdot)$, we have

$$
\begin{aligned}
&\frac12|v_\varepsilon'|^2_{V_0^*}+\varepsilon|v_\varepsilon'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v_\varepsilon|_2^2+\frac{d}{dt}\int_\Omega\Big[\frac{a_3}{4}(v_\varepsilon(\tau)+m(\tau))^4 \\
&+\frac{a_2}{3}(v_\varepsilon(\tau)+m(\tau))^3+\frac{a_1}{2}(v_\varepsilon(\tau)+m(\tau))^2+a_0(v_\varepsilon(\tau)+m(\tau))\Big]dx \\
&\le \frac12|\pi_0[f]|^2_{V_0^*}.
\end{aligned} \tag{3.14}
$$

From the periodic property, we integrate (3.14) over $(0,\omega)$ and then obtain

$$
2\varepsilon\int_0^\omega|v_\varepsilon'(\tau)|_2^2d\tau+\int_0^\omega|v_\varepsilon'(\tau)|^2_{V_0^*}d\tau\le\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau. \tag{3.15}
$$

Combining this inequality with (3.11), we have

$$
\varepsilon\int_0^\omega|v_\varepsilon'(\tau)|_2^2d\tau+\int_0^\omega|v_\varepsilon'(\tau)|^2_{V_0^*}d\tau\le\omega C_1^2\|f\|^2_{L^\infty(0,\omega;H)},
$$

which is (3.7).

Choose any $s,t\in[0,\omega]$ which satisfy $s<t$. Integrating (3.14) on (s,t), we have

$$
\begin{aligned}
&\frac12\int_s^t|v_\varepsilon'(\tau)|^2_{V_0^*}d\tau+\varepsilon\int_s^t|v_\varepsilon'(\tau)|_2^2d\tau+\frac{\kappa}{2}|\nabla v_\varepsilon(t)|_2^2+\int_\Omega\big[\frac{a_3}{4}(v_\varepsilon(t)+m(t))^4 \\
&+\frac{a_2}{3}(v_\varepsilon(t)+m(t))^3+\frac{a_1}{2}(v_\varepsilon(t)+m(t))^2+a_0(v_\varepsilon(t)+m(t))\big]dx \\
&\le\frac12\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau+\frac{\kappa}{2}|\nabla v_\varepsilon(s)|_2^2+\int_\Omega\big[\frac{a_3}{4}(v_\varepsilon(s)+m(s))^4 \\
&\quad+\frac{a_2}{3}(v_\varepsilon(s)+m(s))^3+\frac{a_1}{2}(v_\varepsilon(s)+m(s))^2+a_0(v_\varepsilon(s)+m(s))\big]dx.
\end{aligned} \tag{3.16}
$$

From Young's inequality, we obtain

$$
\begin{aligned}
\frac{a_2}{3}(v_\varepsilon(\tau)+m(\tau))^3 &\le \frac{a_3}{24}(v_\varepsilon(\tau)+m(\tau))^4+\frac{18a_2^4}{a_3^3},\\
\frac{a_1}{2}(v_\varepsilon(\tau)+m(\tau))^2 &\le \frac{a_3}{24}(v_\varepsilon(\tau)+m(\tau))^4+\frac{3a_1^2}{2a_3}\\
a_0(v_\varepsilon(\tau)+m(\tau)) &\le \frac{a_3}{24}(v_\varepsilon(\tau)+m(\tau))^4+\frac{3a_0^{4/3}}{4}\Big(\frac{6}{a_3}\Big)^{1/3}.
\end{aligned} \tag{3.17}
$$

It follows from (3.16) and (3.17) that

$$
\begin{aligned}
&\frac{1}{2}\int_s^t |v_\varepsilon'(\tau)|_{V_0^*}^2 d\tau+\varepsilon\int_s^t |v_\varepsilon'(\tau)|_2^2 d\tau+\frac{\kappa}{2}|\nabla v_\varepsilon(t)|_2^2\\
&+\frac{a_3}{8}\int_\Omega (v_\varepsilon(t)+m(t))^4dx\\
&\le \frac{\kappa}{2}|\nabla v_\varepsilon(s)|_2^2+\frac{3a_3}{8}\int_\Omega (v_\varepsilon(s)+m(s))^4dx\\
&\quad +\frac{1}{2}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau\\
&\quad + A_2.
\end{aligned} \tag{3.18}
$$

Deleting the first two terms on the left-hand side of (3.18) and integrating it on $(0,t)$ with respect to s, we have

$$
\begin{aligned}
&\frac{\kappa t}{2}|\nabla v_\varepsilon(t)|_2^2+\frac{a_3 t}{8}\int_\Omega (v_\varepsilon(t)+m(t))^4dx\\
&\le \frac{\kappa}{2}\int_0^\omega |\nabla v_\varepsilon(s)|_2^2 ds+\frac{3a_3}{8}\int_0^\omega\int_\Omega (v_\varepsilon(s)+m(s))^4\,dx\,ds+\frac{\omega}{2}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau\\
&\quad + A_2\omega.
\end{aligned}
$$

Letting $t=\omega$, one sees that

$$
\begin{aligned}
&\frac{\kappa\omega}{2}|\nabla v_\varepsilon(\omega)|_2^2+\frac{a_3\omega}{8}\int_\Omega (v_\varepsilon(\omega)+m_0)^4dx\\
&\le \frac{\kappa}{2}\int_0^\omega |\nabla v_\varepsilon(s)|_2^2 ds+\frac{3a_3}{8}\int_0^\omega\int_\Omega (v_\varepsilon(s)+m(s))^4\,dx\,ds+\frac{\omega}{2}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau\\
&\quad + A_2\omega,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{\kappa}{2}|\nabla v_\varepsilon(\omega)|_2^2+\frac{a_3}{8}\int_\Omega (v_\varepsilon(\omega)+m_0)^4dx\\
&\le \frac{\kappa}{2\omega}\int_0^\omega |\nabla v_\varepsilon(s)|_2^2 ds+\frac{3a_3}{8\omega}\int_0^\omega\int_\Omega (v_\varepsilon(s)+m(s))^4\,dx\,ds+\frac{1}{2}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau\\
&\quad + A_2.
\end{aligned}
$$

From the periodic property, we have

$$
\begin{aligned}
&\frac{\kappa}{2}|\nabla v_\varepsilon(0)|_2^2 + \frac{a_3}{8}\int_\Omega (v_\varepsilon(0)+m_0)^4 dx \\
&\le \frac{\kappa}{2\omega}\int_0^\omega |\nabla v_\varepsilon(s)|_2^2 ds + \frac{3a_3}{8\omega}\int_0^\omega\int_\Omega (v_\varepsilon(s)+m(s))^4\,dx\,ds \\
&\quad + \frac{1}{2}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau + A_2.
\end{aligned}
\tag{3.19}
$$

Multiplying the equation in (2.9) by v_ε and performing a proper arrangement, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_\varepsilon|_{V_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{dt}|v_\varepsilon|_2^2 + \kappa|\nabla v_\varepsilon|_2^2 \\
&+ a_3\int_\Omega \left[(v_\varepsilon(\tau)+m(\tau))^4 - m(\tau)(v_\varepsilon(\tau)+m(\tau))^3\right] dx \\
&+ a_2\int_\Omega [v_\varepsilon(\tau)+m(\tau)]^3\,dx + [a_1 - a_2 m(\tau)]\int_\Omega [v_\varepsilon(\tau)+m(\tau)]^2\,dx \\
&- a_1 m(\tau)\int_\Omega [v_\varepsilon(\tau)+m(\tau)]\,dx \\
&= \langle F_0^{-1}\pi_0[f], v_\varepsilon\rangle_{V_0,V_0^*} \\
&\le |\pi_0[f]|_{V_0^*}|v_\varepsilon|_{V_0^*} \\
&\le C_1|\pi_0[f]|_{V_0^*}|v_\varepsilon(\tau)+m(\tau)|_2 + C_1|\pi_0[f]|_{V_0^*}|\Omega|^{1/2}|m(\tau)|.
\end{aligned}
\tag{3.20}
$$

From Young's inequality, we obtain

$$
\begin{aligned}
m(\tau)\,[v_\varepsilon(\tau)+m(\tau)]^3 &\le \frac{1}{12}[v_\varepsilon(\tau)+m(\tau)]^4 + \frac{9^3}{4}|m(\tau)|^4, \\
a_2\,[v_\varepsilon(\tau)+m(\tau)]^3 &\le \frac{a_3}{12}[v_\varepsilon(\tau)+m(\tau)]^4 + \frac{a_2^4}{4}\big(\frac{9}{a_3}\big)^3, \\
[a_1 - a_2 m(\tau)]\,[v_\varepsilon(\tau)+m(\tau)]^2 &\le \frac{a_3}{6}[v_\varepsilon(\tau)+m(\tau)]^4 + \frac{3}{2a_3}[a_1 - a_2 m(\tau)]^2 \\
&\le \frac{a_3}{6}[v_\varepsilon(\tau)+m(\tau)]^4 + \frac{3a_1^2}{a_3} + \frac{3a_2^2}{a_3}|m(\tau)|^2, \\
a_1 m(\tau)\,[v_\varepsilon(\tau)+m(\tau)] &\le \frac{a_3}{6}[v_\varepsilon(\tau)+m(\tau)]^2 + \frac{3}{2a_3}a_1^2|m(\tau)|^2 \\
&\le \frac{a_3}{12}[v_\varepsilon(\tau)+m(\tau)]^4 + \frac{a_3}{12} + \frac{3}{2a_3}a_1^2|m(\tau)|^2, \\
|\pi_0[f]|_{V_0^*}|v_\varepsilon(\tau)+m(\tau)|_2 &\le \frac{C_1^2}{4}|\pi_0[f]|_{V_0^*}^2 + |v_\varepsilon(\tau)+m(\tau)|_2^2 \\
&\le \frac{C_1^2}{4}|\pi_0[f]|_{V_0^*}^2 + \frac{a_3}{12}\int_\Omega [v_\varepsilon(\tau)+m(\tau)]^4\,dx + \frac{3}{a_3}|\Omega|
\end{aligned}
$$

and

$$
|\pi_0[f]|_{V_0^*}|\Omega|^{1/2}|m(\tau)| \le \frac{C_1^2}{4}|\pi_0[f]|_{V_0^*}^2 + |\Omega||m(\tau)|^2. \tag{3.21}
$$

In light of (3.12), (3.20) and (3.21), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_\varepsilon|^2_{V_0^*}+\frac{\varepsilon}{2}\frac{d}{dt}|v_\varepsilon|_2^2+\kappa|\nabla v_\varepsilon|_2^2+\frac{a_3}{2}\int_\Omega[v_\varepsilon(\tau)+m(\tau)]^4\,dx\\&\le\frac{C_1^2}{2}|\pi_0[f]|^2_{V_0^*}+A_1.\end{aligned}\tag{3.22}$$

From the periodic property, we integrate (3.22) over $(0,\omega)$ and obtain

$$\begin{aligned}&\kappa\int_0^\omega|\nabla v_\varepsilon(\tau)|_2^2d\tau+\frac{a_3}{2}\int_0^\omega\int_\Omega(v_\varepsilon(\tau)+m(\tau))^4\,dx\,d\tau\\&\le\frac{C_1^2}{2}\int_0^\omega|\pi_0[f]|^2_{V_0^*}d\tau+A_1\omega.\end{aligned}\tag{3.23}$$

Combining (3.19) with (3.23), we have

$$\begin{aligned}&\frac{\kappa}{2}|\nabla v_\varepsilon(0)|_2^2+\frac{a_3}{8}\int_\Omega(v_\varepsilon(0)+m_0)^4dx\\&\le\frac{C_1^2}{2\omega}\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau+A_1+\frac{1}{2}\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau+A_2\\&\le A_1+A_2+\frac{\omega+C_1^2}{2\omega}\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)}.\end{aligned}\tag{3.24}$$

Letting $s=0$ in (3.18) and deleting the first two terms on the left-hand side, we obtain

$$\begin{aligned}&\frac{\kappa}{2}|\nabla v_\varepsilon(t)|_2^2+\frac{a_3}{8}\int_\Omega(v_\varepsilon(t)+m(t))^4dx\\&\le\frac{\kappa}{2}|\nabla v_\varepsilon(0)|_2^2+\frac{3a_3}{8}\int_\Omega(v_\varepsilon(0)+m_0)^4dx+\frac{1}{2}\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau+A_2\\&\le3\Big[\frac{\kappa}{2}|\nabla v_\varepsilon(0)|_2^2+\frac{a_3}{8}\int_\Omega(v_\varepsilon(0)+m_0)^4dx\Big]+\frac{1}{2}\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau+A_2.\end{aligned}\tag{3.25}$$

It follows from (3.24) and (3.25) that for any $t\in[0,\omega]$,

$$\frac{\kappa}{2}|\nabla v_\varepsilon(t)|_2^2+\frac{a_3}{8}\int_\Omega(v_\varepsilon(t)+m(t))^4dx\le3A_1+4A_2+\frac{3C_1^2+4\omega}{2\omega}\int_0^\omega|\pi_0[f(\tau)]|^2_{V_0^*}d\tau,$$

which together with (3.11) yields (3.8).

Multiplying the equation of (2.9) by $-\Delta_N v_\varepsilon$ and integrating it by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_\varepsilon|_2^2+\frac{\varepsilon}{2}\frac{d}{dt}|\nabla v_\varepsilon|_2^2+\kappa|\Delta_N v_\varepsilon|_2^2+3a_3\int_\Omega(v_\varepsilon(\tau)+m(\tau))^2|\nabla v_\varepsilon|^2dx\\&+2a_2\int_\Omega(v_\varepsilon(\tau)+m(\tau))|\nabla v_\varepsilon|^2dx+a_1\int_\Omega|\nabla v_\varepsilon|^2dx\\&=\langle F_0^{-1}\pi_0[f],-\Delta_N v_\varepsilon\rangle_{V_0,V_0^*}\\&\le C_1|\pi_0[f]|_{V_0^*}|\Delta_N v_\varepsilon|_2\\&\le\frac{C_1^2}{2\kappa}|\pi_0[f]|^2_{V_0^*}+\frac{\kappa}{2}|\Delta_N v_\varepsilon|_2^2.\end{aligned}$$

After a proper arrangement, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_\varepsilon|_2^2+\frac{\varepsilon}{2}\frac{d}{dt}|\nabla v_\varepsilon|_2^2+\frac{\kappa}{2}|\Delta_N v_\varepsilon|_2^2+3a_3\int_\Omega(v_\varepsilon(\tau)+m(\tau))^2|\nabla v_\varepsilon|^2dx\\&+2a_2\int_\Omega(v_\varepsilon(\tau)+m(\tau))|\nabla v_\varepsilon|^2dx\\&\le\frac{C_1^2}{2\kappa}|\pi_0[f]|_{V_0^*}^2-a_1\int_\Omega|\nabla v_\varepsilon|^2dx.\end{aligned}\tag{3.26}$$

From Young's inequality, we have

$$2a_2(v_\varepsilon(\tau)+m(\tau))\le 2a_3(v_\varepsilon(\tau)+m(\tau))^2+\frac{a_2^2}{2a_3}.\tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_\varepsilon|_2^2+\frac{\varepsilon}{2}\frac{d}{dt}|\nabla v_\varepsilon|_2^2+\frac{\kappa}{2}|\Delta_N v_\varepsilon|_2^2+a_3\int_\Omega(v_\varepsilon(\tau)+m(\tau))^2|\nabla v_\varepsilon|^2dx\\&\le\Big(\frac{a_2^2}{2a_3}+|a_1|\Big)\int_\Omega|\nabla v_\varepsilon|^2dx+\frac{C_1^2}{2\kappa}|\pi_0[f]|_{V_0^*}^2.\end{aligned}\tag{3.28}$$

With the help of the periodic property, we integrate (3.28) over $(0,\omega)$ and then get that

$$\int_0^\omega|\Delta_N v_\varepsilon(\tau)|_2^2d\tau\le\frac{2}{\kappa}\Big(\frac{a_2^2}{2a_3}+|a_1|\Big)\int_0^\omega\int_\Omega|\nabla v_\varepsilon(\tau)|^2\,dx\,d\tau+\frac{C_1^2}{\kappa^2}\int_0^\omega|\pi_0[f(\tau)]|_{V_0^*}^2d\tau.$$

Substituting (3.8) and (3.11) into the above inequality, we obtain (3.9).

By (3.7) and (3.8), we know that $v_\varepsilon\in W^{1,2}(0,\omega;V_0^*)\cap L^\infty(0,\omega;V_0)$. Therefore, $v_\varepsilon(\tau)+m(\tau)\in W^{1,2}(0,\omega;V^*)\cap L^\infty(0,\omega;V)$. Since

$$W^{1,2}(0,\omega;V^*)\cap L^\infty(0,\omega;V)\hookrightarrow C([0,\omega];L^6(\Omega)),$$

it is clear that there exists a positive constant C_3 such that

$$\begin{aligned}&\|v_\varepsilon(\tau)+m(\tau)\|_{C([0,\omega];L^6(\Omega))}^2\\&\le C_3(\|v_\varepsilon(\tau)+m(\tau)\|_{W^{1,2}(0,\omega;V^*)}^2+\|v_\varepsilon(\tau)+m(\tau)\|_{L^\infty(0,\omega;V)}^2).\end{aligned}\tag{3.29}$$

Now, we establish the estimates for $\|v_\varepsilon(\tau)+m(\tau)\|_{W^{1,2}(0,\omega;V^*)}$ and $\|v_\varepsilon(\tau)+m(\tau)\|_{L^\infty(0,\omega;V)}$, respectively. Since

$$\begin{aligned}\int_0^\omega|v_\varepsilon(\tau)+m(\tau)|_{V^*}^2d\tau&\le\int_0^\omega(|v_\varepsilon(\tau)|_{V_0^*}^2+2|v_\varepsilon(\tau)|_{V_0^*}|m(\tau)|_{V^*}+|m(\tau)|_{V^*}^2)d\tau\\&\le 2(C_1^2C_2^2\int_0^\omega|v_\varepsilon(\tau)|_{V_0}^2d\tau+C_1^2\int_0^\omega|m(\tau)|_2^2d\tau)\\&\le 2\omega C_1^2(C_2^2\|v_\varepsilon(\tau)\|_{L^\infty(0,\omega;V_0)}^2+|\Omega||m(\tau)|^2),\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^\omega|v_\varepsilon(\tau)+m(\tau)|_{V^*}^2\,d\tau&\le 2\omega C_1^2\Big[\frac{2C_2^2}{\kappa}(3A_1+4A_2)+2m_0^2|\Omega|\\&\quad+\Big(\frac{3C_1^4C_2^2+4\omega C_1^2C_2^2}{\kappa}+2\omega^2\Big)\|f\|_{L^\infty(0,\omega;H)}^2\Big].\end{aligned}\tag{3.30}$$

Similarly, we have

$$\begin{aligned}
&\int_0^\omega |v_\varepsilon'(\tau)+m'(\tau)|_{V^*}^2 d\tau \\
&\le \int_0^\omega \Big(|v_\varepsilon'(\tau)|_{V_0^*}^2 + 2|v_\varepsilon'(\tau)|_{V_0^*}|m'(\tau)|_{V^*} + |m'(\tau)|_{V^*}^2\Big)\, d\tau \\
&\le 2\Big(\int_0^\omega |v_\varepsilon'(\tau)|_{V_0^*}^2 d\tau + C_1^2\int_0^\omega |m'(\tau)|_2^2 d\tau\Big).
\end{aligned} \tag{3.31}$$

Moreover,

$$\int_0^\omega |m'(\tau)|_2^2 d\tau = \int_0^\omega \Big|\frac{1}{|\Omega|}\int_\Omega f(x,\tau)dx\Big|_2^2 d\tau \le \omega\|f\|_{L^\infty(0,\omega;H)}^2.$$

Thus, together with (3.7), (3.31) can be written as

$$\int_0^\omega \Big|\frac{d}{dt}(v_\varepsilon(\tau)+m(\tau))\Big|_{V^*}^2 d\tau \le 4\omega C_1^2\|f\|_{L^\infty(0,\omega;H)}^2. \tag{3.32}$$

It follows from (3.30) and (3.32) that

$$\begin{aligned}
&\|v_\varepsilon(\tau)+m(\tau)\|_{W^{1,2}(0,\omega;V^*)}^2 \\
&\le 2\omega C_1^2\Big[\frac{2C_2^2}{\kappa}(3A_1+4A_2) + 2m_0^2|\Omega| \\
&\quad + \Big(\frac{3C_1^4C_2^2 + 4\omega C_1^2C_2^2}{\kappa} + 2\omega^2 + 2\Big)\|f\|_{L^\infty(0,\omega;H)}^2\Big].
\end{aligned} \tag{3.33}$$

Also, since

$$\begin{aligned}
&\|v_\varepsilon(\tau)+m(\tau)\|_{L^\infty(0,\omega;V)}^2 \\
&= \operatorname{ess\,sup}_{\tau\in[0,\omega]}\Big[\int_\Omega |v_\varepsilon(\tau)+m(\tau)|^2dx + \int_\Omega |\nabla(v_\varepsilon(\tau)+m(\tau))|^2dx\Big] \\
&\le 2\operatorname{ess\,sup}_{\tau\in[0,\omega]}\int_\Omega [v_\varepsilon^2(\tau)+|m(\tau)|^2]dx + \operatorname{ess\,sup}_{\tau\in[0,\omega]}\int_\Omega |\nabla v_\varepsilon(\tau)|^2dx \\
&\le (2C_2^2+1)\|v_\varepsilon\|_{L^\infty(0,\omega;V_0)}^2 + 2\operatorname{ess\,sup}_{\tau\in[0,\omega]}\int_\Omega |m(\tau)|^2dx,
\end{aligned}$$

from (3.8) and (3.12) we have

$$\begin{aligned}
&\|v_\varepsilon(\tau)+m(\tau)\|_{L^\infty(0,\omega;V)}^2 \\
&\le \frac{2}{\kappa}(2C_2^2+1)(3A_1+4A_2) + 4m_0^2|\Omega| \\
&+ \Big[\frac{(2C_2^2+1)(3C_1^2+4\omega)}{\kappa}C_1^2 + 4\omega^2\Big]\|f\|_{L^\infty(0,\omega;H)}^2.
\end{aligned} \tag{3.34}$$

Combining (3.33) with (3.34), we obtain

$$\|v_\varepsilon(\tau)+m(\tau)\|_{C([0,\omega];L^6(\Omega))}^2 \le A_3. \tag{3.35}$$

Thus,

$$\|v_\varepsilon(\tau)+m(\tau)\|_{C([0,\omega];L^6(\Omega))}^6 \le A_3^3, \tag{3.36}$$

which is (3.10). The proof is complete. □

Define a set

$$\begin{aligned} Y_1 := \Big\{ &\bar{v} \in L^\infty(0,\omega;V_0) \cap W^{1,2}(0,\omega;H_0) | \bar{v}(0) = \bar{v}(\omega), \\ &\|\bar{v}(\tau) + m(\tau)\|^6_{C([0,\omega];L^6(\Omega))} \le A_3^3 \Big\}. \end{aligned} \tag{3.37}$$

Now, for any $\bar{v} \in Y_1$, we study the problem

$$\begin{gathered} F_0^{-1} v'(\tau) + \varepsilon v'(\tau) - \kappa \Delta_N v(\tau) = -\pi_0[g(\bar{v}(\tau) + m(\tau))] + F_0^{-1}\pi_0[f(\tau)] \\ \text{in } H_0, \quad 0 < \tau < \omega, \\ v(0) = v(\omega) \quad \text{in } H_0. \end{gathered} \tag{3.38}$$

For convenience, we denote the above system by $(E_\varepsilon, \bar{v})$.

Lemma 3.5. *Let $v(t)$ be the solution of $(E_\varepsilon, \bar{v})$. Then the following estimates can be established*

$$\begin{aligned} &\int_0^\omega |v'(\tau)|^2_{V_0^*} d\tau + \varepsilon \int_0^\omega |v'(\tau)|_2^2 d\tau \\ &\le \frac{\omega}{\varepsilon} \left[3A_3^3(a_3^2 + a_2^2 + a_1^2) + |\Omega|(a_2^2 + 2a_1^2) \right] + \omega C_1^2 \|f\|^2_{L^\infty(0,\omega;H)}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} &\int_0^\omega |v(\tau)|_2^2 d\tau \\ &\le \frac{2\omega C_2^4}{\kappa^2} \Big[3A_3^3(a_3^2 + a_2^2 + a_1^2) + |\Omega|(a_2^2 + 2a_1^2) + C_1^4 \|f\|^2_{L^\infty(0,\omega;H)} \Big], \end{aligned} \tag{3.40}$$

$$\begin{aligned} \kappa |\nabla v(t)|_2^2 \le 2\Big(\frac{\omega}{\varepsilon} + \frac{C_2^2}{\kappa}\Big) &\left[3A_3^3(a_3^2 + a_2^2 + a_1^2) + |\Omega|(a_2^2 + 2a_1^2) \right] \\ &+ 2C_1^2 \Big(\omega + \frac{C_1^2 C_2^2}{\kappa}\Big) \|f\|^2_{L^\infty(0,\omega;H)} \end{aligned} \tag{3.41}$$

for $t \in [0,\omega]$ and

$$\begin{aligned} &\int_0^\omega |\Delta_N v(\tau)|_2^2 d\tau \\ &\le \frac{2\omega}{\kappa^2} \Big[3A_3^3(a_3^2 + a_2^2 + a_1^2) + |\Omega|(a_2^2 + 2a_1^2) + C_1^4 \|f\|^2_{L^2(0,\omega;V_0^*)} \Big], \end{aligned} \tag{3.42}$$

where A_3 is the same as that in Lemma 3.4.

Proof. For any $\tau \in [0,\omega]$, we have

$$\begin{aligned} &\int_\Omega \big| \pi_0[a_3(\bar{v}(\tau) + m(\tau))^3] \big|^2 dx \\ &= a_3^2 \int_\Omega (\bar{v}(\tau) + m(\tau))^6 dx - \frac{a_3^2}{|\Omega|} \Big[\int_\Omega (\bar{v}(\tau) + m(\tau))^3 dx \Big]^2 \\ &\le a_3^2 \int_\Omega (\bar{v}(\tau) + m(\tau))^6 dx. \end{aligned} \tag{3.43}$$

Similarly, for any $\tau \in [0,\omega]$, we have

$$\int_\Omega \big| \pi_0[a_2(\bar{v}(\tau) + m(\tau))^2] \big|^2 dx \le a_2^2 \int_\Omega (\bar{v}(\tau) + m(\tau))^4 dx\,, \tag{3.44}$$

$$\int_\Omega |\pi_0[a_1(\bar{v}(\tau) + m(\tau))]|^2 dx \le a_1^2 \int_\Omega (\bar{v}(\tau) + m(\tau))^2 dx. \tag{3.45}$$

It follows from Young's inequality that

$$a_2^2\int_\Omega(\bar v(\tau)+m(\tau))^4dx \le a_2^2\int_\Omega(\bar v(\tau)+m(\tau))^6dx+\frac{4a_2^2}{27}|\Omega|\,, \tag{3.46}$$

$$\begin{aligned} a_1^2\int_\Omega(\bar v(\tau)+m(\tau))^2dx &\le a_1^2\int_\Omega(\bar v(\tau)+m(\tau))^6dx+\frac{2a_1^2}{3^{\frac32}}|\Omega| \\ &< a_1^2\int_\Omega(\bar v(\tau)+m(\tau))^6dx+\frac{2a_1^2}{3}|\Omega|. \end{aligned} \tag{3.47}$$

By (3.46)-(3.47), for any $\tau\in[0,\omega]$, we have

$$\begin{aligned} &\int_\Omega\big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))]\big|^2\,dx \\ &\le 3(a_3^2+a_2^2+a_1^2)\int_\Omega(\bar v(\tau)+m(\tau))^6dx+|\Omega|\Big(\frac{4a_2^2}{9}+2a_1^2\Big) \\ &\le 3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2). \end{aligned} \tag{3.48}$$

Multiplying the equation in (3.38) by v' and integrating by parts, we have

$$\begin{aligned} &|v'|_{V_0^*}^2+\varepsilon|v'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v|_2^2 \\ &=(-\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))+a_0],v')_0 \\ &\quad+\langle F_0^{-1}\pi_0[f],v'\rangle_{V_0,V_0^*} \\ &\le\big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))]\big|_2\,|v'|_2 \\ &\quad+|\pi_0[f]|_{V_0^*}|v'|_{V_0^*} \\ &\le\frac{\varepsilon}{2}|v'|_2^2+\frac{1}{2\varepsilon}\,\big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))]\big|_2^2 \\ &\quad+\frac12|v'|_{V_0^*}^2+\frac12|\pi_0[f]|_{V_0^*}^2, \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac12|v'|_{V_0^*}^2+\frac{\varepsilon}{2}|v'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v|_2^2 \\ &\le\frac{1}{2\varepsilon}\big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2 \\ &\quad+a_1(\bar v(\tau)+m(\tau))]\big|_2^2+\frac12|\pi_0[f]|_{V_0^*}^2. \end{aligned} \tag{3.49}$$

It follows from (3.48) that

$$\begin{aligned} &\frac12|v'|_{V_0^*}^2+\frac{\varepsilon}{2}|v'|_2^2+\frac{\kappa}{2}\frac{d}{dt}|\nabla v|_2^2 \\ &\le\frac{1}{2\varepsilon}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+\frac12|\pi_0[f]|_{V_0^*}^2. \end{aligned} \tag{3.50}$$

Integrating (3.50) on $(0,\omega)$ and from (3.11), we obtain

$$\begin{aligned} &\frac12\int_0^\omega|v'(\tau)|_{V_0^*}^2d\tau+\frac{\varepsilon}{2}\int_0^\omega|v'(\tau)|_2^2d\tau \\ &\le\frac{\omega}{2\varepsilon}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+\frac{\omega C_1^2}{2}\|f\|_{L^\infty(0,\omega;H)}^2, \end{aligned}$$

which is (3.39).

Multiplying the equation of (3.38) by v and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v|_{V_0^*}^2+\frac{\varepsilon}{2}\frac{d}{dt}|v|_2^2+\kappa|\nabla v|_2^2\\
&=(-\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2\\
&\quad +a_1(\bar v(\tau)+m(\tau))+a_0],v)_0+\langle F_0^{-1}\pi_0[f],v\rangle_{V_0,V_0^*}\\
&\le \big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))]\big|_2|v|_2\\
&\quad +|\pi_0[f]|_{V_0^*}|v|_{V_0^*}\\
&\le \frac{\kappa}{2}|\nabla v|_2^2+\frac{C_2^2}{\kappa}\Big|\pi_0[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2\\
&\quad +a_1(\bar v(\tau)+m(\tau))]\Big|_2^2+\frac{C_1^2C_2^2}{\kappa}|\pi_0[f]|_{V_0^*}^2,
\end{aligned}\tag{3.51}$$

where the last two inequality signs follow from Young's inequality and (2.2). Combining (3.51) with (3.48), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v|_{V_0^*}^2+\frac{\varepsilon}{2}\frac{d}{dt}|v|_2^2+\frac{\kappa}{2}|\nabla v|_2^2\\
&\le \frac{C_2^2}{\kappa}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+\frac{C_1^2C_2^2}{\kappa}|\pi_0[f]|_{V_0^*}^2.
\end{aligned}\tag{3.52}$$

Integrating (3.52) on $(0,\omega)$, we obtain

$$\begin{aligned}
\frac{\kappa}{2}\int_0^\omega|\nabla v(\tau)|_2^2d\tau &\le \frac{\omega C_2^2}{\kappa}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\
&\quad +\frac{C_1^2C_2^2}{\kappa}\|\pi_0[f]\|_{L^2(0,\omega;V_0^*)}^2.
\end{aligned}\tag{3.53}$$

Therefore, from (2.2) and (3.11), we have

$$\begin{aligned}
\frac{\kappa}{2}\int_0^\omega|v(\tau)|_2^2d\tau &\le \frac{\kappa C_2^2}{2}\int_0^\omega|\nabla v(\tau)|_2^2d\tau\\
&\le \frac{\omega C_2^4}{\kappa}\Big\{\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+C_1^4\|f\|_{L^\infty(0,\omega;H)}^2\Big\},
\end{aligned}$$

which is (3.40).

For any $s,t\in[0,\omega]$ satisfying $s<t$, we integrate (3.50) on (s,t) and obtain

$$\begin{aligned}
&\frac{1}{2}\int_s^t|v'(\tau)|_{V_0^*}^2d\tau+\frac{\varepsilon}{2}\int_s^t|v'(\tau)|_2^2d\tau+\frac{\kappa}{2}|\nabla v(t)|_2^2\\
&\le \frac{\omega}{2\varepsilon}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+\frac{1}{2}\int_0^\omega|\pi_0[f(\tau)]|_{V_0^*}^2d\tau+\frac{\kappa}{2}|\nabla v(s)|_2^2.
\end{aligned}\tag{3.54}$$

Deleting the first two terms on the left-hand side of (3.54) and integrating it on $(0,t)$ with respect to s, we have

$$\begin{aligned}
\frac{\kappa t}{2}|\nabla v(t)|_2^2 &\le \frac{\omega^2}{2\varepsilon}\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]+\frac{\omega}{2}\int_0^\omega|\pi_0[f(\tau)]|_{V_0^*}^2d\tau\\
&\quad +\frac{\kappa}{2}\int_0^\omega|\nabla v(s)|_2^2ds.
\end{aligned}$$

In view of (3.53), we obtain

$$\begin{aligned}\frac{\kappa t}{2}|\nabla v(t)|_2^2 &\le \omega\Big(\frac{\omega}{2\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+\Big(\frac{\omega}{2}+\frac{C_1^2C_2^2}{\kappa}\Big)\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)}.\end{aligned}\tag{3.55}$$

Let $t=\omega$, then (3.55) can be rewritten as

$$\begin{aligned}\frac{\kappa\omega}{2}|\nabla v(\omega)|_2^2 &\le \omega\Big(\frac{\omega}{2\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+\Big(\frac{\omega}{2}+\frac{C_1^2C_2^2}{\kappa}\Big)\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)},\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\kappa}{2}|\nabla v(\omega)|_2^2 &\le \Big(\frac{\omega}{2\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+\Big(\frac{1}{2}+\frac{C_1^2C_2^2}{\kappa\omega}\Big)\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)}.\end{aligned}\tag{3.56}$$

It follows from the periodic property that

$$\begin{aligned}\frac{\kappa}{2}|\nabla v(0)|_2^2 &\le \Big(\frac{\omega}{2\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+\Big(\frac{1}{2}+\frac{C_1^2C_2^2}{\kappa\omega}\Big)\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)}.\end{aligned}\tag{3.57}$$

Choosing $s=0$ in (3.54), by (3.57) we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^t|v'(\tau)|^2_{V_0^*}d\tau+\frac{\varepsilon}{2}\int_0^t|v'(\tau)|_2^2d\tau+\frac{\kappa}{2}|\nabla v(t)|_2^2\\ &\le\Big(\frac{\omega}{\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+\Big(1+\frac{C_1^2C_2^2}{\kappa\omega}\Big)\|\pi_0[f]\|^2_{L^2(0,\omega;V_0^*)}.\end{aligned}\tag{3.58}$$

Dropping the first two terms on the left-hand side of (3.58) and from (3.11), we obtain

$$\begin{aligned}\frac{\kappa}{2}|\nabla v(t)|_2^2 &\le \Big(\frac{\omega}{\varepsilon}+\frac{C_2^2}{\kappa}\Big)\left[3A_3^3(a_3^2+a_2^2+a_1^2)+|\Omega|(a_2^2+2a_1^2)\right]\\ &\quad+C_1^2\Big(\omega+\frac{C_1^2C_2^2}{\kappa}\Big)\|f\|^2_{L^\infty(0,\omega;H)},\end{aligned}$$

which is (3.41).

Now, multiplying the equation of (3.38) by $-\Delta_N v$ and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v|_2^2+\frac{\varepsilon}{2}\frac{d}{dt}|\nabla v|_2^2+\kappa|\Delta_N v|_2^2\\ &\le\left|\pi_0\left[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))\right]\right|_2|\Delta_N v|_2\\ &\quad+|\pi_0[f]|_{V_0^*}|\Delta_N v|_{V_0^*}\\ &\le\frac{1}{\kappa}\left|\pi_0\left[a_3(\bar v(\tau)+m(\tau))^3+a_2(\bar v(\tau)+m(\tau))^2+a_1(\bar v(\tau)+m(\tau))\right]\right|_2^2\\ &\quad+\frac{C_1^2}{\kappa}|\pi_0[f]|^2_{V_0^*}+\frac{\kappa}{2}|\Delta_N v|_2^2.\end{aligned}$$

After a proper arrangement and with the aid of (3.48), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|v|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}|\nabla v|_2^2 + \frac{\kappa}{2}|\Delta_N v|_2^2 \\ &\le \frac{1}{\kappa}\left[3A_3^3(a_3^2+a_2^2+a_1^2) + |\Omega|(a_2^2+2a_1^2)\right] + \frac{C_1^2}{\kappa}|\pi_0[f]|_{V_0^*}^2. \end{aligned} \tag{3.59}$$

Integrating (3.59) on $(0,\omega)$, we have

$$\begin{aligned} \frac{\kappa}{2}\int_0^\omega |\Delta_N v(\tau)|_2^2 dx \le \frac{\omega}{\kappa}&\left[3A_3^3(a_3^2+a_2^2+a_1^2) + |\Omega|(a_2^2+2a_1^2)\right] \\ &+ \frac{C_1^2}{\kappa}\int_0^\omega |\pi_0[f(\tau)]|_{V_0^*}^2 d\tau, \end{aligned}$$

i.e.,

$$\int_0^\omega |\Delta_N v(\tau)|_2^2 dx \le \frac{2\omega}{\kappa^2}\Big[3A_3^3(a_3^2+a_2^2+a_1^2) + |\Omega|(a_2^2+2a_1^2) + C_1^4\|f\|_{L^\infty(0,\omega;H)}^2\Big].$$

Therefore, we obtain (3.42). The proof is complete. □

Now, we prove Theorem 2.3 by the Schauder fixed point theorem.

Proof of Theorem 2.3. Based on Lemma 3.5, we define a set

$$Y_2 := \{\bar v \in Y_1 : \sup_{t\in[0,w]} \kappa|\nabla \bar v|_2^2 + \varepsilon|\bar v|_{W^{1,2}(0,\omega;H_0)}^2 + |\bar v|_{W^{1,2}(0,\omega;V_0^*)}^2 \le M\},$$

where

$$\begin{aligned} M :&= \Big[3\omega C_1^2 + \frac{2C_1^4C_2^2}{\kappa} + \frac{2\omega C_1^4C_2^4}{\kappa^2}(\varepsilon + C_1^2)\Big]\|f\|_{L^\infty(0,\omega;H)}^2 \\ &\quad + \Big[\frac{3\omega}{\varepsilon} + \frac{2C_2^2}{\kappa} + \frac{2\omega C_2^4}{\kappa^2}(\varepsilon + C_1^2)\Big]\left[3A_3^3(a_3^2+a_2^2+a_1^2) + |\Omega|(a_2^2+2a_1^2)\right]. \end{aligned}$$

We can see that Y_2 is a non-empty compact convex subset of $L^2(0,\omega;H_0)$. Since for any $\tau\in[0,\omega]$, we have

$$|\pi_0[f(\tau)]|_{H_0}^2 = \int_\Omega |f(x,\tau)|^2 dx - \frac{1}{|\Omega|}\Big(\int_\Omega f(x,\tau)dx\Big)^2 \le \|f\|_{L^\infty(0,\omega;H)}^2.$$

Thus, it follows from (3.48) and (H1) that

$$-\pi_0[a_3(\bar v(\tau)+m(\tau))^3 + a_2(\bar v(\tau)+m(\tau))^2 + a_1(\bar v(\tau)+m(\tau))] + \pi_0[f] \in L^\infty(0,\omega;H_0).$$

As a result, by Theorem 3.1, there exists a unique solution v_ε for each $\bar v\in Y_2$. From Lemma 3.5, it is clear that $v_\varepsilon\in Y_2$. Hence, the mapping S defined by $S(\bar v) = v_\varepsilon$ maps Y_2 into itself.

Next, we show that v_ε is a solution of (3.38) and that S is continuous in Y_2 with respect to the topology of $L^2(0,\omega;H_0)$. Let $\{\bar v_n\}$ be any convergent sequence in Y_2 with respect to the topology of $L^2(0,\omega;H_0)$. We denote the limit of $\{\bar v_n\}$ by $\bar v$. Let $\{v_n\}$ be the sequence of solutions corresponding to $\{\bar v_n\}$. It follows from lemma 3.5 that $v_n (n = 1,2,\dots)$ satisfy (3.39) and (3.40). Thus, we can find a v_ε and a subsequence of $\{v_n\}$ which is denoted by $\{v_{n_k}\}$ such that

$$v_{n_k} \to v_\varepsilon \quad \text{weakly in } W^{1,2}(0,\omega;H_0). \tag{3.60}$$

Therefore, by (3.41),

$$v_{n_k} \to v_\varepsilon \quad \text{in } L^2(0,\omega;H_0), \tag{3.61}$$

$$v'_{n_k} \to v'_\varepsilon \quad \text{weakly in } L^2(0,\omega;H_0). \tag{3.62}$$

Since $W^{1,2}(0,\omega;H_0)\cap L^\infty(0,\omega;V_0)\hookrightarrow C([0,\omega];H_0)$ [15, Section 8, Corollary 5] and the embedding is compact, it is clear that

$$v_{n_k}\to v_\varepsilon \quad \text{in } C([0,\omega];H_0). \tag{3.63}$$

Thus,

$$v_{n_k}(0)\to v_\varepsilon(0),\quad v_{n_k}(\omega)\to v_\varepsilon(\omega)\quad \text{in } H_0,$$

which implies v_ε satisfies the periodic condition

$$v_\varepsilon(0)=v_\varepsilon(\omega). \tag{3.64}$$

Since F_0^{-1} is linear and selfadjoint, it holds for any $\eta\in D(\Delta_N)$ that

$$\begin{aligned}\int_0^\omega (F_0^{-1}v'_{n_k}(\tau)-F_0^{-1}v'_\varepsilon(\tau),\eta)_0d\tau &=\int_0^\omega (F_0^{-1}(v'_{n_k}(\tau)-v'_\varepsilon(\tau)),\eta)_0d\tau\\ &=\int_0^\omega (v'_{n_k}(\tau)-v'_\varepsilon(\tau),F_0^{-1}\eta)_0d\tau.\end{aligned} \tag{3.65}$$

Therefore, from (3.62), we know that

$$F_0^{-1}v'_{n_k}\to F_0^{-1}v'_\varepsilon \quad \text{weakly in } L^2(0,\omega;H_0) \text{ as } k\to\infty. \tag{3.66}$$

Also, for any $\eta\in D(\Delta_N)$, we have

$$\begin{aligned}\Big|\int_0^\omega (-\Delta_N v_{n_k}-(-\Delta_N v_\varepsilon),\eta)_0d\tau\Big| &=\Big|\int_0^\omega (v_{n_k}-v_\varepsilon,-\Delta_N\eta)_0d\tau\Big|\\ &\le \|v_{n_k}-v_\varepsilon\|_{L^2(0,\omega;H_0)}|\Delta_N\eta|_2\omega^{1/2}.\end{aligned}$$

In view of (3.61), we obtain

$$-\Delta_N v_{n_k}\to -\Delta_N v_\varepsilon \quad \text{weakly in } L^2(0,\omega;H_0) \text{ as } k\to\infty. \tag{3.67}$$

For any $v(\tau)\in L^2(0,\omega;H_0)$, let $\psi(v(\tau))=\pi_0[a_3(v(\tau)+m(\tau))^3]$. Then for any $v_1(\tau),v_2(\tau)\in L^2(0,\omega;H_0)$,

$$\begin{aligned}&\int_0^\omega (\psi(v_1(\tau))-\psi(v_2(\tau)),v_1(\tau)-v_2(\tau))_0d\tau\\ &=a_3\int_0^\omega\int_\Omega [v_1(\tau)-v_2(\tau)]^2\Big\{\Big[(v_1(\tau)+m(\tau))+\frac12(v_2(\tau)+m(\tau))\Big]^2\\ &\quad+\frac34[v_2(\tau)+m(\tau)]^2\Big\}dx\,d\tau\ge 0.\end{aligned} \tag{3.68}$$

Therefore, ψ is a monotone operator in $L^2(0,\omega;H_0)$. Since $a_3(v(\tau)+m(\tau))^3$ is continuous with respect to $v(\tau)$, it is easy to prove that $\psi(\cdot)$ is hemicontinuous [1, Chapter II, Definition 1.3] in $L^2(0,\omega;H_0)$. In addition, it is clear to see that ψ is everywhere defined in $L^2(0,\omega;H_0)$. Thus, by [1, Chapter II, Theorem 1.3], $\psi(\cdot)$ is maximal monotone in $L^2(0,\omega;H_0)$.

It follows from the definition of Y_2 and (3.43) that

$$\|\pi_0[a_3(\bar v_{n_k}+m(\tau))^3]\|^2_{L^2(0,\omega;H_0)}\le a_3^2A_3^3\omega.$$

Thus, there exist a $\gamma(\tau)\in L^2(0,\omega;H_0)$ and a subsequence of $\{\bar v_{n_k}\}$ which is still denoted by $\{\bar v_{n_k}\}$ such that

$$\pi_0[a_3(\bar v_{n_k}(\tau)+m(\tau))^3]=\psi(\bar v_{n_k}(\tau))\to\gamma(\tau)\quad \text{weakly in } L^2(0,\omega;H_0) \tag{3.69}$$

as $k \to \infty$. It follows from (3.69), the maximal monotonicity of ψ and the [17, Theorem A], we obtain $\psi(\bar{v}) = \gamma$, i.e.,

$$\pi_0[a_3(\bar{v}_{n_k} + m(\tau))^3] \to \pi_0[a_3(\bar{v} + m(\tau))^3] \quad \text{weakly in } L^2(0,\omega;H_0). \tag{3.70}$$

From [15, Section 8, Corollary 5], we know that $W^{1,2}(0,\omega;H_0) \cap L^\infty(0,\omega;V_0) \hookrightarrow C([0,\omega];L^4(\Omega) \cap H_0)$ and the embedding is compact. Thus,

$$v_{n_k} \to v_\varepsilon \quad \text{in } C([0,\omega];L^4(\Omega) \cap H_0). \tag{3.71}$$

Since

$$\begin{aligned}
& a_2^2 \int_0^\omega \int_\Omega \left[(\bar{v}_{n_k}(\tau) + m(\tau))^2 - (\bar{v}(\tau) + m(\tau))^2\right]^2 \,dx\,d\tau \\
&= a_2^2 \int_0^\omega \int_\Omega (\bar{v}_{n_k}(\tau) - \bar{v}(\tau))^2 (\bar{v}_{n_k}(\tau) + \bar{v}(\tau) + 2m(\tau))^2 \,dx\,d\tau \\
&\le a_2^2 \int_0^\omega \Big[\int_\Omega (\bar{v}_{n_k}(\tau) - \bar{v}(\tau))^4 dx\Big]^{1/2} \Big[\int_\Omega (\bar{v}_{n_k}(\tau) + \bar{v}(\tau) + 2m(\tau))^4 dx\Big]^{1/2} d\tau \\
&\le a_2^2 \|\bar{v}_{n_k}(\tau) - \bar{v}(\tau))\|^2_{C([0,\omega];L^4(\Omega)\cap H_0)} \int_0^\omega \Big[\int_\Omega (\bar{v}_{n_k}(\tau) + \bar{v}(\tau) + 2m(\tau))^4 dx\Big]^{1/2} d\tau,
\end{aligned}$$

where the last second inequality sign follows from the Hölder inequality. From (3.71), we see that

$$a_2(\bar{v}_{n_k}(\tau) + m(\tau))^2 \to a_2(\bar{v}(\tau) + m(\tau))^2 \quad \text{in } L^2(0,\omega;H_0) \text{ as } k \to \infty.$$

Then, it is easy to prove that

$$\pi_0[a_2(\bar{v}_{n_k}(\tau) + m(\tau))^2] \to \pi_0[a_2(\bar{v}(\tau) + m(\tau))^2] \quad \text{in } L^2(0,\omega;H_0) \tag{3.72}$$

as $k \to \infty$. Similarly, (3.63) implies

$$\pi_0[a_1(\bar{v}_{n_k}(\tau) + m(\tau))] \to \pi_0[a_1(\bar{v}(\tau) + m(\tau))] \quad \text{in } L^2(0,\omega;H_0) \tag{3.73}$$

as $k \to \infty$.

Consequently, with the help of (3.62), (3.66), (3.67), (3.70), (3.72) and (3.73), for any $\eta \in D(\Delta_N)$, we take $k \to \infty$ on both sides of the equality

$$\begin{aligned}
& \int_0^\omega (F_0^{-1} v'_{n_k}(\tau), \eta)_0 d\tau + \int_0^\omega (\varepsilon v'_{n_k}(\tau), \eta)_0 d\tau + \int_0^\omega (-\kappa \Delta_N v_{n_k}(\tau), \eta)_0 d\tau \\
&= \int_0^\omega (\pi_0[g(\bar{v}_{n_k}(\tau) + m(\tau))], \eta)_0 d\tau + \int_0^\omega (F_0^{-1} f, \eta)_0 d\tau,
\end{aligned}$$

and then

$$\begin{aligned}
& \int_0^\omega (F_0^{-1} v'_\varepsilon(\tau), \eta)_0 d\tau + \int_0^\omega (\varepsilon v'_\varepsilon(\tau), \eta)_0 d\tau + \int_0^\omega (-\kappa \Delta_N v_\varepsilon(\tau), \eta)_0 d\tau \\
&= \int_0^\omega (\pi_0[g(\bar{v}(\tau) + m(\tau))], \eta)_0 d\tau + \int_0^\omega (F_0^{-1} f, \eta)_0 d\tau.
\end{aligned}$$

This implies that $v_\varepsilon = S(\bar{v})$ is a unique solution of (2.9). As a result, from (3.61), S is continuous in Y_2 with respect to the topology of $L^2(0,\omega;H_0)$. By the Schauder fixed theorem, we can see that S has at least one fixed point in Y_2. The proof is complete. □

4. Proof of main results

In this section, based on Lemma 3.4, we prove Theorem 2.2 by taking the limit as $\varepsilon \to 0$.

Proof of Theorem 2.2. By Lemma 3.4, we can see that the constants on the right-hand side of (3.7)-(3.10) are independent of ε. Thus, it follows from (3.7) and (3.8) that

$$v_\varepsilon \in W^{1,2}(0,\omega;V_0^*) \cap L^\infty(0,\omega;V_0), \tag{4.1}$$

and there exists a $v \in W^{1,2}(0,\omega;V_0^*) \cap L^\infty(0,\omega;V_0)$ such that

$$v_\varepsilon \to v \quad \text{weakly in } W^{1,2}(0,\omega;V_0^*) \text{ as } \varepsilon \to 0, \tag{4.2}$$

$$v_\varepsilon \to v \quad \text{weakly star in } L^\infty(0,\omega;V_0) \text{ as } \varepsilon \to 0, \tag{4.3}$$

$$v_\varepsilon \to v \quad \text{in } L^2(0,\omega;H_0) \text{ as } \varepsilon \to 0, \tag{4.4}$$

$$v_\varepsilon' \to v' \quad \text{weakly in } L^2(0,\omega;V_0^*) \text{ as } \varepsilon \to 0. \tag{4.5}$$

Since $W^{1,2}(0,\omega;V_0^*)\cap L^\infty(0,\omega;V_0) \hookrightarrow C([0,\omega];H_0)$ [15, Section 8, Corollary 5] and the embedding is compact, it follows from (4.2) and (4.3) that

$$v_\varepsilon \to v \quad \text{in } C([0,\omega];H_0) \text{ as } \varepsilon \to 0.$$

Thus,

$$v_\varepsilon(0) \to v(0), \quad v_\varepsilon(\omega) \to v(\omega) \quad \text{in } H_0,$$

which implies

$$v(0) = v(\omega). \tag{4.6}$$

Similarly as (3.65), for any $\eta \in D(\Delta_N)$, we have

$$\int_0^\omega \langle F_0^{-1} v_\varepsilon'(\tau) - F_0^{-1} v'(\tau), \eta\rangle_{V_0,V_0^*} d\tau = \int_0^\omega \langle v_\varepsilon'(\tau) - v'(\tau), F_0^{-1}\eta\rangle_{V_0^*,V_0} d\tau.$$

By (4.5), we obtain

$$F_0^{-1} v_\varepsilon' \to F_0^{-1} v' \quad \text{weakly in } L^2(0,\omega;V_0) \text{ as } \varepsilon \to 0. \tag{4.7}$$

Furthermore, for any $\eta \in D(\Delta_N)$, we have

$$\begin{aligned}\int_0^\omega (-\Delta_N v_\varepsilon(\tau) - (-\Delta_N v(\tau)), \eta)_0 d\tau &= \int_0^\omega (v_\varepsilon(\tau) - v(\tau), -\Delta_N \eta)_0 d\tau \\ &\le \|v_\varepsilon - v\|_{L^2(0,\omega;H_0)} |\Delta_N \eta|_2 \omega^{1/2}.\end{aligned}$$

It follows from (4.4) that

$$-\Delta_N v_\varepsilon \to -\Delta_N v \quad \text{weakly in } L^2(0,\omega;H_0) \text{ as } \varepsilon \to 0. \tag{4.8}$$

From (3.10) and the similar arguments as (3.70), we obtain

$$\pi_0[a_3(v_\varepsilon(\tau) + m(\tau))^3] \to \pi_0[a_3(v(\tau) + m(\tau))^3] \quad \text{weakly in } L^2(0,\omega;H_0) \tag{4.9}$$

as $\varepsilon \to 0$.

By [15, Section 8, Corollary 5], we see that both $W^{1,2}(0,\omega;V_0^*)\cap L^\infty(0,\omega;V_0) \hookrightarrow C([0,\omega];L^4(\Omega)\cap H_0)$ and $W^{1,2}(0,\omega;V_0^*)\cap L^\infty(0,\omega;V_0) \hookrightarrow C([0,\omega];H_0)$ are compact. Therefore, being similar as (3.72) and (3.73), we have

$$\pi_0[a_2(v_\varepsilon(\tau) + m(\tau))^2] \to \pi_0[a_2(v(\tau) + m(\tau))^2] \quad \text{in } L^2(0,\omega;H_0) \text{ as } k \to \infty \tag{4.10}$$

and

$$\pi_0[a_1(v_\varepsilon(\tau) + m(\tau))] \to \pi_0[a_1(v(\tau) + m(\tau))] \quad \text{in } L^2(0,\omega;H_0) \text{ as } k \to \infty. \tag{4.11}$$

With the help of (4.7)-(4.11), for any $\eta \in D(\Delta_N)$, we take $\varepsilon \to 0$ on both sides of the equation

$$\begin{aligned} &\varepsilon \int_0^\omega \langle v'_\varepsilon(\tau), \eta \rangle_{V_0^*, V_0} d\tau \\ &= \int_0^\omega \Big(F_0^{-1} f(\tau) - \pi_0[g(v_\varepsilon(\tau) + m(\tau))] + \kappa \Delta_N v_\varepsilon(\tau), \eta \Big)_0 d\tau \\ &\quad - \int_0^\omega \langle F_0^{-1} v'_\varepsilon(\tau), \eta \rangle_{V_0, V_0^*} d\tau, \end{aligned}$$

and then get that for any $\eta \in D(\Delta_N)$,

$$\begin{aligned} 0 &= \int_0^\omega (F_0^{-1} f(\tau) - \pi_0[g(v(\tau) + m(\tau))] + \kappa \Delta_N v(\tau), \eta)_0 d\tau \\ &\quad - \int_0^\omega \langle F_0^{-1} v'(\tau), \eta \rangle_{V_0, V_0^*} d\tau \end{aligned}$$

holds, which together with (4.6) implies that v is a solution of the problem

$$\begin{gathered} F_0^{-1} v'(\tau) - \kappa \Delta_N v(\tau) + \pi_0[g(v(\tau) + m(\tau))] = F_0^{-1} f(\tau), \quad 0 < \tau < \omega, \\ v(0) = v(\omega). \end{gathered}$$

As a result, from the equivalence of (1.2) and (2.8), we know that $u(x,\tau) = v(x,\tau) + m(\tau)$ is the solution of (1.2)-(1.4). $\square$

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